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On sharp maximal inequalities for stochastic processes

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TOPIC I: Sharp maximal inequalities for continuous time processes

§1. Introduction. The main method for obtaining a sharp maximal inequalities

Let $X = (X_t)_{t \geqslant 0}$ be a process on $(\Omega, \mathcal{F}, \mathsf{P})$ with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$, $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$. For any Markov time τ w.r.t. $(\mathcal{F}_t)_{t \geqslant 0}$ the inequalities

$$\boxed{\mathsf{E}\left(\sup_{0 \leqslant t \leqslant \tau} X_t\right) \leqslant C_X \cdot f(\mathsf{E} g(\tau))} \qquad (*)$$

are called **maximal inequalities** for X. Here C_X is a constant, $f(\cdot)$ and $g(\cdot)$ are some functions.

Markov times $\tau = \tau(\omega)$ usually belong to the set

$$\mathfrak{M} = \{\tau - \text{Markov time w.r.t. } (\mathcal{F}_t)_{t \geqslant 0}, \ \mathsf{E}\tau < \infty\}.$$

The inequality (*) is called **sharp maximal inequality** if there exist a non-trivial Markov time $\widehat{\tau} \in \mathfrak{M}$ such that $\mathsf{E}\left(\sup_{0 \leqslant t \leqslant \widehat{\tau}} X_t\right) = C_X \cdot f(\mathsf{E} g(\widehat{\tau}))$.

Examples of maximal inequalities for some well-known processes include (Graversen, Peskir, Shiryaev 1998–2001):

- for **geometric Brownian motion** $X_t = \exp(\sigma B_t + (\mu - \sigma^2/2)t)$ with $\mu < 0$, $\sigma > 0$:

$$\mathsf{E}\left(\max_{\mathbf{0} \leqslant \mathbf{t} \leqslant \tau} \mathbf{X_t}\right) \leqslant \mathbf{1} - \frac{\sigma^{\mathbf{2}}}{\mathbf{2}\mu} + \frac{\sigma^{\mathbf{2}}}{\mathbf{2}\mu} \exp\left(-\frac{(\sigma^{\mathbf{2}} - \mathbf{2}\mu)^{\mathbf{2}}}{\mathbf{2}\sigma^{\mathbf{2}}} \mathsf{E}\tau - \mathbf{1}\right);$$

- for **Ornstein-Uhlenbeck process** $(X_t)_{t \geqslant 0}$ with $dX_t = -\beta X_t dt + dB_t$, $\beta > 0$:

$$\frac{\mathbf{C_1}}{\sqrt{\beta}} \mathsf{E}\sqrt{\ln(1 + \beta\tau)} \leqslant \mathbf{E}\left(\max_{\mathbf{0} \leqslant \mathbf{t} \leqslant \tau} |\mathbf{X_t}|\right) \leqslant \frac{\mathbf{C_2}}{\sqrt{\beta}} \mathsf{E}\sqrt{\ln(1 + \beta\tau)},$$

  where $C_1, C_2 > 0$ are some universal constants;

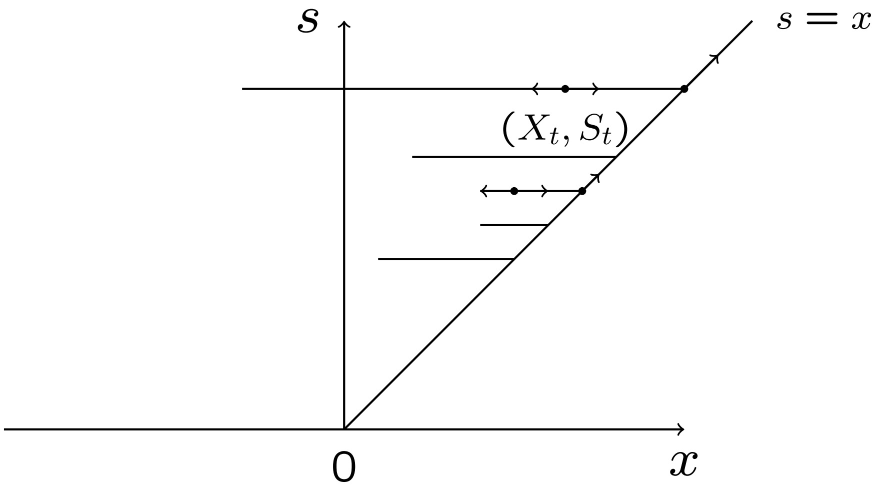
- for **"bang-bang process"** $(X_t)_{t \geqslant 0}$ with $dX_t = -\mu\, \mathrm{sgn}(X_t) dt + dB_t$, $\mu > 0$:

$$\mathsf{E}\left(\max_{\mathbf{0} \leqslant \mathbf{t} \leqslant \tau} |\mathbf{X_t}| \right) \leqslant \mathbf{G}_\mu(\mathsf{E}\tau),$$

where $G_\mu(x) = \inf_{c>0} \left(cx + \frac{1}{2\mu} \ln \left(1 + \frac{\mu}{c} \right) \right)$.

Assume that $X = (X_t)_{t \geqslant 0}$ is the Markov process. For given measurable functions $L = L(x)$ and $K = K(x)$ we define

$$I_t = \int_0^t L(X_s)ds, \qquad S_t = \max_{0 \leqslant s \leqslant t} K(X_s), \qquad t \geqslant 0.$$



Consider the following optimal stopping problem:

$$\mathbf{V}_*(\mathbf{c}) = \sup_\tau \mathsf{E}\left(\mathbf{F}(\mathbf{I}_\tau, \mathbf{X}_\tau, \mathbf{S}_\tau) - \mathbf{c}\,\mathbf{G}(\mathbf{I}_\tau, \mathbf{X}_\tau, \mathbf{S}_\tau)\right), \qquad (1)$$

where F, G are given measurable functions, $\tau \in \mathfrak{M}$, $c > 0$ is a parameter.

Suppose we solved the problem (1) and found the function $V_*(c)$. Then for any τ and c we have

$$\mathsf{E}F(I_\tau, X_\tau, S_\tau) \leqslant V_*(c) + c\,\mathsf{E}G(I_\tau, X_\tau, S_\tau))$$

Taking the infimum on both sides by $c > 0$ we obtain the inequality

$$\mathbf{\mathsf{E}F(I_\tau, X_\tau, S_\tau) \leqslant H(\mathsf{E}G(I_\tau, X_\tau, S_\tau)) := \inf_{c>0} (V_*(c) + c\,\mathsf{E}G(I_\tau, X_\tau, S_\tau))} \tag{2}$$

which is true for any Markov time $\tau \in \mathfrak{M}$. If infimum is minimum and it is achieved on some $c_* > 0$ then inequality (2) is **sharp**.

The corresponding solution $\tau_*(c)$ of problem (1) when $c = c_*$ is a stopping time on which (2) becomes an equality.

Consider the particular case $F(x,y,z) = z$, $G(x,y,z) = x$, $L(x) = c(x)$, $K(x) = x$. The function $c = c(x)$ is assumed to be positive and continuous and it is called **cost for observations**. We obtain the following optimal stopping problem:

$$\boxed{V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s}\left(S_\tau - \int_0^\tau c(X_t)dt\right),} \tag{3}$$

where

- $\mathsf{E}_{s,x}$, $s \geqslant x$ is expectation under the measure $\mathsf{P}_{x,s} = \mathsf{Law}(X, S \,|\, \mathsf{P}, X_0 = x, S_0 = s)$

- τ is the optimal stopping time such that $\mathsf{E}_{x,s}\left(\int_0^\tau c(X_t)dt\right) < \infty$

In addition we assume that $X = (X_t)_{t \geqslant 0}$ is a diffusion process and it is a solution of stochastic differential equation

$$\boxed{dX_t = b(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = 0,}$$

where $B = (B_t)_{t \geqslant 0}$ is the Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$. Diffusion coefficient $\sigma = \sigma(x) > 0$ and drift coefficient $b = b(x)$ are **continuous**.

We need to know a **scale function** $R = R(x)$ and a **speed measure** $m = m(x)$ in order to obtain a solution of the problem (3). It is well known that in the case of diffusion process X we have

$$R(x) = \int^{x} \mathbf{exp}\left(- \int^{y} \frac{2b(u)}{\sigma^2(u)} du \right) dy, \qquad x \in \mathbb{R},$$

$$m(dx) = \frac{2dx}{R'(x)\sigma^2(x)}$$

From the general optimal stopping theory we may decompose the **state space** $E = \{(x,s) \in \mathbb{R}^2 : x \leqslant s, s \geqslant 0\}$ of the process (X,S), $S_t = (\max_{u \leqslant t} X_u) \vee s$ by

$$\boxed{E = C_* \cup D_*,}$$

where

- $C_* = \{(x,s) \in E : V_*(x,s) > s\}$ is a **continuation set**. If $(x,s) \in C_*$ we need to continue our observations;
- $D_* = \{(x,s) \in E : V_*(x,s) = s\}$ is a **stopping set**. If $(x,s) \in D_*$ we need to stop our observations

Therefore if we start in C_* we need to stop at the first time when the process (X,S) reaches D_*. In other words, $\tau_* = \inf\{t \geqslant 0 : (X_t, S_t) \in D_*\}$.

Proposition 1. ***The diagonal*** $\{(x,s) \in E : x = s\}$ ***does not belong to the continuation set*** C_***.**

Reduce the problem $V_*(x,s) = \sup_\tau \mathsf{E}_{x,s}\left(S_\tau - \int\limits_0^\tau c(X_t)dt\right)$ to the optimal stopping problem in standard formulation. Consider the process

$$A_t = a + \int_0^t c(X_u)du, \quad a \geqslant 0$$

and observe that $Z_t = (A_t, X_t, S_t)$, $t > 0$, $Z_0 = (a, x, s)$ is a Markov process. Define the function $\widetilde{G}(a,x,s) = s - a$ and observe that the initial problem takes the form

$$\widetilde{\mathbf{V}}_*(\mathbf{a},\mathbf{x},\mathbf{s}) = \sup_\tau \mathsf{E}_{\mathbf{a},\mathbf{x},\mathbf{s}} \widetilde{\mathbf{G}}(\mathbf{Z}_\tau),$$

where times τ are such that $\mathsf{E}A_\tau < \infty$. However since $\widetilde{V}_*(a,x,s) = V_*(x,s) - a$ it is sufficient to find the function $V_*(x,s)$ i.e. to solve **2-dimensional optimal stopping problem** for (X,S).

The infinitesimal operator of the process $Z = (Z_t)_{t \geqslant 0}$ equals

$$\mathbb{L}_Z = c(x)\frac{\partial}{\partial a} + \mathbb{L}_X = c(x)\frac{\partial}{\partial a} + b(x)\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2}{\partial x^2} \text{ if } x < s.$$

Since the cost for observations $c(x)$ is positive we should not allow the process X to decrease too fast when s is fixed. It means that for given s **there exist a point $g(s)$** such that we should stop the observations when (X, S) achieves a point $(g(s), s)$. In other words

$$\boxed{\tau_* = \inf\{t > 0 : X_t \leqslant g(S_t)\}}$$

The unknown function $g = g(s)$ is called the **boundary of the stopping set** D_*.

The function $V_*(x, s)$, $g(s) < x \leqslant s$ is a solution of the system

$$(\mathbb{L}_X V)(x,s) = c(x) \qquad \text{if } g(s) < x < s, \tag{4}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \qquad (\textbf{normal reflection}), \tag{5}$$

$$V(x,s)|_{x=g(s)+} = s \qquad (\textbf{instanteneous stopping}), \tag{6}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \qquad (\textbf{smooth fit}), \tag{7}$$

which is called a **Stefan problem with moving boundary $g = g(s)$**.

Explain the meaning of each equation (4)-(7).

- According to the general optimal stopping theory, $\mathbb{L}_Z V(x,s) = 0$ when $(x,s) \in C_*$. Thus we get the equation (4);
- The **instanteneous stopping** condition (6) follows from the fact that $V(x,s) = s$ when $(x,s) \in D_*$;
- The **smooth fit** condition (7) means that the derivative of the function $V(x,s)$ is contintinuous on the boundary of C_* and D_*;
- Clarify the **normal reflection** condition.

Applying the Ito formula for semimartingales

$$df(X_t, S_t) = f_x^{'}(X_t, S_t)dX_t + f_s^{'}(X_t, S_t)dS_t + \frac{1}{2} f_{xx}^{''}(X_t, S_t)d\langle X, X\rangle_t$$

to the process $(f(X_t, S_t))_{t \geqslant 0}$, taking the expectation $\mathsf{E}_{s,s}$ on both sides and multiplying on t^{-1} we get

$$\frac{\mathsf{E}_{s,s} f(X_t, S_t) - f(s,s)}{t} = \mathsf{E}_{s,s}\left(\frac{1}{t}\int_0^t \mathbb{L}_X f(X_u, S_u)du\right) +$$

$$\mathsf{E}_{s,s}\left(\frac{1}{t}\int_0^t \frac{\partial f}{\partial s}(X_u, S_u)dS_u\right) \longrightarrow \mathbb{L}_X f(s,s) + \frac{\partial f}{\partial s}(s,s)\left(\lim_{t\downarrow 0}\frac{\mathsf{E}_{s,s}(S_t - s)}{t}\right)$$

as $t \downarrow 0$. Since the diffusion coefficient $\sigma > 0$ as $t \downarrow 0$ we have

$$\frac{1}{t}\mathsf{E}_{s,s}(S_t - s) \to \infty$$

Therefore the condition $\mathbf{f'_s(s,s) = 0}$ assures us that the limit $\mathbb{L}_X f(s,s) + \frac{\partial f}{\partial s}(s,s)\left(\lim_{t\downarrow 0}\frac{\mathsf{E}_{s,s}(S_t - s)}{t}\right)$ is finite.

Find the functions $V(x,s)$ and $g(s)$ – the solutions of system (4)-(7). Denote

$$\tau_g = \inf\{t > 0\colon X_t \leqslant g(S_t)\}, \quad \tau_{g(s),s} = \inf\{t > 0\colon X_t \notin (g(s), s)\}$$

and consider the function

$$V_g(x,s) = \mathsf{E}_{x,s}\left(S_{\tau_g} - \int\limits_0^{\tau_g} c(X_t)dt\right).$$

Using the strong Markov property of X w.r.t. time $\tau_{g(s),s}$ when $x \in (g(s), s)$ we have

$$V_g(x,s) = s\,\mathsf{P}_{x,s}(X_{\tau_{g(s),s}} = g(s)) + V_g(s,s)\mathsf{P}_{x,s}(X_{\tau_{g(s),s}} = s) - \mathsf{E}_{x,s}\int\limits_0^{\tau_{g(s),s}} c(X_t)dt =$$

$$= s\,\frac{R(s)-R(x)}{R(s)-R(g(s))} + V_g(s,s)\frac{R(x)-R(g(s))}{R(s)-R(g(s))} -$$

$$\int\limits_{g(s)}^{s} G_{g(s),s}(x,y)c(y)m(dy),$$

where $G_{a,b}(x,y)$ is the **Green function** of X on the segment $[a,b]$:

$$G_{a,b}(x,y) = \begin{cases} \dfrac{(R(b)-R(x))(R(y)-R(a))}{R(b)-R(a)} & \textbf{if } a \leqslant y \leqslant x, \\[2ex] \dfrac{(R(b)-R(y))(R(x)-R(a))}{R(b)-R(a)} & \textbf{if } x \leqslant y \leqslant b. \end{cases}$$

Rewrite the expression for $V_g(x,s)$ in the following form:

$$V_g(s,s) - s = \frac{R(s)-R(g(s))}{R(x)-R(g(s))}\left(V_g(x,s) - s + \int\limits_{g(s)}^{s} G_{g(s),s}(x,y)c(y)m(dy)\right)$$

Suppose that $V_g(x,s)$ satisfies the **smooth fit** condition. Then

$$\lim_{x \downarrow g(s)} \frac{V_g(x,s) - s}{R(x) - R(g(s))} = \frac{1}{R'(g(s))} \frac{\partial V_g}{\partial x}(x,s)\Big|_{x=g(s)+} = 0,$$

$$\lim_{x \downarrow g(s)} \frac{R(s) - R(g(s))}{R(x) - R(g(s))} \int_{g(s)}^{s} G_{g(s),s}(x,y)c(y)m(dy) =$$

$$\int_{g(s)}^{s} (R(s) - R(y))c(y)m(dy).$$

Therefore we have

$$V_g(s,s) = s + \int_{g(s)}^{s} (R(s) - R(y))c(y)m(dy),$$

Finally we obtain

$$V_g(x,s) = s + \int_{g(s)}^{x} (R(x) - R(y))c(y)m(dy), \tag{8}$$

for all $g(s) \leqslant x \leqslant s$.

Now suppose that the function $V_g(x,s)$ is given by (8). Then it is easy to show that $V_g(x,s)$ is a solution of **Stefan problem** (4)-(7) if and only if the boundary $g = g(s)$ belongs to C^1 and satisfies the equation

$$g^{'}(s) = \frac{\sigma^2(g(s))R^{'}(g(s))}{2c(g(s))(R(s) - R(g(s))}. \tag{9}$$

Observe that the equation (9) has a whole family of solutions. We need to specify the criteria which enables us to choose the solution $g_* = g_*(s)$ – a **boundary of the stopping set** D_*.

We call the solution $g(s)$ of the equation (9) an **admissible solution** if $g(s) < s$ for all $s \geqslant 0$.

Theorem [maximality principle]. *The boundary* $g_* = g_*(s)$ *of the stopping set* D_* *in the problem*

$$V_*(x, s) = \sup_{\tau} \mathsf{E}_{x,s} \left(S_\tau - \int_0^\tau c(X_t)dt \right) \qquad (*)$$

is a ***maximal admissible solution*** *of the differential equation (9).*

Theorem. *Consider the stopping problem (*) for diffusion process* $X = (X_t)_{t \geqslant 0}$ *such that* $dX_t = b(X_t)dt + \sigma(X_t)dB_t$. *Supremum is taken by all Markov times* τ *such that*

$$\mathsf{E}_{x,s}\left(\int_0^{\tau} c(X_t)dt\right) < \infty. \tag{10}$$

Assume that there exist the maximal admissible solution $g_*(s)$ *of* (9). *Then*

1) *The value function* $V_*(x,s)$ *in problem (*) is finite and can be determined on* E *by*

$$V_*(x,s) = \begin{cases} s, & \textbf{if } x \leqslant g_*(s), \\ s + \int\limits_{g_*(s)}^{x} (R(x) - R(y))c(y)m(dy), & \textbf{if } g_*(s) \leqslant x \leqslant s. \end{cases}$$

2) *The Markov time* $\tau_* = \inf\{t > 0\colon X_t \leqslant g_*(S_t)\}$ *is optimal in problem (*) if it satisfies the condition* (10);

3) *If there exist an optimal stopping time* σ *in problem (*) such that* $\mathsf{E}_{x,s}\left(\int\limits_0^{\sigma} c(X_t)dt\right) < \infty$ *then* $\mathsf{P}_{x,s}(\tau_* \leqslant \sigma) = 1$ *for all* (x, s) *and time* τ_* *is also optimal in problem (*).*

If the equation (9) *doesn't have a maximal admissible solution then* $V_*(x, s) = +\infty$ ***for all*** (x, s) *and there is no optimal stopping time in problem (*).*

Theorem [verification theorem]. *Assume that for the solution* $\widehat{V} = \widehat{V}(x, s)$ *of Stefan problem* (4)-(7) *the following statements are true:*

(i) $\widehat{V}(x, s) \geqslant s, \quad (x, s) \in E$;

(ii) $\widehat{V}(x, s) = \mathsf{E}_{x,s}\left(S_{\tau_g} - \int_0^{\tau_g} c(X_t)dt\right)$, $(x, s) \in E$ for some Markov time $\tau_g = \inf\{t \geqslant 0 \colon X_t \leqslant g(S_t)\}$ satisfying (10);

(iii) $\widehat{V}(x, s) \geqslant \mathsf{E}_{x,s}\widehat{V}(X_\tau, S_\tau)$ for any Markov time τ satisfying (10).

Then $\widehat{V}(x, s)$ *coincides with the value function* $V_*(x, s)$ *in problem (*) and* τ_g *is optimal.*

§2. Maximal inequalities for standard Brownian motion and it's modulus. Martingale and «Stefan problem» approaches

Consider the **standard Brownian motion** $B = (B_t)_{t \geqslant 0}$, $B_0 = 0$. This was the first process for which sharp maximal inequalities were established.

- **"square root inequality"**

$$\boxed{\mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} B_t\right) \leqslant \sqrt{\mathsf{E}\tau}} \tag{11}$$

- **"square root of two inequality"**

$$\boxed{\mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} |B_t|\right) \leqslant \sqrt{2\mathsf{E}\tau}} \tag{12}$$

Inequalities (11) and (12) are also called **Dubins-Jacka-Schwarz-Shiryaev inequalities**.

Denote $S_t(B) = \max_{0 \leqslant u \leqslant t} B_u$ and $S_t(|B|) = \max_{0 \leqslant u \leqslant t} |B_u|$.

Martingale approach. First proof the inequality (11). Consider a stochastic process

$$\boxed{Z_t = c((S_t(B) - B_t)^2 - t) + \frac{1}{4c}, \; t \geqslant 0}$$

when $c > 0$. Due to Levy theorem $\mathrm{Law}(S(B) - B) = \mathrm{Law}(|B|)$ and the process $B_t^2 - t$ is a martingale. Therefore $(Z_t)_{t \geqslant 0}$ **is also martingale** w.r.t. natural filtration of B.

It is easy to see that $(\sqrt{c}x - 1/(2\sqrt{c}))^2 \geqslant 0$. From this inequality it follows that $x - ct \leqslant c(x^2 - t) + 1/(4c)$ for all $x \in \mathbb{R}$. Thus for any $\tau \in \mathfrak{M}$ we get

$$\mathsf{E}(S_{\tau\wedge t}(B) - c\tau \wedge t) = \mathsf{E}(S_{\tau\wedge t}(B) - B_{\tau\wedge t} - c\tau \wedge t) \leqslant \mathsf{E}Z_{\tau\wedge t} = \mathsf{E}Z_0 = \frac{1}{4c}$$

Taking the limit as $t \to \infty$ from Doob's optional sampling theorem we have $\mathsf{E}S_\tau(B) \leqslant c\mathsf{E}\tau + 1/(4c)$. Taking an infimum on $c > 0$ on both sides we obtain (11).

Prove that inequality $\mathsf{E}S_\tau(B) \leqslant \sqrt{\mathsf{E}\tau}$ is **sharp**. For each $a > 0$ consider the time

$$\tau_a = \inf\{t \geqslant 0\colon S_t(B) - B_t = a\}$$

We see that $\mathsf{E}S_{\tau_a}(B) = \mathsf{E}(S_{\tau_a}(B) - B_{\tau_a}) = a$. Since $\mathrm{Law}(\tau_a) = \mathrm{Law}(\inf\{t \geqslant 0\colon |B_t| = a\})$ from Wald identities we get $a^2 = \mathsf{E}B_{\tau_a}^2 = \mathsf{E}\tau_a$.

Corollary. *For any continuous local martingale $M = (M_t)_{t \geqslant 0}$, $M_0 = 0$ we have*

$$\boxed{\mathsf{E}\left(\max_{0 \leqslant t \leqslant T} M_t\right) \leqslant \sqrt{\mathsf{E}\langle M\rangle_T},} \tag{13}$$

for any $T > 0$. Here $(\langle M\rangle_t)_{t \geqslant 0}$ is a quadratic characteristic of M.

This inequality follows from (11) and Dambis-Dubins-Schwarz theorem. Indeed, $\mathsf{E}(\max_{t \leqslant T} M_t) = \mathsf{E}(\max_{t \leqslant T} B_{\langle M\rangle_t}) = \mathsf{E}(\max_{t \leqslant \langle M\rangle_T} B_t) \leqslant \sqrt{\mathsf{E}\langle M\rangle_T}$.

Prove the inequality $\mathsf{E}S_\tau(|B|) \leqslant \sqrt{2\mathsf{E}\tau}$. Consider a continuous martingale

$$U_t = \mathsf{E}(|B_\tau| - \mathsf{E}|B_\tau| \,|\, \mathcal{F}^B_{t\wedge\tau}),\ t \geqslant 0$$

Applying (13) to $\max_{t\leqslant T} U_t$ and taking $T \to +\infty$ we get $\mathsf{E}(\max_{t\geqslant 0} U_t) \leqslant \sqrt{\mathsf{E}(|B_\tau| - \mathsf{E}|B_\tau|)^2}$. Using this inequality we estimate $\mathsf{E}S_\tau(|B|)$ by

$$\mathsf{E}\left(\max_{0\leqslant t\leqslant\tau} |B_t|\right) = \mathsf{E}\left(\max_{t\geqslant 0} |B_{t\wedge\tau}|\right) = \mathsf{E}\left(\max_{t\geqslant 0} |\mathsf{E}(B_\tau \,|\, \mathcal{F}^B_{t\wedge\tau})|\right) \leqslant$$
$$\mathsf{E}\left(\max_{t\geqslant 0} \mathsf{E}(|B_\tau| \,|\, \mathcal{F}^B_{t\wedge\tau})\right) = \mathsf{E}\left(\max_{t\geqslant 0} U_t\right) + \mathsf{E}|B_\tau| \leqslant \sqrt{\mathsf{E}(|B_\tau| - \mathsf{E}|B_\tau|)^2} +$$
$$\mathsf{E}|B_\tau| = \sqrt{\mathsf{E}\tau - (\mathsf{E}|B_\tau|)^2} + \mathsf{E}|B_\tau| \leqslant \sqrt{2\mathsf{E}\tau}.$$

In order to get the last inequality in this series we used a simple inequality $\sqrt{A - x^2} + x \leqslant \sqrt{2A}$ when $0 < x < \sqrt{A}$.

Now show that inequality $\mathsf{E}S_\tau(|B|) \leqslant \sqrt{2\mathsf{E}\tau}$ is **sharp**. Consider the time

$$\widehat{\tau}_a = \inf\{t \geqslant 0 \colon S_t(|B|) - |B_t| = a\}$$

It turns out that $\mathsf{E}\widehat{\tau}_a = 2a^2$ and $\mathsf{E}(\max_{t\leqslant\widehat{\tau}_a} |B_t|) = 2a$.

«Stefan problem» approach. Basically the proof of (11) and (12) is the application of the main theorem of §1 to the problem

$$V_*(x,s) = \sup_\tau \mathsf{E}_{x,s}\left(S_\tau - \int_0^\tau c(X_t)dt\right)$$

in the case when $c(X_t) \equiv c > 0$, $X_t = B_t$ or $X_t = |B_t|$.

First, prove the inequality $\mathsf{E}S_\tau(B) \leqslant \sqrt{\mathsf{E}\tau}$. In the case of Brownian motion $R(x) = x$, $m(dx) = 2dx$, $x \in \mathbb{R}$. According to the theorem the equation for boundary is

$$g^{'}(s) = \frac{1}{2c(s - g(s))}$$

The maximal admissible solution of this equation is $g_*(s) = s - 1/(2c)$.

Therefore the **value function** $V_*(x,s) = \sup_{t \leqslant \tau} \mathsf{E}_{x,s}(S_\tau(B) - c\tau)$ when $0 \leqslant s - x \leqslant 1/(2c)$ equals

$$V_*(x,s) = s + 2c \int\limits_{g(s)}^{x} (x-y)dy = c(x-s)^2 + x + \frac{1}{4c}$$

Since we need the value $V_*(0,0)$ for any $\tau \in \mathfrak{M}$ we get

$$\mathsf{E}S_\tau(B) \leqslant \inf_{c>0}\{V_*(0,0) + c\mathsf{E}\tau\} = \inf_{c>0}\{1/(4c) + c\mathsf{E}\tau\} = \sqrt{\mathsf{E}\tau}$$

However we cannot apply directly the method from §1 in the case of $X_t = |B_t|$ and obtain the inequality $\mathsf{E}S_\tau(|B|) \leqslant \sqrt{2\mathsf{E}\tau}$. The reason is that we cannot represent $X_t = |B_t|$ in the form $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with continuous b and σ. But we can consider the problem

$$W_*(x,s) = \sup_\tau \mathsf{E}_{x,s}\left(s \vee \max_{0 \leqslant t \leqslant \tau} |x + B_t| - c\tau\right)$$

and reduce it to the Stefan problem.

Infinitesimal operator of $|B|$ equals $L = \frac{1}{2}\frac{d^2}{dx^2}$, $x > 0$ with endpoint $x = 0$. Thus Stefan problem in our case is

$$\begin{cases} \dfrac{\partial^2 W}{\partial x^2}(x,s) = 2c, & x \neq 0,\ g(s) < x \leqslant s, \\ \dfrac{\partial W}{\partial x}(0+, s) = 0, & s\colon\ g(s) < 0; \\ \dfrac{\partial W}{\partial s}(x,s)\Big|_{x=s-} = 0;\ W(x,s)|_{x=g(s)+} = s;\ \dfrac{\partial W}{\partial x}(x,s)\Big|_{x=g(s)+} = 0. \end{cases}$$

The solution of this system is the function

$$W_*(x,s) = \begin{cases} s, & s - x \geqslant \dfrac{1}{2c}, \\ c(x-s)^2 + x + \dfrac{1}{4c}, & s \geqslant 1/(2c),\ s - x \leqslant 1/(2c), \\ cx^2 + \dfrac{1}{2c}, & 0 \leqslant s \leqslant \dfrac{1}{2c} \end{cases}$$

Since $W_*(0,0) = 1/(2c)$ for each $\tau \in \mathfrak{M}$ we have $\mathsf{E}S_\tau(|B|) \leqslant \inf_{c>0}\{1/(2c) + c\mathsf{E}\tau\} = \sqrt{2\mathsf{E}\tau}$.

§3. Maximal inequalities for skew Brownian motion. Solution to the corresponding Stefan problem

The process $X^\alpha = (X_t^\alpha)_{t\geqslant 0}$ defined on probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is called a **skew Brownian motion** if it satisfies the stochastic equation

$$\boxed{X_t^\alpha = X_0^\alpha + B_t + (2\alpha - 1)L_t^0(X^\alpha),} \tag{14}$$

where $L^0 = (L_t^0(X^\alpha))_{t\geqslant 0}$ c $L_0^0(X^\alpha) = 0$ is the local time of X^α in zero.

The skew Brownian motion with parameter $\alpha = 1/2$ has the same distribution as **standard Brownian motion**, with parameter $\alpha = 1$ – as the **modulus of standard Brownian motion**.
Denote by $W^\alpha = (W_t^\alpha)_{t\geqslant 0}$ the unique strong solution of (14) such that $W_0^\alpha = 0$.

Consider the optimal stopping problem

$$\boxed{V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s}\left(s \vee \max_{0\leqslant t\leqslant \tau}(x+W_t^\alpha) - c\tau\right)} \tag{15}$$

with constant cost for observations $c > 0$. We cannot directly apply the methods from §1 since $X_t = x + W_t^\alpha$ cannot be represented in the form $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with continuous $b(\cdot)$ and $\sigma(\cdot)$. However we can write the analogue of Stefan problem (4)-(7) in the case of optimal stopping problem.

The infinitesimal operator for X equals $L = \frac{1}{2}\frac{d^2}{dx^2}$ and defined for functions

$$\{f : f^{''} \textbf{ exists on } \mathbb{R}\setminus\{0\},\ f^{''}(0+) = f^{''}(0-),\ \lim_{x\to\infty} f(x) = 0$$
$$\textbf{and } \alpha f^{'}(0+) = (1-\alpha) f^{'}(0-)\}$$

Therefore we get the Stefan problem for value function

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2}(x,s) = 2c, \quad x \neq 0,\ g(s) < x \leqslant s, \\ \alpha \dfrac{\partial V}{\partial x}(0+,s) = (1-\alpha)\dfrac{\partial V}{\partial x}(0-,s), \quad s\colon g(s) < 0; \\ \dfrac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0;\ V(x,s)|_{x=g(s)+} = s;\ \dfrac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \end{cases}$$

The solution of this system is given in the following

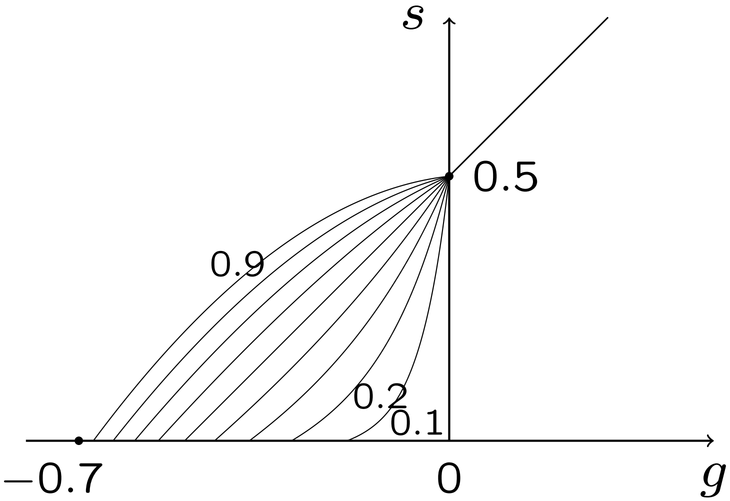
Theorem 1. *The optimal stopping time* τ_c *in the problem* (15) *exists and equals*

$$\boxed{\tau_* = \inf\{t \geqslant 0 : X_t \leqslant g(S_t)\}}$$

The mapping $g = g(s)$, $s \geqslant 0$ *is given by*

$$s = \begin{cases} g + 1/(2c), & \textbf{if } g \geqslant 0, \\ \dfrac{\beta^2 - 1}{2c\beta^2} e^{2c\beta g} + \dfrac{g}{\beta} + \dfrac{1}{2c\beta^2}, & \textbf{if } g < 0, \end{cases}$$

parameter $\beta = (1 - \alpha)/\alpha$.



The boundary $s = s(g)$ of the stopping set when $c = 1$ and $\alpha = 0.1, 0.2, \dots, 0.9$.

If we consider the sets $D_* = \{(x,s) \in E : x \leqslant g(s)\}$, $C_* = E \setminus D_*$ *then the value function equals*

$$V_*(x,s) = \begin{cases} s + c(x - g(s))^2, & (x,s) \in C_*,\ x \geqslant 0,\ s \geqslant \dfrac{1}{2c} \\ & \textbf{or}\ x < 0,\ s < \dfrac{1}{2c}, \\ s + c(x - g(s))^2 + 2c(1-\beta)xg(s), & (x,s) \in C_*, x \geqslant 0, s < \dfrac{1}{2c}, \\ s, & (x,s) \in D_* \end{cases}$$

The proof of the theorem is based on finding the solution to Stefan problem. Particularly the equation for boundary $g = g(s)$ is

$$g^{'}(s) = \begin{cases} \dfrac{1}{2c(s - g(s))}, & s \colon g(s) \geqslant 0, \\ \dfrac{1}{2c(\beta s - g(s))}, & s \colon g(s) < 0 \end{cases}$$

The general solution of this equation is $s(g) = a_0 e^{2cg} + g + 1/(2c)$ when $g \geqslant 0$ and $s(g) = b_0 e^{2c\beta g} + g/\beta + 1/(2c\beta^2)$ when $g < 0$.

In order to prove that the solution of Stefan problem $V(x,s)$ coincides with the value function $V_*(x,s) = \sup_\tau \mathsf{E}_{x,s}\left(s \vee \max_{0 \leqslant t \leqslant \tau}(x + W_t^\alpha) - c\tau\right)$ we use the following analogue of **Ito formula**:

$$\begin{aligned}
\widehat{V}(X_t, S_t) = \quad & \widehat{V}(X_0, S_0) + \int\limits_0^t \widehat{V}_x'(X_u, S_u)dB_u + \int\limits_0^t \widehat{V}_s'(X_u, S_u)dS_u + \\
& \frac{2\alpha - 1}{2}\int\limits_0^t (\widehat{V}_x'(0+, S_u) + \widehat{V}_x'(0-, S_u))dL_u^0 + \frac{1}{2}\int\limits_0^t (\widehat{V}_x'(0+, S_u) \\
& -\widehat{V}_x'(0-, S_u))dL_u^0 + \frac{1}{2}\int\limits_0^t \widehat{V}_{xx}''(X_u, S_u)\mathbb{I}(X_u \neq 0)du
\end{aligned}$$

Once we know the value $V_*(0,0)$ it is possible to obtain the maximal inequalities.

Theorem 2 (**Lyulko'2012**). *For any Markov time* $\tau \in \mathfrak{M}$ *and for any* $\alpha \in (0,1)$ *the following inequality holds:*

$$\mathsf{E}\left(\max_{0\leqslant t\leqslant\tau} W_t^{\alpha}\right) \leqslant M_\alpha\sqrt{\mathsf{E}\tau}, \tag{16}$$

where $M_\alpha = \alpha(1+A_\alpha)/(1-\alpha)$ *and* A_α *is the unique solution of the equation*

$$A_\alpha e^{A_\alpha+1} = \frac{1-2\alpha}{\alpha^2},$$

such that $A_\alpha > -1$.

The inequality (16) *is* ***sharp*** *i.e. for any* $T>0$ *there exist a Markov time* τ *with* $\mathsf{E}\tau = T$ *such that*

$$\mathsf{E}\left(\max_{0\leqslant t\leqslant\tau} W_t^{\alpha}\right) = M_\alpha\sqrt{\mathsf{E}\tau}.$$

The inequalities like (16) can be obtained not only for maximum $\max_{0 \leqslant t \leqslant \tau} W_t^\alpha$. Thus in **[Zhitlukhin'2012]** there were stated the following inequalities for **range** of skew Brownian motion:

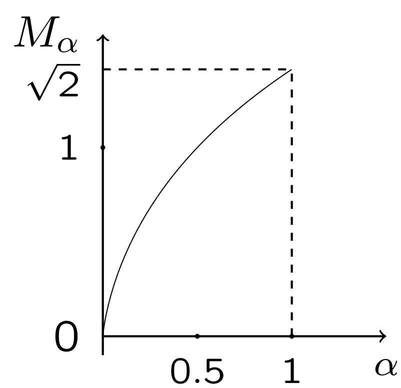
$$\mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} W_t^\alpha - \min_{0 \leqslant t \leqslant \tau} W_t^\alpha\right) \leqslant \sqrt{K_\alpha \mathsf{E}\tau},$$

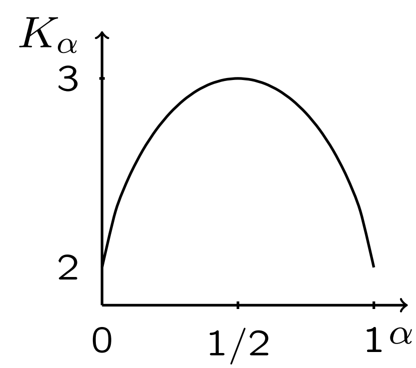
where $K_\alpha = C_\alpha + C_{1-\alpha}$,

$$C_\alpha = \frac{\alpha}{1-\alpha}\left(\frac{\alpha D_\alpha^2}{1-\alpha} - 2D_\alpha - 2\alpha \int_{D_\alpha}^0 \frac{\alpha x + \alpha - 1}{(2\alpha - 1)e^x - \alpha} dx\right)$$

and D_α is the unique negative solution of the equation

$$(2\alpha - 1)\alpha^{-2} e^{D_\alpha} - 1 = D_\alpha$$





§4. Maximal inequalities for Bessel processes. Solution to the corresponding Stefan problem

A continuous nonnegative Markov process $X = (X_t(x))_{t \geqslant 0}$, $x \geqslant 0$ is called a **Bessel process of dimension** $\gamma \in \mathbb{R}$ ($X \in \mathrm{Bes}^{\gamma}(x)$) if it's infinitesimal operator equals

$$\boxed{\mathbb{L}_X = \frac{1}{2}\left(\frac{\gamma - 1}{x}\frac{d}{dx} + \frac{d^2}{dx^2}\right)}$$

The endpoint $x = 0$ is called **trap** if $\gamma \leqslant 0$, **instantaneously reflecting** if $\gamma \in (0, 2)$ and **entrance** if $\gamma \geqslant 2$.

In the case $\alpha = n \in \mathbb{N}$ the Bessel process can be realized as a **radial part of n-dimensional Brownian motion**

$$X_t(x) = \left(\sum_{i=1}^{n} (B_t^i + a_i)^2 \right)^{1/2},$$

where $a = (a_1, a_2, \dots, a_n)$ is a vector in $\mathbb{R}^n$ with norm $x = \sqrt{a_1^2 + \dots + a_n^2}$. $B^1, B^2, \dots, B^n$ are independent Brownian motions starting from zero. The Bessel process of dimension $\gamma = 1$ is a **modulus of standard Brownian motion $x + |B_t|$.**

Consider the optimal stopping problem

$$V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s} \left(s \vee \max_{0 \leqslant t \leqslant \tau} X_t(x) - c\tau \right) \qquad (*)$$

where Markov times $\tau \in \mathfrak{M}$.

Theorem 3. *Let $X \in \text{Bes}^{\gamma}(x)$ where the dimension $\gamma \in \mathbb{R}$ and $c > 0$. The optimal stopping time τ_* in problem (*) exists and equals*

$$\tau_* = \inf\{t \geqslant 0 \colon (X_t, S_t) \in D_*\}$$

with $X_t = X_t(x)$, $S_t = S_t(x,s) = s \vee \max_{0 \leqslant u \leqslant t} X_u$ *and stopping set* $D_* = \{(x,s) \colon s_* \leqslant s, x \leqslant g_*(s)\}$ *where* $g_* = g_*(s)$ *is the unique nonnegative solution of the equation*

$$\frac{2c}{\gamma - 2} g^{'}(s) g(s) \left(1 - \left(\frac{g(s)}{s}\right)^{\gamma-2}\right) = 1 \tag{17}$$

such that $g(s) \leqslant s$ *when* $s \geqslant 0$ *and*

$$\lim_{s \to \infty} \frac{g_*(s)}{s} = 1,$$

and s_* ***is the root of the equation*** $g_*(s) = 0$. *When* $\gamma = 2$ *the equation* (17) *has the form* $2cg^{'}(s)g(s)\ln(s/g) = 1$.

Moreover if we denote

$$C_*^1 = \{(x,s) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon s > s_*,\ g_*(s) < x \leqslant s\},$$
$$C_*^2 = \{(x,s) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon 0 \leqslant x \leqslant s \leqslant s_*\}$$

and define a continuation set by $C_* = C_*^1 \cup C_*^2$ *then depending on the value of parameter* γ *the value function* $V_*(x,s)$ *equals*

if $\alpha > 0$

$$V_*(x,s) = \begin{cases} s, & (x,s) \in D_*, \\ s + \dfrac{c}{\gamma}(x^2 - g_*^2(s)) + \dfrac{2cg_*^2(s)}{\gamma(\gamma-2)}\left(\left(\dfrac{g_*(s)}{x}\right)^{\gamma-2} - 1\right), & (x,s) \in C_*^1, \\ \dfrac{c}{\gamma}x^2 + s_*, & (x,s) \in C_*^2; \end{cases}$$

if $\alpha = 0$

$$V_*(x,s) = \begin{cases} s, & (x,s) \in D_*, \\ s + \frac{c}{2}(g_*^2(s) - x^2) + cx^2 \ln \frac{x}{g_*(s)}, & (x,s) \in C_*; \end{cases}$$

if $\alpha < 0$

$$V_*(x,s) = \begin{cases} s, & (x,s) \in D_*, \\ s + \frac{c}{\gamma}(x^2 - g_*^2(s)) + \frac{2cg_*^2(s)}{\gamma(\gamma-2)} \left(\left(\frac{g_*(s)}{x} \right)^{\gamma-2} - 1 \right), & (x,s) \in C_*. \end{cases}$$

Using this theorem we can obtain the maximal inequalities for Bessel processes.

- if $\gamma \leqslant 0$ then the point $x = 0$ is a **trap**. Therefore $X_t(x) \equiv 0$ if $t \geqslant 0$ and maximal inequalities do not make sense
- if $\gamma > 0$ then from theorem it follows that $V_*(0,0) = s_*$. Denote $V_*(x,s) = V_c^\gamma(x,s)$, $s_* = s_c(\gamma)$

Since Bessel processes are **self-similar**

$$\boxed{\mathrm{Law}(X_t(x),\, t \geqslant 0) = \mathrm{Law}(c^{-1/2} X_{ct}(c^{1/2} x))}$$

the value function $V_c^\gamma(x,s)$ is also self-similar, i.e. $cV_c^\gamma(x,s) = V_1^\gamma(cx, cs)$. Hence $s_c(\gamma) = s_1(\gamma)/c$. Therefore we get the inequalties

$$\mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} X_t(0)\right) \leqslant \inf_{c>0}\{V_*(0,0) + c\mathsf{E}\tau\} =$$

$$\inf_{c>0}\{s_1(\gamma)/c + c\mathsf{E}\tau\} = \sqrt{4 s_1(\gamma)\mathsf{E}\tau}$$

Theorem 4 (**Dubins-Shepp-Shiryaev'1993**). *Let $X \in$ Bes$^\gamma(0)$, $\gamma > 0$. Then for any Markov time $\tau \in \mathfrak{M}$ the following sharp maximal inequality holds:*

$$\boxed{\mathsf{E}\left(\max_{0\leqslant t\leqslant\tau} X_t(0)\right) \leqslant \sqrt{4s_1(\gamma)\mathsf{E}\tau},}$$

where $s_1(\gamma)$ is the root of equation $g_*(s) = 0$ such that

$$\frac{s_1(\gamma)}{\gamma} \longrightarrow \frac{1}{4}$$

as $\gamma \uparrow \infty$.

Observe that in the case $\gamma = 1$ we have $s_1(1) = 1/2$ and therefore we get the maximal inequality for modulus of standard Brownian motion $\mathsf{E}\left(\max_{0\leqslant t\leqslant\tau} |B_t|\right) \leqslant \sqrt{2\mathsf{E}\tau}$.

§5. Doob maximal inequalities

Theorem 5. *Let* $M = (M_t)_{t \geqslant 0}$ ***be a local martingale*** *on a filtered probability space* $(\Omega, (\mathcal{F}_t)_{t \geqslant 0}, \mathsf{P})$. *Then for any* $p > 0$ *there exist a universal constants* c_p *и* C_p *such that*

$$c_p \mathsf{E}([M]_\tau^{p/2}) \leqslant \mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} |M_t|^p\right) \leqslant C_p \mathsf{E}([M]_\tau^{p/2}), \tag{18}$$

where $([M]_t)_{t \geqslant 0}$ *is called a* ***quadratic variation*** *of* M.

The inequalities (18) are called **Burkholder-Davis-Gundy inequalities**. In the case when $M_t = B_t$ **is standard Brownian motion** we get

$$c_p \mathsf{E}\tau^{p/2} \leqslant \mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} |B_t|^{p/2}\right) \leqslant C_p \mathsf{E}\tau^{p/2}, \tag{19}$$

Note that if $p \neq 2$ the exact values of the constants c_p and C_p when inequalities (19) become **sharp** are still not known.

Some particular cases of Burkholder-Davis-Gundy inequalities:

- **Davis inequalities** $(p = 1)$:

$$c_1 \mathsf{E}\sqrt{\tau} \leqslant \mathsf{E}\left(\max_{0\leqslant t\leqslant \tau} |B_t|\right) \leqslant C_1 \mathsf{E}\sqrt{\tau}$$

- **Doob inequalities** $(p = 2)$:

$$c_2 \mathsf{E}\tau \leqslant \mathsf{E}\left(\max_{0\leqslant t\leqslant \tau} B_t^2\right) \leqslant C_2 \mathsf{E}\tau$$

Consider the case $p = 1$. One of the possible ways to obtain the exact values of c_1, C_1 is to solve the optimal stopping problem

$$V(c) = \sup_{\tau} \mathsf{E}\left(\max_{0\leqslant t\leqslant \tau} |B_t| - c\sqrt{\tau}\right), \tag{20}$$

where $c > 0$, τ is the Markov time such that $\mathsf{E}\sqrt{\tau} < \infty$.

The problem (20) can be formulated in a standard way for **3-dimensional Markov process**

$$Z_t = (t, X_t, S_t),\ X_t = |B_t|,\ S_t = \max_{u \leqslant t} |B_u|$$

But this problem is **nonlinear** and we cannot decrease it's dimensionality. The same situation happens when $p \neq 2$.

In the case $p = 2$ the corresponding optimal stopping problem

$$\sup_{\tau} \mathsf{E}(\max_{t \leqslant \tau} B_t^2 - c\tau)$$

is **linear** and we can get the solution explicitly. As a consequence we obtain the **Doob maximal inequalities**

$$\boxed{\mathsf{E}\tau \leqslant \mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} B_t^2\right) \leqslant 4\mathsf{E}\tau,} \qquad (21)$$

where τ is the Markov time such that $\mathsf{E}\tau < \infty$.

Prove the inequality (21) and show that it is **sharp**. Denote $S_t(B^2) = \max_{0 \leqslant u \leqslant t} B_u^2$. The lower bound for $\mathsf{E} S_\tau(B^2)$ follows from the Wald identity:

$$\mathsf{E} S_\tau(B^2) \geqslant \mathsf{E} B_\tau^2 = \mathsf{E}\tau$$

To show that this inequality is sharp it is enough to consider the time $\tau_*(T) = \inf\{t \geqslant 0 \colon |B_t| = \sqrt{T}\}$. Then $\mathsf{E}\tau_*(T) = \mathsf{E} B^2_{\tau_*(T)} = T$ and $\mathsf{E} S_{\tau_*(T)}(B^2) = T$.

In order to prove the upper bound $\mathsf{E}\left(\max_{0 \leqslant t \leqslant \tau} B_t^2\right) \leqslant 4\mathsf{E}\tau$ consider the sequence of stopping times

$$\sigma_{\lambda,\varepsilon} = \inf\{t > 0 \colon \max_{0 \leqslant s \leqslant t} |B_s| - \lambda |B_t| \geqslant \varepsilon\},$$

where $\lambda, \varepsilon > 0$. It is known that $\mathsf{E}(\sigma_{\lambda,\varepsilon})^{p/2} < \infty$ if and only if $\lambda < p/(p-1)$.

Therefore if $\lambda \in (0,2)$ we have

$$\mathsf{E}\left(\max_{0\leqslant t\leqslant \sigma_{\lambda,\varepsilon}} B_t^2\right) = \lambda^2 \mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2 + 2\lambda\varepsilon \mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}| + \varepsilon^2 \leqslant K\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2 \quad (22)$$

for some constant $K > 0$. Divide the both sides of (22) on $\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2$ and take $\lambda \uparrow 2$. Since $\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2 = \mathsf{E}\sigma_{\lambda,\varepsilon} \to \infty$ and $\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|/\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2 \leqslant 1/\sqrt{\mathsf{E}\sigma_{\lambda,\varepsilon}} \to 0$ if $\lambda \uparrow 2$ then from (22) we get

$$K \geqslant \lambda^2 + 2\lambda\varepsilon \frac{\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|}{\mathsf{E}B_{|\sigma_{\lambda,\varepsilon}}|^2} + \frac{\varepsilon^2}{\mathsf{E}|B_{\sigma_{\lambda,\varepsilon}}|^2} \longrightarrow 4.$$

Therefore $K = 4$ is the best possible constant in the upper bound for $\mathsf{E}S_\tau(B^2)$.

TOPIC II: Sharp maximal inequalities for discrete time processes

§1. Maximal inequalities for modulus of simple symmetric Random walk

In this section time t will take discrete values i.e. $t = n = 0, 1, 2, \ldots$
Consider the simple symmetric Random walk
$X_n = S_n = \xi_1 + \ldots + \xi_n$, $X_0 = S_0 = 0$, where $\xi_1, \ldots, \xi_n, \ldots$ are i.i.d. random variables, $\mathsf{P}(\xi_1 = 1) = \mathsf{P}(\xi_1 = -1) = 1/2$

Denote the current maximums of X and $|X|$ by $M_n(S) = \max_{0 \leqslant k \leqslant n} S_k$ and $M_n(|S|) = \max_{0 \leqslant k \leqslant n} |S_k|$.

In order to obtain the maximal inequalities for $(S_n)_{n\geqslant 0}$ and $(|S_n|)_{n\geqslant 0}$ consider the following optimal stopping problems:

$$V_*(c) = \sup_{\tau\in\mathfrak{M}} \mathsf{E}\left(\max_{0\leqslant k\leqslant\tau} S_k - c\tau\right) \qquad (*)$$

and

$$W_*(c) = \sup_{\tau\in\mathfrak{M}} \mathsf{E}\left(\max_{0\leqslant k\leqslant\tau} |S_k| - c\tau\right) \qquad (**)$$

For any nonnegative integer l define the stopping times

$$\tau_l = \begin{cases} \inf\{k > n : M_k(|S|) - |S_k| = l\}, & \textbf{if } m - s < l, \\ n, & \textbf{if } m - s \geqslant l \end{cases}$$

$$\sigma_l = \begin{cases} \inf\{k > n : S_k \neq 0,\, M_k(|S|) - |S_k| = l\}, & \textbf{if } m - s < l, \\ n, & \textbf{if } m - s \geqslant l \end{cases}$$

and a function $Q_l = Q_l(n,s,m,c)$ such that

$$Q_l(n,s,m,c) = \sup_{\tau \in \mathfrak{M}_l} \mathsf{E}_{s,m}\left(M_\tau(|S|) - c\tau\right),$$

where the set of stopping times equals $\mathfrak{M}_l = \{\tau_l,\, \sigma_l : l \in \mathbb{Z}_+\}$.

If the conditions

1) $Q_l(n,s,m,c) \geqslant m - cn,$

2) $Q_l(n,s,m,c) \geqslant \mathsf{E}Q_l(n+1, s+\xi_{n+1}, \max\{m,\, s+\xi_{n+1}\}, c)$ **(excessivity)**

are satisfied then $Q_l(n,s,m,c) = \sup_{\tau \geqslant n} \mathsf{E}_{s,m}\left(M_\tau(|S|) - c\tau\right)$ i.e. the supremum on all stopping times is achieved on the stopping times of the special form τ_l and σ_l. Namely if $l \in [1/(2c) - 1/2, 1/(2c)]$ then supremum is achieved on τ_l. If $l \in [1/(2c) - 1, 1/(2c) - 1/2]$ then supremum is achieved on σ_l.

Take an arbitrary $l \in \mathbb{N}$ and compute $\mathsf{E}\tau_l$ and $\mathsf{E}M_{\tau_l}(|S|)$. Represent τ_l as a sum $\tau_l = \tau^{(1)} + \tau^{(2)}$ where

$$\tau^{(1)} = \inf\{k \geqslant 0 : |S_k| = l\},$$

$$\tau^{(2)} = \inf\{k \geqslant 0 : \max_{0 \leqslant i \leqslant k} (S_{i+\tau^{(1)}} - S_{\tau^{(1)}}) - (S_{k+\tau^{(1)}} - S_{\tau^{(1)}}) = l\}$$

Due to Wald identities for Random walk we have $\mathsf{E}\tau^{(1)} = \mathsf{E}S^2_{\tau^{(1)}} = l^2$. Also note that the distribution law of $\tau^{(2)}$ coincides with distribution law of the time $\inf\{k \geqslant 0 : M_k(S) - S_k = l\}$. This Markov time can be represented as a sum of $M_{\tau^{(2)}}(S) + 1$ i.i.d. random variables with distribution of $\tau_{-l,1} = \inf\{k \geqslant 0 : S_k = -l \text{ or } S_k = 1\}$.
Therefore since $\mathsf{E}M_{\tau^{(2)}}(S) = \mathsf{E}(M_{\tau^{(2)}}(S) - S_{\tau^{(2)}}) = l$ we get

$$\boxed{\mathsf{E}\tau^{(2)} = (\mathsf{E}M_{\tau^{(2)}} + 1)\mathsf{E}\tau_{-l,1} = l(l+1)}$$

Here we used Wald identities $\mathsf{E}S_{\tau_{-l,1}} = 0$, $\mathsf{E}S^2_{\tau_{-l,1}} = \mathsf{E}\tau_{-l,1}$ in order to prove that $\mathsf{E}\tau_{-l,1} = l$.

Finally we have $\mathsf{E}\tau_l = \mathsf{E}\tau^{(1)} + \mathsf{E}\tau^{(2)} = l^2 + l(l+1) = l(2l+1)$ and $\mathsf{E}M_{\tau_l}(|S|) = \mathsf{E}(\max_{0\leqslant k\leqslant \tau^{(1)}} |S_k|) + \mathsf{E}(\max_{0\leqslant k\leqslant \tau^{(2)}} S_k) = 2l$ i.e.

$$\begin{cases} \mathsf{E}\tau_l = l(2l+1), \\ \mathsf{E}M_{\tau_l}(|S|) = 2l \end{cases}$$

From this system we find that $\mathsf{E}M_{\tau_l}(|S|) = (\sqrt{8\mathsf{E}\tau_l + 1} - 1)/2$.

Theorem 6 (Dubins-Schwarz'1988). *For any Markov time $\tau \in \mathfrak{M}$ the following* ***sharp*** *maximal inequality holds:*

$$\mathsf{E}\left(\max_{0\leqslant n\leqslant \tau} |S_n|\right) \leqslant \frac{\sqrt{8\mathsf{E}\tau + 1} - 1}{2} \qquad (23)$$

If we consider the Markov time

$$\tau_* = \inf\{n \geqslant 0 : \max_{0\leqslant k\leqslant n} |S_k| - |S_n| = N\}$$

for any $N \in \mathbb{N}$ then (23) becomes an equality.

§2. Maximal inequalities for simple symmetric Random walk

Consider the optimal stopping problem

$$V_*(c) = \sup_{\tau \in \mathfrak{M}} \mathsf{E}\left(\max_{0 \leqslant k \leqslant \tau} S_k - c\tau\right) \qquad (*)$$

Theorem 7. *The optimal stopping time* $\tau_*(c)$ *and value function* $V_*(c)$ *in problem* $(*)$ *equal*

$$\tau_*(c) = \begin{cases} \inf\{k \geqslant 0 : \left|S_k - \frac{1}{2}\right| = \left\lfloor \frac{1}{2c} + \frac{1}{2} \right\rfloor - \frac{1}{2}\}, & \textbf{\textit{if}} \ \left\lfloor \frac{1}{2c} + \frac{1}{2} \right\rfloor \geqslant \frac{1}{2c}, \\ \inf\{k \geqslant 0 : \left|S_k - \frac{1}{2}\right| = \left\lfloor \frac{1}{2c} + \frac{1}{2} \right\rfloor + \frac{1}{2}\}, & \textbf{\textit{if}} \ \left\lfloor \frac{1}{2c} + \frac{1}{2} \right\rfloor < \frac{1}{2c}. \end{cases}$$

$$V_*(c) = \begin{cases} \left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor - c\left(\left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor - \dfrac{1}{2}\right)^2 + \dfrac{c}{4} - 1, & \textbf{\textit{if}} \ \left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor \geqslant \dfrac{1}{2c}, \\ \left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor - c\left(\left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor + \dfrac{1}{2}\right)^2 + \dfrac{c}{4}, & \textbf{\textit{if}} \ \left\lfloor \dfrac{1}{2c} + \dfrac{1}{2} \right\rfloor < \dfrac{1}{2c}, \end{cases}$$

where $\lfloor x \rfloor$ *is the integer part of* x*.*

Proof. According to the **discrete version of Levy theorem** [Fujita, Mischenko]

$$\textbf{Law}\,(\max S - S,\, \max S) = \textbf{Law}\left(\left|S - \frac{1}{2}\right| - \frac{1}{2},\, L(S)\right),$$

where $L(S) = (L_n(S))_{n \geqslant 0}$, $L_n(S)$ is the number of crossings of the level $1/2$ by Random walk on $[0, n]$.
Rewriting the problem $(*)$ and using Wald identities we have

$$\mathsf{E}(M_\tau(S) - c\tau) = \mathsf{E}(M_\tau(S) - S_\tau) - c\mathsf{E}S_\tau^2 = \mathsf{E}\left(|S_\tau - 1/2| - 1/2 - cS_\tau^2 - 1/2\right)$$

Since $S_\tau^2 = (S_\tau - 1/2)^2 + S_\tau - 1/4$ we can rewrite the last expression

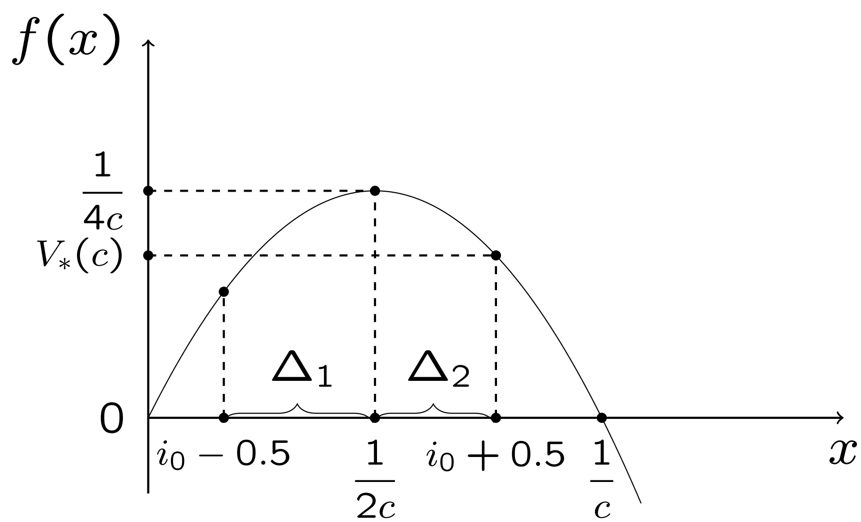
$$\mathsf{E}\left(|S_\tau - 1/2| - cS_\tau^2 - 1/2\right) = \mathsf{E}\left(|S_\tau - 1/2| - c|S_\tau - 1/2|^2\right) + c/4 - 1/2 \tag{24}$$

Observe that the resulting expression does not depend on τ explicitly, there is only dependence on $|S_\tau - 1/2|$. That's why the method we use is called **the method of space change.**

Consider the function $f(x) = x - cx^2$, $x \geqslant 0$. It attains a maximum at the point $c_0 = 1/(2c)$ and therefore $x - cx^2 \leqslant f(\frac{1}{2c}) = 1/(4c)$. Hence from (24) we get

$$\sup_{\tau \in \mathfrak{M}} \mathsf{E} \left(\max_{0 \leqslant n \leqslant \tau} S_n - c\tau \right) \leqslant \frac{1}{4c} + \frac{c}{4} - \frac{1}{2}$$

However this inequality can be not sharp if $\dfrac{1}{2c}$ does not belong to the values set $E = \{k + 1/2\}_{k \geqslant 0}$ of the process $|S - 1/2|$.



Nevertheless it is clear that the maximum of $|S_\tau - 1/2| - c|S_\tau - 1/2|^2$ is attained at the closest point to $1/(2c)$ i.e. at the point $i_0 = \left\lfloor \frac{1}{2} + \frac{1}{2c} \right\rfloor$. The values of optimal stopping time $\tau_*(c)$ and value function $V_*(c)$ depend on the relation between 2 distances $\Delta_1 = 1/(2c) - i_0 + 1/2$ and $\Delta_2 = i_0 + 1/2 - 1/(2c)$:

$$\tau_*(c) = \begin{cases} \inf\{k \geqslant 0 : \left|S_k - \frac{1}{2}\right| = i_0 - \frac{1}{2}\}, & \textbf{if } \Delta_1 \leqslant \Delta_2, \\ \inf\{k \geqslant 0 : \left|S_k - \frac{1}{2}\right| = i_0 + \frac{1}{2}\}, & \textbf{if } \Delta_1 > \Delta_2 \end{cases}$$

$$V_*(c) = \begin{cases} f(i_0 - \dfrac{1}{2}) + \dfrac{c}{4} - \dfrac{1}{2}, & \textbf{if } \Delta_1 \leqslant \Delta_2, \\ f(i_0 + \dfrac{1}{2}) + \dfrac{c}{4} - \dfrac{1}{2}, & \textbf{if } \Delta_1 > \Delta_2 \end{cases}$$

Theorem 8. *For any Markov time* $\tau \in \mathfrak{M}$ *the following inequality holds:*

$$\boxed{\mathsf{E}\left(\max_{0 \leqslant n \leqslant \tau} S_n\right) \leqslant \frac{\sqrt{4\mathsf{E}\tau + 1} - 1}{2}} \tag{25}$$

If for any $N \in \mathbb{N}$ *we consider the Markov time*

$$\tau_* = \inf\{n \geqslant 0 : \max_{0 \leqslant k \leqslant n} S_k - S_n = N\}$$

then (25) *becomes an equality.*

Proof. Use the inequality (24) which we already proved:

$$\mathsf{E}\left(\max_{0 \leqslant n \leqslant \tau} S_n\right) \leqslant \inf_{c>0}\left\{c\left(\mathsf{E}\tau + \frac{1}{4}\right) + \frac{1}{4c} - \frac{1}{2}\right\} = \frac{\sqrt{4\mathsf{E}\tau + 1} - 1}{2}$$

which gives us exactly (25).

Now show that (25) is **sharp**. Due to the discrete version of Levy theorem the time $\tau_* = \inf\{n \geqslant 0 : \max_{0 \leqslant k \leqslant n} S_k - S_n = N\}$ coincides by distribution with

$$\inf\{n \geqslant 0 : |S_n - 1/2| - 1/2 = N\} =$$
$$\inf\{n \geqslant 0 : S_n = -N \text{ or } S_n = N+1\} = \tau_{-N,N+1}$$

Using Wald identities we can check that

$$\mathsf{E}\tau_* = \mathsf{E}\tau_{-N,N+1} = N(N+1)$$

On the other hand

$$\mathsf{E}M_{\tau_*} = \mathsf{E}(M_{\tau_*} - S_{\tau_*}) = N = \frac{\sqrt{4N(N+1)+1} - 1}{2}$$

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