TECHNISCHE UNIVERSITÄT DRESDEN

# AFFINE PROCESSES WITH STOCHASTIC DISCONTINUITIES

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joint work with M. Keller-Ressel and T. Schmidt

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$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$ a filtered probability space with an adapted process $X$. The jumps of the process $X$ can be exhausted by

- predictable, i.e. the limit of a sequence of stopping times (announcing times)
- accessible, i.e. $\exists\, (\tau_n)_{n\in\mathbb{N}}$ stopping times:

$$\mathbb{P}\left(\text{"jump time"} = \tau_n \text{ for some } n\right) = 1$$

- totally inaccessible, for all predictable times $\tau$

$$\mathbb{P}\left(\text{"jump time"} = \tau\right) = 0$$

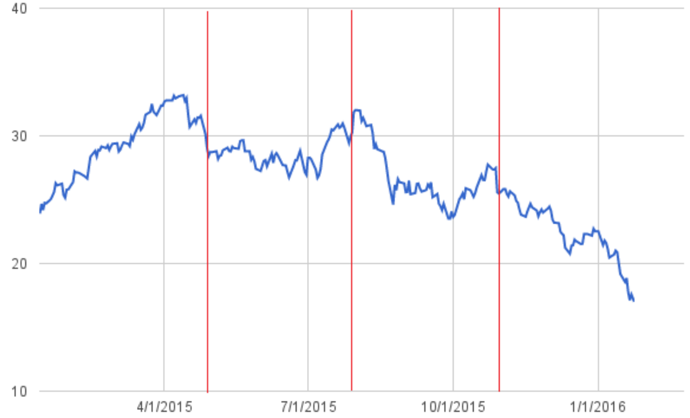


Figure: Daily closing price of the Deutsche Bank stock starting january 1st 2015

# Introduction –A Glance at Financial Time Series

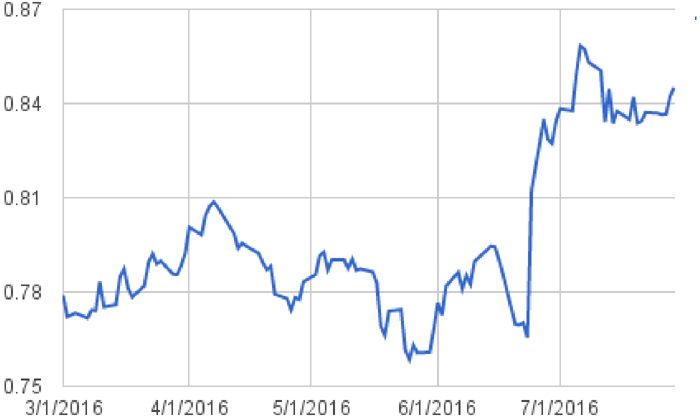


Figure: EUR GBP exchange rate starting from march 1st 2016

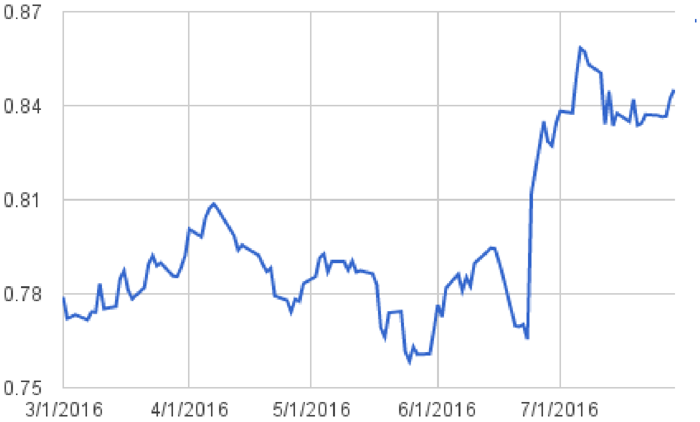


Figure: EUR GBP exchange rate starting from march 1st 2016

I still believe in (predictable) jumps

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space, and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

**Definition**

A $d-$dimenisonal semimartingale $X$ on $D$ (e.g.$\mathbb{R}^m_{\geq} \times \mathbb{R}^n$) is called an affine semimartingale if there exist $\mathbb{C}$ and $\mathbb{C}^d$-valued (deterministic) functions $\phi_s(t,u)$ and $\psi_s(t,u)$, respectively, such that

$$\mathbb{E}\left[\mathrm{e}^{\langle u, X_t \rangle} | \mathcal{F}_s\right] = \exp\left(\phi_s\left(t,u\right) + \langle \psi_s\left(t,u\right), X_s \rangle\right)$$

hold for all $u \in \mathcal{U}$,$0 \leq s \leq t$ and $x \in D$. If $\phi_s\left(t,u\right) = \phi_{t-s}\left(u\right)$ and $\psi_s\left(t,u\right) = \psi_{t-s}\left(u\right)$ for all $u \in i\mathbb{R}^d$,$0 \leq s \leq t$ the process $X$ is called time homogeneous

$$\mathcal{U} := \left\{ u \in \mathbb{C}^d : \langle \Re u, x \rangle \leq 0 \text{ for all } x \in D \right\}$$

Let $X$ be an affine Semimartinglae satisfying the following

**Assumption (Full support)**

*Let $X$ satisfy* $\mathrm{conv}(\mathrm{supp}(X_t) = D$ *for all* $t > 0$

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**Assumption (Quasi-Regularity)**

*Let $\phi$, $\psi$ satisfy*

1. *$\phi$ and $\psi$ are of finite variation in $s$ and both cádlág in $s$ and $t$.*
2. *For all $0 < s \leq t$ the functions*

$$u \mapsto \phi_{s-}(t,u) \quad \text{and} \quad \psi_{t-}(t,u)$$

*are continuous on $\mathcal{U}$*

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- in contrast to [Filipovic (2005)] and [Duffie et al.(2003)] we don't require stochastic continuity
- let $J = \{s \in \mathbb{R}_{\geq 0} | \mathbb{P}(\Delta X_s \neq 0) > 0\}$

## Theorem 1

*There exist parameters* $A, (\gamma_i, \beta_i, \alpha_i, \mu_i)_{i\in\{0,\cdots,d\}}$, $A : \mathbb{R}_\geq \to \mathbb{R}_\geq$, *increasing and cádlág,* $\gamma_i : \mathbb{R}_\geq \times \mathcal{U} \to \mathcal{U}$, $\beta_i : \mathbb{R}_\geq \to \mathbb{R}^d$, $\alpha_i : \mathbb{R}_\geq \to \mathcal{S}_+^d\left(R^d\right)$ *and Borel measures* $\left(\mu_i\left(t,\cdot\right)\right)_i$ *on* $D \setminus \{0\}$ *s.t.* $\int_{D\setminus\{0\}} (1+|x|)\, \mu_i\left(t, dx\right) \leq \infty$ *for all* $t \in \mathbb{R}_\geq$

1. *The Semimartingale characteristics* $(B, C, \nu)$ *of* $X$ *are of affine form :*

$$B_t = \int_0^t \beta_0\left(s\right) + \left\langle \bar{\beta}\left(s\right), X_{s-}\right\rangle \mathrm{d}A_s$$

$$C_t = \int_0^t \alpha_0\left(s\right) + \left\langle \bar{\alpha}\left(s\right), X_{s-}\right\rangle \mathrm{d}A_s$$

$$\nu^c\left(\mathrm{d}s, \mathrm{d}x\right) = \left(\mu_0\left(s, \mathrm{d}x\right) + \left\langle X_{s-}, \bar{\mu}\left(s, \mathrm{d}x\right)\right\rangle\right) \mathrm{d}A_s$$

$$\int_D e^{\langle u,\xi\rangle} \nu^d\left(\omega, \{t\}, \mathrm{d}\xi\right) = \exp\left(\gamma_0\left(t, u\right) + \left\langle \bar{\gamma}, X_{t-}\right\rangle\right)$$

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1. *The Semimartingale characteristics* $(B, C, \nu)$ *of* $X$ *are of affine form*
2. *The continuous parts of* $\phi$ *and* $\psi$ *satisfy Riccati equations*

$$\frac{\mathrm{d}\phi^c_s (t, u)}{\mathrm{d}A^c_s} = -R_0 (s, \psi_{s-} (t, u)), \quad \phi_t (t, u) = 0$$

$$\frac{\mathrm{d}\psi^c_s (t, u)}{\mathrm{d}A^c_s} = -\bar{R} (s, \psi_{s-} (t, u)), \quad \psi_t (t, u) = u$$

$$R_i (s, u) = \langle \beta_i (s), u \rangle + \frac{1}{2} \langle u, \alpha_i (s) \cdot u \rangle + \int_D \left( \mathrm{e}^{\langle x, u \rangle} - 1 - \langle u, h(x) \rangle \right) \mu_i (s, \mathrm{d}x)$$

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2. *The continuous parts of* $\phi$ *and* $\psi$ *satisfy Riccati equations*
3. *The Jumps of* $\phi$ *and* $\psi$ *are determined by* $\gamma$

$$\begin{aligned} \Delta\phi_s(t, u) &= -\gamma_0(s, \psi_s(t, u)) \\ \Delta\psi_s(t, u) &= -\bar{\gamma}(s, \psi_s(t, u)), \quad s \in J \end{aligned}$$

we call the equations in (2) together with those in (3) generalized measure Riccati equations

For $X$ with semimartingale triplet $(B, C, \nu)$ we define a complex valued random measure on $[0, t]$ by (with $\psi_{s-} := \psi_{s-}(t, u)$),

$$G(ds, \omega, t, u) := \langle \psi_{s-}, dB_s(\omega) \rangle + \frac{1}{2} \langle \psi_{s-}, dC_s(\omega) \psi_{s-} \rangle + \int_D e^{\langle \psi_{s-}, \xi \rangle} - 1 - \langle \psi_{s-}, h(\xi) \rangle \nu^c(\omega, dt, d\xi)$$

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with an application of Itôs formula:

$$d\phi^c_{s-}(t, u) + \left\langle X_{s-}(\omega), d\psi^c_{s-}(t, u) \right\rangle = -G(ds, \omega, t, u)$$

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with an application of Itôs formula:

$$\underbrace{\begin{pmatrix} 1 & X_{s-}^{x^0}(\omega) \\ \vdots & \vdots \\ 1 & X_{s-}^{x_d}(\omega) \end{pmatrix}}_{\Theta_{s-}(\omega)} \cdot \underbrace{\begin{pmatrix} d\phi_{s-}^{c}(t, u) \\ d\psi_{s-}^{c,1}(t, u) \\ \vdots \\ d\psi_{s-}^{c,d}(t, u) \end{pmatrix}}_{d\Psi_{s-}^{c}(t,u)} = -\underbrace{\begin{pmatrix} G_0(ds; \omega, t, u) \\ \vdots \\ G_d(ds; \omega, t, u) \end{pmatrix}}_{\mathcal{G}(ds;\omega,t,u)}$$

Invert $\Theta_{s-}(\omega)$ (possible for $(s,\omega) \in [\tau-\varepsilon,\tau] \times E$, $\mathbb{P}(E) > 0$):

$$d\Psi_s^c(t,u) = \Theta_{s-}(\omega)^{-1} \cdot \mathcal{G}(ds;\omega,t,u)$$

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We can disintegrate the characteristic triplet of $X$:

$$\begin{aligned} B_t^c &= \int_0^t b_s d\mathcal{A}_s \\ C_t &= \int_0^t c_s d\mathcal{A}_s \\ \nu^c(\omega, dt, dx) &= K_{\omega,t} d\mathcal{A}_t(\omega) \end{aligned}$$

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Take $\omega^* \in E$ set $A_s = \mathcal{A}_s(\omega^*)$

$$d\Psi_s^c(t,u) \ll dA_s$$

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Take $\omega^* \in E$ set $A_s = \mathcal{A}_s(\omega^*)$

$$d\Psi_s^c(t,u) \ll dA_s$$

define

$$\beta(s) := \Theta_{s-}(\omega^*)^{-1} \cdot b_s(\omega^*)$$
$$\alpha(s) := \Theta_{s-}(\omega^*)^{-1} \cdot c_s(\omega^*)$$
$$\mu(s,dx) := \Theta_{s-}(\omega^*)^{-1} \cdot K_s(\omega^*,dx).$$

On the canonical state space $D := \mathbb{R}^m_{\geq} \times \mathbb{R}^n$:

## Theorem 2

*Let $A$ be non-decreasing and cádlág and let $(\alpha, \beta, \nu, \gamma)$ be some strongly admissible parameters w.r.t. $A$. Then there exists a unique quasi-regular affine semimartingale $X$ (starting at $X_0 \in D$) with*

$$\mathbb{E}\left[\mathrm{e}^{\langle u, X_t \rangle} | \mathcal{F}_s\right] = \exp\left(\phi_s\left(t, u\right) + \langle \psi_s\left(t, u\right), X_s \rangle\right)$$

*Where $\phi$ and $\psi$ satisfy the generalized measure Riccati equations*

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*Where $\phi$ and $\psi$ satisfy the generalized measure Riccati equations*

$$\begin{aligned}
\frac{\mathrm{d}\phi^c_s(t,u)}{\mathrm{d}A^c_s} &= -R_0(s, \psi_{s-}(t,u)), \quad \phi_t(t,u) = 0 \\
\frac{\mathrm{d}\psi^c_s(t,u)}{\mathrm{d}A^c_s} &= -\bar{R}(s, \psi_{s-}(t,u)), \quad \psi_t(t,u) = u \\
\Delta\phi_s(t,u) &= -\gamma_0(s, \psi_s(t,u)) \\
\Delta\psi_s(t,u) &= -\bar{\gamma}(s, \psi_s(t,u)), \quad s \in J.
\end{aligned}$$

## Definition (From Duffie et al. (2003))

The parameters $(\alpha, \beta, \mu)$ are called *admissible*, if

- $\alpha_0 \in \mathrm{Sem}^d$ with $\alpha_{0;\mathcal{I}\mathcal{I}} = 0$,
- $\alpha_i \in \mathrm{Sem}^d$ with $\alpha_{i;\mathcal{I}\setminus i,\mathcal{I}\setminus i} = 0$,
- $\beta_0 \in D$,
- $\bar{\beta}_{\mathcal{I}\mathcal{J}} = 0$ and $\beta_{i;\mathcal{I}\setminus i} \in \mathbb{R}^{d-1}_{\geq 0}$ for all $i \in \mathcal{I}$,
- $\mu_i = 0$ for all $t \geq 0$ for $i \in \mathcal{J}$
- for $i \in \mathcal{I} \cup \{0\}$, $\mu_i$ is a Borel measure on $D \setminus \{0\}$ satisfying $\mathcal{M}_i(D \setminus \{0\}) < \infty$ with

  $$\mathcal{M}_i(d\xi) := \left(\left\langle h_{\mathcal{I}\setminus i}(\xi), 1\right\rangle + \left\| h_{\mathcal{J}\setminus i}(\xi)\right\|^2\right) \mu_i(d\xi)$$

  and the continuous truncation function $h \colon \mathbb{R}^d \to [-1, 1]^d$

  $$h_k(\xi) = \begin{cases} 0, \xi_k = 0 \\ (1 \wedge |\xi|)\frac{\xi_k}{|\xi_k|}, \text{otherwise} \end{cases}$$

Let $A$ be non-decreasing and cádlág.

**Definition**

$(\alpha, \beta, \mu)$ are called strong admissibility w.r.t. A, if $(\alpha(t), \beta(t), \mu(t, \cdot))$ are admissible for $A$-a.e. $t$ and additionally

- $(\alpha(t), \beta(t), \mathcal{M}(t, D \setminus \{0\}))_{t \geq 0}$ are locally integrable with respect to $A$,
- $\bar{\gamma}$ is of Lévy-Khintchine form for $t \in J = \{t \in \mathbb{R}_{\geq} | \Delta A \neq 0\}$, i.e. for $i = 1, \cdots, d$

$$\begin{aligned} \gamma_i(t, u) &= \langle \beta_i(t), u \rangle + \frac{1}{2} \langle u, \alpha_i(t) \cdot u \rangle \\ &\quad + \int_D \left( \mathrm{e}^{\langle x, u \rangle} - 1 - \langle u, h(x) \rangle \right) \mu_i(t, \mathrm{d}x) \end{aligned}$$

- $\gamma_0(t, \cdot)$, $t \in J$ is a log-characteristic function of a random variable supported on $D$ (and locally summable in t locally uniformly on $\mathcal{U}$)

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Affine processes and applications in finance.
*Ann. Appl. Probab.*, 13:984–1053, 2003.

D. Filipović.
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*Stochastic Process. Appl.*, 115:639–659, 2005.

D. Filipović.
*Term-Structure Models: A Graduate Course.*
Springer, 2009.

M. Keller-Ressel.
*Affine Processes: Theory and Applications to Finance.*
PhD thesis, TU Vienna, 2008.

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Dynamic Defaultable Term Structure Modelling beyond the Intensity Paradigm.
*Mathematical Finance*, 2016.

Let $X$ be a stochastically continuous affine semimartingale on $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ with characteristics $\phi$ and $\psi$, e.g. Brownian motion, and let $\{t_1, \dots, t_N\}$ some time points, $a^i \in \mathbb{R}^d$ and $b^i \in \mathbb{R}^d$ s.t. $a^i + b^i \cdot x \in D$ for all $x \in D$, $i = 1, \dots, N$.

$$\tilde{X}_t = X_t + \sum_{i=1}^{N} \mathbb{1}_{\{t \geq t_i\}} \left( a^i + b^i \cdot X_{t_i} \right)$$

is an affine semimartingale with characteristics $\tilde{\phi}$ and $\tilde{\psi}$ given via the recursion, for $s \leq t_{k-l} \leq \cdots \leq t_k \leq t$ and $u \in i\mathbb{R}^d$

$$\phi^0(u) = \phi_{t_k,t}(u), \qquad \phi^{i+1} = \phi^i(u) + \phi_{t_{k-i-1},t_{k-i}}\left(\psi^i(u) + u \cdot b^{k-i}\right) + \left\langle u, a^{k-i} \right\rangle$$

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then $\tilde{\psi}(s,t,u) = \psi_{s,t_{k-l}}\left(\psi^l(u) + u \cdot b^{k-l}\right)$ and similar for $\tilde{\phi}(s,t,u)$.