TIME-HOMOGENEOUS AFFINE STATE PROCESSES FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

STEFAN TAPPE

Abstract. The goal of this paper is to clarify for which starting points the state processes of a stochastic partial differential equation with an affine realization are time-homogeneous. We will illustrate our results by means of the HJMM equation from mathematical finance.

1. Introduction

In the paper [24] we have clarified when a semilinear stochastic partial differential equation (SPDE) of the form

$$\left\{ \begin{array}{rcl} dr_t & = & (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 & = & h_0 \end{array} \right. \tag{1.1}$$

in the spirit of [5] driven by a $\mathbb{R}^n$-valued Wiener process W (for some positive integer $n \in \mathbb{N}$) with an affine realization admits affine and admissible state processes. Denoting by H the state space of (1.1), which we assume to be a separable Hilbert space, this means that for each starting point $h_0 \in \mathfrak{I}$ (where $\mathfrak{I} \subset H$ denotes the set of initial points) we can express the weak (in fact, even strong) solution r to (1.1) with $r_0 = h_0$ locally as

$$r = \psi + X \tag{1.2}$$

with a deterministic curve $\psi : \mathbb{T} \to H$, where $\mathbb{T} = [0, \delta]$ for some $\delta = \delta(h_0) > 0$, and a finite dimensional (typically time-inhomogeneous) affine process X having values in the state space $\mathfrak{C} \oplus U$ with a finite dimensional proper cone $\mathfrak{C} \subset H$ and a finite dimensional subspace $U \subset H$, which makes the SPDE (1.1) analytically rather tractable.

In order to make the SPDE (1.1) even more tractable, the goal of this paper is to determine the set of all initial points $h_0 \in \mathfrak{I}$ for which the corresponding affine state processes are time-homogeneous. Actually, as we would like to emphasize already at this point, we will even look for all such initial points $h_0 \in \mathfrak{J}$ from a larger set $\mathfrak{J} \supset \mathfrak{I}$. We we explain and derive details about this larger set later.

There is a substantial literature about affine realizations for SPDEs, in particular for the HJMM equation from mathematical finance. Here we use the name HJMM equation, as it is the Heath-Jarrow-Morton (HJM) model from [16] with Musiela parametrization presented in [4]. The existence of finite dimensional realizations (FDRs) – which are à priori more general than affine realizations – for the HJMM equation driven by Wiener processes has intensively been studied in the literature, and we refer to [3, 2, 13, 14] and references therein, and to [1] for a survey. As shown in [13], the existence of a FDR for the Wiener process driven HJMM equation implies the existence of an affine realization. The existence of affine realizations has

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been studied in [21] for the HJMM equation driven by Wiener processes, in [22, 19] for the HJMM equation driven by Lévy processes, and in [23] for general SPDEs driven by Lévy processes.

Affine processes have found growing interest due to their analytical tractability, in particular regarding applications in the field of mathematical finance. We mention, for example, the papers [7, 8, 6, 10, 11].

In [24] we have clarified when a SPDE with an affine realization admits affine and admissible state processes. Given this situation, the goal of this paper is to investigate for which starting points the corresponding affine state processes are even time-homogeneous.

Let us illustrate the essential geometric ideas and outline our main findings. Concerning the precise assumptions on the parameters (A, α, σ) of the SPDE (1.1) and the precise definitions of the concepts used in the sequel, we refer to Section 2. We suppose that the set $\mathfrak{I} \subset H$ of initial points admits a decomposition

$$\mathfrak{I} = \partial \mathfrak{I} \oplus (\mathfrak{C} \oplus U) \tag{1.3}$$

with a subset $\partial \mathfrak{I} \subset H$, which we call the boundary of $\mathfrak{I}$, and that the Hilbert space H admits the direct sum decomposition $H = G \oplus V$, where $G := \overline{\langle \partial \mathfrak{I} \rangle}$, and where $V := C \oplus U$ and $C := \langle \mathfrak{C} \rangle$, the linear space generated by the cone. Furthermore, we suppose that $\partial \mathfrak{I} \cap \mathcal{D}(A)$ is open in $G \cap \mathcal{D}(A)$ with respect to the graph norm

$$\|h\|_{\mathcal{D}(A)} = \sqrt{\|h\|_H^2 + \|Ah\|_H^2}, \quad h \in \mathcal{D}(A), \tag{1.4}$$

where $\mathcal{D}(A)$ denotes the domain of the linear operator $A : \mathcal{D}(A) \subset H \to H$ appearing in (1.1). We will comment on this particular assumption later. Furthermore, we assume that the SPDE (1.1) has an affine realization generated by $\mathfrak{C} \oplus U$ with initial points $\mathfrak{I}$ and with affine and admissible state processes. Then we have

$$\mathfrak{I} \subset \mathcal{D}(A), \tag{1.5}$$

$$\sigma(\mathfrak{I}) \subset V^n, \tag{1.6}$$

and for each $g \in \partial \mathfrak{I}$ we have

$$\beta_g(v) \in V, \quad v \in \mathfrak{C} \oplus U, \tag{1.7}$$

$$v \mapsto \Pi_V \beta(g + v) : \mathfrak{C} \oplus U \to V \text{ is affine and inward pointing,} \tag{1.8}$$

$$v \mapsto \sigma(g + v) : \mathfrak{C} \oplus U \to V^n \text{ is square-affine and parallel,} \tag{1.9}$$

where $\beta : \mathcal{D}(A) \to H$ is defined as $\beta := A + \alpha$, and where we use the notation $\beta_g(v) := \beta(g + v) - \beta(g)$. According to [24], conditions (1.5)–(1.9) are necessary, and essentially also sufficient, for the existence of an affine realization with affine and admissible state processes.

Now, let $h_0 \in \mathfrak{I}$ be arbitrary, and denote by $h_0 = g_0 + v_0$ its decomposition according to (1.3). Then there exist a time interval $\mathbb{T}$ of the form $\mathbb{T} = [0, \delta]$ for some $\delta = \delta(h_0) > 0$ and a foliation – that is, a collection of affine spaces – $(\mathcal{M}_t)_{t \in \mathbb{T}}$ generated by $\mathfrak{C} \oplus U$ with $h_0 \in \mathcal{M}_0$, which is invariant for the SPDE (1.1). Denoting by $\psi : \mathbb{T} \to G$ the unique parametrization of the foliation $(\mathcal{M}_t)_{t \in \mathbb{T}}$ with values in G, we can express the foliation as

$$\mathcal{M}_t = \{\psi(t)\} \oplus (\mathfrak{C} \oplus U) \quad \text{for all } t \in \mathbb{T}.$$

Denoting by r the strong solution to the SPDE (1.1) with $r_0 = h_0$, the invariance of the foliation means that $r_t \in \mathcal{M}_t$ for each $t \in \mathbb{T}$. In the prevailing situation, we can say even more. Namely, we have the representation (1.2), where the $\mathfrak{C} \oplus U$-valued affine state process X is the strong solution to the SDE

$$\left\{ \begin{array}{rcl} dX_t & = & \Pi_V \beta(\psi(t) + X_t)dt + \sigma(\psi(t) + X_t)dW_t \\ X_0 & = & v_0. \end{array} \right. \tag{1.10}$$

So far, all described results can be found in [24]. Inspecting the SDE (1.10), we see that the affine process X is time-homogeneous if the parametrization ψ is constant – in other words $\psi(t) = g_0$ for all $t \in \mathbb{T}$ – and that this condition is essentially also necessary for X being time-homogeneous. From a geometric point of view, this means that the foliation $(\mathcal{M}_t)_{t \in \mathbb{T}}$ only consists of a single leaf, or equivalently, that the affine space $\{g_0\} \oplus (\mathfrak{C} \oplus U)$ is invariant for the SPDE (1.1). In this case, we obtain the global parametrization $\psi : \mathbb{R}_+ \to G$ given by $\psi(t) = g_0$ for all $t \in \mathbb{R}_+$. Writing the SPDE (1.1) as

$$\left\{ \begin{array}{rcl} dr_t & = & \beta(r_t)dt + \sigma(r_t)dW_t \\ r_0 & = & h_0, \end{array} \right.$$

and noting conditions (1.5)–(1.9), by geometric considerations (see Proposition 2.13 for the precise statement) it follows that the affine space $\{g_0\} \oplus (\mathfrak{C} \oplus U)$ is invariant for the SPDE (1.1) if and only if we have

$$\beta(g_0) \in V.$$

This gives rise to define the so-called *singular set* $\mathfrak{S} \subset \mathcal{D}(A)$ as

$$\mathfrak{S} := \beta^{-1}(V), \tag{1.11}$$

and to consider all starting points from the intersection

$$\mathfrak{I} \cap \mathfrak{S}. \tag{1.12}$$

In the context of the HJMM equation, the singular set $\mathfrak{S}$ was introduced in [13, 14] in a framework using convenient analysis on Fréchet spaces, and it was shown in the aforementioned papers that the singular is closed and nowhere dense, which explains its name. We will derive an analogous result in our framework; see Remark 5.6 below.

Recall that we have assumed that $\partial \mathfrak{I} \cap \mathcal{D}(A)$ is open in $G \cap \mathcal{D}(A)$ with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$. Now, it is time comment on this assumption. By condition (1.5), it just means that $\partial \mathfrak{I}$ is open in $G \cap \mathcal{D}(A)$, and this ensures that for the chosen starting point $h_0 \in \mathfrak{I}$ with decomposition $h_0 = g_0 + v_0$, where $g_0 \in \partial \mathfrak{I}$ and $v_0 \in \mathfrak{C} \oplus U$ according to (1.3), we find some $\delta = \delta(h_0) > 0$ and a local parametrization $\psi : \mathbb{T} \to G$, where $\mathbb{T} = [0, \delta]$, of an invariant foliation $(\mathcal{M}_t)_{t \in \mathbb{T}}$ such that $\psi(0) = g_0$ and

$$\psi(t) \in \partial \mathfrak{I} \quad \text{for all } t \in \mathbb{T}. \tag{1.13}$$

Condition (1.13) is required in order to ensure that the solution X to the SDE (1.10) is an affine process, as conditions (1.7)–(1.9) are only satisfied for each $g \in \partial \mathfrak{I}$. In the particular situation $h_0 \in \mathfrak{I} \cap \mathfrak{S}$ we obtain a global parametrization, namely simply the constant mapping $\psi : \mathbb{R}_+ \to G$ given by $\psi(t) = g_0$ for all $t \in \mathbb{R}_+$. Consequently, if we only consider starting points from $\mathfrak{I} \cap \mathfrak{S}$, then we do not need that $\partial \mathfrak{I}$ is open, and the representation (1.2) for the strong solution r to the SPDE (1.1) with $r_0 = h_0$ holds globally; that is, we can take the time interval $\mathbb{T} = \mathbb{R}_+$ and simply write the representation (1.2) as

$$r = g_0 + X. \tag{1.14}$$

This insight motivates us to come back to the idea which we have mentioned earlier, and to consider – instead of $\mathfrak{I} \cap \mathfrak{S}$ as in (1.12) – an intersection of the form

$$\mathfrak{I} \cap \mathfrak{S} \tag{1.15}$$

with a subset $\mathfrak{J} \subset H$ being larger that $\mathfrak{I}$. More precisely, concerning $\mathfrak{J}$ we demand the following:

1.1. **Assumption.** *We suppose that the subset $\mathfrak{J}$ satisfies the following conditions:*

(1) *We have $\mathfrak{I} \subset \mathfrak{J}$.*
(2) *It admits a decomposition of the form*

$$\mathfrak{J} = \partial\mathfrak{J} \oplus (\mathfrak{C} \oplus U) \tag{1.16}$$

with a subset $\partial\mathfrak{J} \subset G$, which we call the boundary of $\mathfrak{J}$.
(3) *Conditions (1.5) and (1.6) are fulfilled with $\mathfrak{I}$ replaced by $\mathfrak{J}$, and conditions (1.7)–(1.9) are fulfilled for each $g \in \partial\mathfrak{J}$.*

Let $\mathfrak{J} \subset H$ be a subset satisfying Assumption 1.1. As we will show (see Proposition 2.11), the intersection (1.15) has the direct sum decomposition

$$\mathfrak{J} \cap \mathfrak{S} = (\partial\mathfrak{J} \cap \mathfrak{S}) \oplus (\mathfrak{C} \oplus U). \tag{1.17}$$

In our first main result (see Theorem 3.1), we will show that for every starting point $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ we can express the strong solution r to (1.1) with $r_0 = h_0$ globally as (1.14), where $g_0 \in \partial\mathfrak{J} \cap \mathfrak{S}$ stems from the decomposition $h_0 = g_0 + v_0$ according to (1.17), and where the $\mathfrak{C} \oplus U$-valued time-homogeneous affine process X is a strong solution to the SDE

$$\begin{cases} dX_t &= \beta(g_0 + X_t)dt + \sigma(g_0 + X_t)dW_t \\ X_0 &= v_0. \end{cases} \tag{1.18}$$

In view of Theorem 3.1, it arises the question, which choices of the subset $\mathfrak{J}$ are possible such that Assumption 1.1 is fulfilled. This question is answered by our second main result (see Theorem 4.1), which states the following:

(1) The set $\mathfrak{J} := \overline{\mathfrak{I}}^{\mathcal{D}(A)}$ satisfies Assumption 1.1, and in this case the decomposition (1.16) is given by $\partial\mathfrak{J} = \overline{\partial\mathfrak{I}}^{\mathcal{D}(A)}$.
(2) For every subset $\mathfrak{J} \subset H$ satisfying Assumption 1.1 we have

$$\partial\mathfrak{J} \subset G \cap (\Pi_V \beta)^{-1}(\mathfrak{C} \oplus U). \tag{1.19}$$

Let us indicate some consequences and additional results:

- The choice $\mathfrak{J} := \overline{\mathfrak{I}}^{\mathcal{D}(A)}$ is always possible, but it is minimal if we additionally demand that $\mathfrak{J}$ is closed with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$.
- Defining the subset $\mathfrak{J}$ according to the decomposition (1.16) with boundary

$$\partial\mathfrak{J} := G \cap (\Pi_V \beta)^{-1}(\mathfrak{C} \oplus U), \tag{1.20}$$

then Assumption 1.1 is generally not satisfied. However, if it is satisfied (as it will be the case in Sections 5–9), then the choice of $\mathfrak{J}$ is maximal. Moreover, in any case $\mathfrak{J}$ is closed with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$, and the intersection $\mathfrak{J} \cap \mathfrak{S}$ is given by the decomposition (1.17) with

$$\partial\mathfrak{J} \cap \mathfrak{S} = G \cap \beta^{-1}(\mathfrak{C} \oplus U). \tag{1.21}$$

The remainder of this paper is organized as follows. In Section 2 we provide the mathematical framework and some preliminary results. In Section 3 we present our first main result and its consequences, and in Section 4 we present our second main result and its consequences. Then, in Section 5 we consider the situation where the drift has a particular structure depending on the volatility, and in Section 6 we deal with the HJMM equation, which turns out to be a particular case. In Section 7 we study affine realizations for the HJMM equation generated by a subspace, which includes the Hull-White extension of the Vasiček model and the Ho-Lee model, in Section 8 we study one-dimensional realizations for the HJMM equation generated by a cone, which includes the Hull-White extension of the Cox-Ingersoll-Ross model, and in Section 9 we treat a two-dimensional example where the cone consists of a one-dimensional proper cone and a one-dimensional subspace.

2. The mathematical framework and preliminary results

In this section, we provide the mathematical framework and some preliminary results. For further details about SPDEs of the type (1.1) we refer to [5], [20], [15] or [18]. Concerning the definitions and explanations of the notions regarding affine realizations used in the sequel, we refer to [24]; in particular Appendix A therein. Let H be a separable Hilbert space and let $A : \mathcal{D}(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup on H. Let $\alpha : H \to H$ and $\sigma : H \to H^n$ (for some positive integer $n \in \mathbb{N}$) be continuous mappings.

2.1. **Remark.** *We call a filtered probability space $\mathbb{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ satisfying the usual conditions a stochastic basis. As in* [24]*, the concepts of strong, weak and mild solutions to (1.1) are understood in a martingale sense (cf.* [5, Chap. 8]*), that is, we do not fix a stochastic basis $\mathbb{B}$ in advance, but rather call a pair (r, W) – where r is a continuous, adapted process and W a $\mathbb{R}^n$-valued standard Wiener process on some stochastic basis $\mathbb{B}$ – a strong, weak or mild solution to (1.1), if the process r has the respective property.*

Recall that in Section 1 we have introduced the mapping $\beta : \mathcal{D}(A) \to H$ by setting $\beta := A + \alpha$, and the graph norm $\| \cdot \|_{\mathcal{D}(A)}$ by (1.4).

2.2. **Remark.** *Note that, by the continuity of α and σ, the mappings*

$$\beta : (\mathcal{D}(A), \| \cdot \|_{\mathcal{D}(A)}) \to (H, \| \cdot \|_H),$$
$$\sigma|_{\mathcal{D}(A)} : (\mathcal{D}(A), \| \cdot \|_{\mathcal{D}(A)}) \to (H^n, \| \cdot \|_{H^n})$$

are continuous, too.

Recall that we have defined the singular set $\mathfrak{S} \subset \mathcal{D}(A)$ by (1.11).

2.3. **Remark.** *By Remark 2.2 the singular set $\mathfrak{S}$ is closed with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$, because V is a closed subset of H.*

Furthermore, as indicated in Section 1, let $\mathfrak{C} \subset H$ be a finite dimensional proper convex cone and let $U \subset H$ be a finite dimensional subspace such that $C \cap U = \{0\}$, where $C = \langle \mathfrak{C} \rangle$. We assume that the subspace $V = C \oplus U$ satisfies $d \geq 1$, where $d := \dim V$. We set $m := \dim C$. Let $\mathfrak{I} \subset H$ be a nonempty subset, which we call the set of initial points. We assume that it admits a decomposition of the form (1.3) with a subset $\partial\mathfrak{I} \subset H$, which we call the boundary of $\mathfrak{I}$. We assume that the Hilbert space admits the direct sum decomposition $H = G \oplus V$, where $G := \overline{\langle \partial\mathfrak{I} \rangle}$. In the sequel, we denote by $\Pi_G : H \to G$ and $\Pi_V : H \to V$ the corresponding projections.

2.4. **Assumption.** *We assume (1.5) and (1.6), and that for each $g \in \partial\mathfrak{I}$ we have (1.7)–(1.9).*

2.5. **Assumption.** *We suppose that $\partial\mathfrak{I} \cap \mathcal{D}(A)$ is open in $G \cap \mathcal{D}(A)$ with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$.*

2.6. **Assumption.** *We suppose that $\alpha : H \to H$ is Lipschitz continuous with respect to $\| \cdot \|_H$, that $\alpha(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and that $\alpha|_{\mathcal{D}(A)} : \mathcal{D}(A) \to \mathcal{D}(A)$ is Lipschitz continuous with respect to $\| \cdot \|_{\mathcal{D}(A)}$.*

2.7. **Proposition.** *Suppose that Assumptions 2.4–2.6 are fulfilled. Then the SPDE (1.1) has an affine realization generated by $\mathfrak{C} \oplus U$ with initial points $\mathfrak{I}$ and with affine and admissible state processes.*

Proof. See [24, Thm. 3.6]. □

2.8. **Remark.** *Let us point out the following observations:*

(1) *Even without imposing Assumptions 2.5 and 2.6, conditions (1.5)–(1.9) are necessary for the existence of an affine realization with affine and admissible state process; see* [24, Thm. 3.6 and Rem. 3.7].
(2) *Condition (1.5) implies that* $\partial\mathfrak{I}, C, U \subset \mathcal{D}(A)$.
(3) *By the continuity of* σ*, condition (1.6) is equivalent to* $\sigma(\overline{\mathfrak{I}}) \subset V^n$*, which we have assumed in* [24].

Let $B \subset \mathcal{D}(A)$ be a subset. Besides the closure $\overline{B} \subset H$ with respect to the Hilbert space norm $\| \cdot \|_H$, we will also consider the closure $\overline{B}^{\mathcal{D}(A)} \subset \mathcal{D}(A)$ with respect to the graph norm $\| \cdot \|_{\mathcal{D}(A)}$.

2.9. **Remark.** *We have* $B \subset \overline{B}^{\mathcal{D}(A)} \subset \overline{B}$*. In particular, if* B *is closed with respect to* $\| \cdot \|_H$*, then it is also closed with respect to* $\| \cdot \|_{\mathcal{D}(A)}$.

For the rest of this section, we will provide the precise statements of the geometric ideas presented in Section 1. Let $\mathfrak{I} \subset H$ be a subset satisfying Assumption 1.1.

2.10. **Lemma.** *For each* $g \in \partial\mathfrak{I}$ *the following statements are equivalent:*

(i) *We have* $g \in \partial\mathfrak{I} \cap \mathfrak{S}$.
(ii) *We have* $g + v \in \mathfrak{I} \cap \mathfrak{S}$ *for all* $v \in \mathfrak{C} \oplus U$.
(iii) *There exists* $v \in \mathfrak{C} \oplus U$ *such that* $g + v \in \mathfrak{I} \cap \mathfrak{S}$.

Proof. (i) $\Rightarrow$ (ii): Let $v \in \mathfrak{C} \oplus U$ be arbitrary. By the decomposition (1.16) we have $g + v \in \mathfrak{I}$. Furthermore, since $g \in \mathfrak{S}$, by (1.7) we obtain

$$\beta(g + v) = \beta(g) + \beta_g(v) \in V,$$

showing that $g + v \in \mathfrak{S}$.
(ii) $\Rightarrow$ (iii): This implication is obvious.
(iii) $\Rightarrow$ (i): Since $g + v \in \mathfrak{S}$, by (1.7) we obtain

$$\beta(g) = \beta(g + v) - \beta_g(v) \in V,$$

showing that $g \in \mathfrak{S}$. □

2.11. **Proposition.** *We have the decomposition (1.17).*

Proof. Let $h \in \mathfrak{I} \cap \mathfrak{S}$ be arbitrary. By the decomposition (1.16) of $\mathfrak{I}$ there are unique elements $g \in \partial\mathfrak{I}$ and $v \in \mathfrak{C} \oplus U$ such that $h = g + v$. By Lemma 2.10 we obtain $g \in \partial\mathfrak{I} \cap \mathfrak{S}$, and hence $h \in (\partial\mathfrak{I} \cap \mathfrak{S}) \oplus (\mathfrak{C} \oplus U)$.

Conversely, let $h \in (\partial\mathfrak{I} \cap \mathfrak{S}) \oplus (\mathfrak{C} \oplus U)$ be arbitrary. Then there are unique elements $g \in \partial\mathfrak{I} \cap \mathfrak{S}$ and $v \in \mathfrak{C} \oplus U$ such that $h = g + v$. By Lemma 2.10 we obtain $h \in \mathfrak{I} \cap \mathfrak{S}$. □

2.12. **Lemma.** *For each* $g_0 \in \partial\mathfrak{I}$ *the following statements are equivalent:*

(i) *We have* $g_0 \in \partial\mathfrak{I} \cap \mathfrak{S}$.
(ii) *We have* $\beta(g_0 + v) \in V$ *for all* $v \in \mathfrak{C} \oplus U$.
(iii) *We have*

$$\beta(g_0 + v) \in V, \quad v \in \mathfrak{C} \oplus U, \tag{2.1}$$

$$v \mapsto \beta(g_0 + v) : \mathfrak{C} \oplus U \to V \text{ is inward pointing,} \tag{2.2}$$

$$\sigma(g_0 + v) \in V^n, \quad v \in \mathfrak{C} \oplus U, \tag{2.3}$$

$$v \mapsto \sigma(g_0 + v) : \mathfrak{C} \oplus U \to V^n \text{ is parallel.} \tag{2.4}$$

Proof. (i) $\Leftrightarrow$ (ii): This equivalence follows from Lemma 2.10 and Proposition 2.11.
(ii) $\Leftrightarrow$ (iii): This equivalence follows by taking into account properties (1.5)–(1.9). □

2.13. **Proposition.** *We have the identity*

$$\partial\mathfrak{I} \cap \mathfrak{S} = \{g_0 \in \partial\mathfrak{I} : \{g_0\} \oplus (\mathfrak{C} \oplus U) \text{ is invariant for (1.1)}\}.$$

Proof. Let $g_0 \in \partial\mathfrak{I}$ be arbitrary. Then the affine space $\{g_0\} \oplus \mathfrak{C} \oplus U$ is invariant for the SPDE (1.1) if and only if we have (2.1)–(2.4), and by Lemma 2.12 this is the case if and only if $g_0 \in \partial\mathfrak{I} \cap \mathfrak{S}$. □

2.14. **Corollary.** *The following statements are true:*

(1) *The set $\mathfrak{I} \cap \mathfrak{S}$ is invariant for the SPDE (1.1).*

(2) *We have the identity*

$$\mathfrak{I} \cap \mathfrak{S} = \{h_0 \in \mathfrak{I} : \{h_0\} + (\mathfrak{C} \oplus U) \text{ is invariant for (1.1)}\}.$$

Proof. This is an immediate consequence of Propositions 2.11 and 2.13. □

3. The first main result and its consequences

In this section, we present and prove our first main result. The general mathematical framework is that of Section 2.

3.1. **Theorem.** *Let $\mathfrak{I} \subset H$ be a subset such that Assumption 1.1 is fulfilled. Let $h_0 \in \mathfrak{I} \cap \mathfrak{S}$ be arbitrary, and denote by $h_0 = g_0 + v_0$ its decomposition according to (1.17). Then the following statements are true:*

(1) *The SDE (1.18) has a strong solution X, which is a $\mathfrak{C} \oplus U$-valued time-homogeneous affine process.*

(2) *The process (1.14) is a strong solution to the SPDE (1.1) with $r_0 = h_0$.*

Proof. Due to Assumption 1.1, conditions (1.5) and (1.6) are fulfilled with $\mathfrak{I}$ replaced by $\mathfrak{J}$, and conditions (1.7)–(1.9) are fulfilled with $g = g_0$. Therefore, by [6] the SDE (1.18) has a strong solution, which is a $\mathfrak{C} \oplus U$-valued time-homogeneous affine process. By (1.14) and (1.18), the process r satisfies

$$\begin{aligned} r_t = g_0 + X_t &= g_0 + v_0 + \int_0^t \beta(g_0 + X_s)ds + \int_0^t \sigma(g_0 + X_s)dW_s \\ &= h_0 + \int_0^t \big(Ar_s + \alpha(r_s)\big)ds + \int_0^t \sigma(r_s)dW_s, \quad t \in \mathbb{R}_+, \end{aligned}$$

showing that it is a strong solution to the SPDE (1.1) with $r_0 = h_0$. □

Next, we would like to express the strong solution r to the SPDE (1.1) with respect to a given basis. For this purpose, suppose that $e := \dim G_0 < \infty$, where $G_0 := \langle \partial\mathfrak{I} \cap \mathfrak{S} \rangle$, let $\mu = (\mu_1, \ldots, \mu_e)$ be a basis of G_0, and let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a basis of $\mathfrak{C} \oplus U$, which means that λ is a basis of V, and we have

$$\mathfrak{C} = \langle \lambda_1, \ldots, \lambda_m \rangle^+ := \bigg\{ \sum_{i=1}^m \alpha_i \lambda_i : \alpha_1, \ldots, \alpha_m \geq 0 \bigg\}.$$

We define the coordinate mapping $\kappa_\lambda : V \to \mathbb{R}^d$ as the following isomorphism. For each $v \in V$ we set $\kappa_\lambda(v) := b$, where $b \in \mathbb{R}^d$ denotes the unique vector such that $v = \sum_{i=1}^d b_i \lambda_i$. For what follows, the vectors $e_1, \ldots, e_d \in \mathbb{R}^d$ denote the unit vectors in $\mathbb{R}^d$.

3.2. **Lemma.** *For every linear operator $\ell \in L(H, \mathbb{R}^d)$ the following statements are equivalent:*

(i) *We have $v = \sum_{i=1}^d \ell_i(v)\lambda_i$ for all $v \in V$.*

(ii) *We have $\ell(\lambda_i) = e_i$ for all $i = 1, \ldots, d$.*

(iii) *We have $\ell|_V = \kappa_\lambda$.*

Proof. (i) ⇒ (ii): Let $j = 1, \dots, d$ be arbitrary. Inserting $v = \lambda_j$, we obtain

$$\sum_{i=1}^d \ell_i(\lambda_j)\lambda_i = \lambda_j,$$

and hence $\ell(\lambda_j) = e_j$.
(ii) ⇒ (iii): Let $v \in V$ be arbitrary, and define the vector $b \in \mathbb{R}^d$ as $b := \kappa_\lambda(v)$. Then we have

$$\ell(v) = \ell\bigg(\sum_{i=1}^d b_i\lambda_i\bigg) = \sum_{i=1}^d b_i \ell(\lambda_i) = \sum_{i=1}^d b_i e_i = b = \kappa_\lambda(v),$$

showing that $\ell|_V = \kappa_\lambda$.
(iii) ⇒ (i): Let $v \in V$ be arbitrary, and define the vector $b \in \mathbb{R}^d$ as $b := \kappa_\lambda(v)$. Then we have

$$v = \sum_{i=1}^b b_i\lambda_i = \sum_{i=1}^d \ell_i(v)\lambda_i,$$

completing the proof. □

3.3. **Proposition.** *Let $\mathfrak{J} \subset H$ be a subset such that Assumption 1.1 is fulfilled. Let $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ be arbitrary, and denote by*

$$h_0 = \sum_{i=1}^e a_i\mu_i + \sum_{j=1}^d b_j\lambda_j \tag{3.1}$$

its representation with respect to the basis (μ, λ). Let $\ell \in L(H, \mathbb{R}^d)$ be a linear operator such that $G = \ker(\ell)$ and $\ell|_V = \kappa_\lambda$. Then the following statements are true:

(1) *The SDE*

$$\begin{cases} dY_t &= \ell(\beta(\sum_{i=1}^e a_i\mu_i + \sum_{j=1}^d Y_t^j\lambda_j))dt \\ & \quad +\ell(\sigma(\sum_{i=1}^e a_i\mu_i + \sum_{j=1}^d Y_t^j\lambda_j))dW_t \\ Y_0 &= b, \end{cases} \tag{3.2}$$

has a strong solution Y, which is a $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$-valued time-homogeneous affine process.

(2) *The process*

$$r = \sum_{i=1}^e a_i\mu_i + \sum_{j=1}^d Y^j\lambda_j, \tag{3.3}$$

is a strong solution to the SPDE (1.1) with $r_0 = h_0$.

(3) *We have $Y = \ell(r)$.*

Proof. By (3.1), the decomposition $h_0 = g_0 + v_0$ according to (1.17) is given by

$$g_0 = \sum_{i=1}^e a_i\mu_i \quad \text{and} \quad v_0 = \sum_{j=1}^d b_j\lambda_j. \tag{3.4}$$

Therefore, by Theorem 3.1 the SDE (1.18) has a strong solution X, which is a $\mathfrak{C} \oplus U$-valued time-homogeneous affine process, and the process (1.14) is a strong solution to the SPDE (1.1) with $r_0 = h_0$. Hence, the process $Y := \ell(X)$ is a

$\mathbb{R}_+^m \times \mathbb{R}^{d-m}$-valued time-homogeneous affine process. Noting the assumption $\ell|_V = \kappa_\lambda$, by Lemma 3.2 we obtain

$$X = \sum_{i=1}^d \ell_i(X)\lambda_i = \sum_{i=1}^d Y^i \lambda_i,$$

showing that Y is a strong solution to the SDE (3.2). Furthermore, in view of (1.14) and (3.4), we obtain the representation (3.3) of r, and noting the assumption $G = \ker(\ell)$, by (1.14) we obtain $Y = \ell(X) = \ell(g_0 + X) = \ell(r)$. □

4. The second main result and its consequences

In this section, we present and prove our second main result. The general mathematical framework is that of Section 2. Regarding the notions used in the following proof, we refer to [24, Appendix A].

4.1. **Theorem.** *The following statements are true:*

(1) *The set $\mathfrak{J} := \overline{\mathfrak{I}}^{\mathcal{D}(A)}$ satisfies Assumption 1.1, and its decomposition (1.16) is given by $\partial\mathfrak{J} = \overline{\partial\mathfrak{I}}^{\mathcal{D}(A)}$.*
(2) *For every subset $\mathfrak{J} \subset H$ satisfying Assumption 1.1 we have (1.19).*

Proof. By the definition of $\mathfrak{J}$ we have $\mathfrak{I} \subset \mathfrak{J}$. Next, note that

$$\overline{\mathfrak{I}}^{\mathcal{D}(A)} = \overline{\partial\mathfrak{I}}^{\mathcal{D}(A)} \oplus \overline{\mathfrak{C} \oplus U}^{\mathcal{D}(A)}.$$

Since $\mathfrak{C} \oplus U$ is closed with respect to $\|\cdot\|_H$, by Remark 2.9 we have the decomposition (1.16) with $\partial\mathfrak{J} = \overline{\partial\mathfrak{I}}^{\mathcal{D}(A)}$. Moreover, we have (1.5) with $\mathfrak{I}$ replaced by $\mathfrak{J}$ by the definition of $\mathfrak{I}$, and we have (1.6) with $\mathfrak{I}$ replaced by $\mathfrak{J}$, because σ is continuous and V is closed. Now, let $g \in \partial\mathfrak{J}$ be arbitrary. Then, there is a sequence $(g_n)_{n\in\mathbb{N}} \subset \partial\mathfrak{I}$ such that $\|g_n - g\|_{\mathcal{D}(A)}$ for $n \to \infty$. By Remark 2.2, for all $v \in \mathfrak{C} \oplus U$ we obtain

$$\beta_g(v) = \beta(g+v) - \beta(g) = \lim_{n\to\infty} \big(\beta(g_n+v) - \beta(g_n)\big) = \lim_{n\to\infty} \beta_{g_n}(v) \in V,$$

showing (1.7). Furthermore, by Remark 2.2, for all $v \in \mathfrak{C} \oplus U$ and all $\eta \in \mathfrak{C}$ with $\langle v, \eta \rangle_V = 0$ we obtain

$$\begin{aligned} \langle \eta, \Pi_V \beta(g+v) \rangle_V &= \lim_{n\to\infty} \langle \eta, \Pi_V \beta(g_n+v) \rangle_V \geq 0 \\ \text{and} \quad \langle \eta, \sigma_k(g+v) \rangle_V &= \lim_{n\to\infty} \langle \eta, \sigma_k(g_n+v) \rangle_V = 0, \quad k = 1, \ldots, n, \end{aligned}$$

showing that $v \mapsto \Pi_V \beta(g+v)$ is inward pointing and that $v \mapsto \sigma(g+v)$ is parallel. Furthermore, there are sequences $(\beta_1^n)_{n\in\mathbb{N}} \subset V$ and $(\beta_2^n)_{n\in\mathbb{N}} \subset L(V)$ such that for each $n \in \mathbb{N}$ we have

$$\Pi_V \beta(g_n + v) = \beta_1^n + \beta_2^n(v), \quad v \in \mathfrak{C} \oplus U.$$

In particular, setting $v = 0$, for each $n \in \mathbb{N}$ we obtain

$$\beta_1^n = \Pi_V \beta(g_n), \tag{4.1}$$

and, by taking account (1.7), for all $n \in \mathbb{N}$ and all $v \in \mathfrak{C} \oplus U$ we obtain

$$\beta_2^n(v) = \Pi_V \beta(g_n+v) - \beta_1^n = \Pi_V\big(\beta(g_n+v) - \beta(g_n)\big) = \beta(g_n+v) - \beta(g_n). \tag{4.2}$$

We define the element $\beta_1 \in V$ and the mapping $\beta_2 : V \to V$ as

$$\begin{aligned} \beta_1 &:= \lim_{n\to\infty} \beta_1^n \quad \text{and} \\ \beta_2(v) &:= \lim_{n\to\infty} \beta_2^n(v), \quad v \in V. \end{aligned}$$

By Remark 2.2 and relations (4.1), (4.2), these limits exist, we have $\beta_2 \in L(V)$ and

$$\beta_1 = \Pi_V \beta(g),$$
$$\beta_2(v) = \beta(g+v) - \beta(g), \quad v \in \mathfrak{C} \oplus U.$$

Therefore, we obtain

$$\Pi_V \beta(g+v) = \lim_{n\to\infty} \Pi_V \beta(g_n + v) = \lim_{n\to\infty} \big(\beta_1^n + \beta_2^n(v)\big) = \beta_1 + \beta_2(v), \quad v \in \mathfrak{C} \oplus U,$$

showing that $v \mapsto \Pi_V \beta(g+v)$ is affine, which proves (1.8). There are sequences $(T_1^n)_{n\in\mathbb{N}} \subset L(V)$ and $(T_2^n)_{n\in\mathbb{N}} \subset L(V, L(V))$ such that for each $n \in \mathbb{N}$ we have

$$\sigma^2(g_n + v) = T_1^n + T_2^n(v), \quad v \in \mathfrak{C} \oplus U.$$

In particular, for each $n \in \mathbb{N}$ we obtain

$$T_1^n = \sigma^2(g_n), \tag{4.3}$$
$$T_2^n(v) = \sigma^2(g_n + v) - \sigma^2(g_n), \quad v \in \mathfrak{C} \oplus U. \tag{4.4}$$

We define the linear operator $T_1 \in L(V)$ and the mapping $T_2 : V \to L(V)$ as

$$T_1 := \lim_{n\to\infty} T_1^n \quad \text{and}$$
$$T_2(v) := \lim_{n\to\infty} T_2^n(v), \quad v \in V.$$

Since the mapping

$$V^n \to L(V), \quad \sigma \mapsto \sigma^2 = \hat{\sigma}\hat{\sigma}^*$$

is continuous, by Remark 2.2 and relations (4.3), (4.4) these limits exists, we have $T_2 \in L(V, L(V))$ and

$$T_1 = \sigma^2(g),$$
$$T_2(v) = \sigma^2(g+v) - \sigma^2(g), \quad v \in \mathfrak{C} \oplus U.$$

Therefore, we obtain

$$\sigma^2(g+v) = \lim_{n\to\infty} \sigma^2(g_n + v) = \lim_{n\to\infty} \big(T_1^n + T_2^n(v)\big) = T_1 + T_2(v), \quad v \in \mathfrak{C} \oplus U,$$

showing that $v \mapsto \sigma(g_0 + v)$ is square-affine, which proves (1.9). Consequently, Assumption 1.1 is fulfilled with $\mathfrak{J} := \overline{\mathfrak{J}}^{\mathcal{D}(A)}$, which proves the first statement.

For the proof of the second statement, let $g_0 \in \partial\mathfrak{J}$ be arbitrary. Since $v \mapsto \Pi_V \beta(g_0 + v) : \mathfrak{C} \oplus U \to V$ is affine and inward pointing, by [24, Prop. A.10] we obtain $\Pi_V \beta(g_0) \in \mathfrak{C} \oplus U$, showing the inclusion (1.19). □

For the rest of this section, we define the subset $\mathfrak{J} \subset H$ defined according to the decomposition (1.16) with boundary $\partial\mathfrak{J}$ given by (1.20).

4.2. **Corollary.** *The subset $\mathfrak{J}$ is closed with respect to the graph norm $\|\cdot\|_{\mathcal{D}(A)}$, and we have identity (1.21).*

Proof. Noting (1.16) and (1.20), the subset $\mathfrak{J}$ is closed by Remark 2.2, and identity (1.21) follows from the definition (1.11) of the singular set $\mathfrak{S}$. □

4.3. **Proposition.** *Suppose that $\alpha(g) = 0$ for all $g \in G \cap \mathcal{D}(A)$. Then, for every direct sum decomposition $\mathcal{D}(A) = \ker(A) \oplus F$ with a closed subspace $F \subset \mathcal{D}(A)$ we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = G \cap \Big(\ker(A) \oplus \big(F \cap A^{-1}(\mathfrak{C} \oplus U)\big)\Big). \tag{4.5}$$

Proof. By assumption we have $\beta(g) = Ag$ for all $g \in G \cap \mathcal{D}(A)$. Consequently, by identity (1.21) from Corollary 4.2 we obtain

$$\begin{aligned}\partial\mathfrak{J} \cap \mathfrak{S} &= G \cap \beta^{-1}(\mathfrak{C} \oplus U) = G \cap A^{-1}(\mathfrak{C} \oplus U) \\ &= G \cap \Big(\ker(A) \oplus \big(F \cap A^{-1}(\mathfrak{C} \oplus U)\big)\Big),\end{aligned}$$

completing the proof. □

5. SPDEs with drift depending on the volatility

In this section, we investigate the situation for SPDEs with drift term having a particular structure depending on the volatility. The general mathematical framework is that of Section 2. The only difference is that we do not specify the set $\mathfrak{I}$ of initial points in advance. Instead of that, let $G \subset H$ be a closed subspace such that $H = G \oplus V$.

5.1. **Assumption.** *We suppose that the following conditions are fulfilled:*

1. *We have* $\sigma(H) \subset V^n$.
2. *The mapping* $\sigma^2 : H \to L(V)$ *is Lipschitz continuous.*
3. *There is a linear operator* $S \in L(L(V), H)$ *with* $\operatorname{ran}(S) \subset \mathcal{D}(A)$ *such that* $\alpha = S\sigma^2$.

5.2. **Assumption.** *We suppose that for each* $g \in G$ *we have (1.9), the mapping*

$$g \mapsto \sigma_g^2 : G \to L(V, L(V)) \tag{5.1}$$

is constant, and we have

$$Ac + S(\sigma^2(c) - \sigma^2(0)) \in (\mathfrak{C} \oplus \langle c \rangle) \oplus U, \quad c \in \partial\mathfrak{C}, \tag{5.2}$$

$$Au \in U, \quad u \in U. \tag{5.3}$$

In condition (5.2), the set $\partial\mathfrak{C}$ denotes the edges of the cone $\mathfrak{C}$; see [24, Appendix A]. We define the set $\mathfrak{I} \subset H$ of initial points according to the decomposition (1.3), where the boundary $\partial\mathfrak{I}$ is given by

$$\partial\mathfrak{I} := G \cap (\Pi_V \beta)^{-1}(\operatorname{Int} \mathfrak{C} \oplus U). \tag{5.4}$$

Furthermore, we define the subset $\mathfrak{J} \subset H$ according to the decomposition (1.16) with boundary $\partial\mathfrak{J}$ given by (1.20).

5.3. **Proposition.** *Suppose that Assumptions 5.1 and 5.2 are fulfilled. Then the following statements are true:*

1. *The subset* $\mathfrak{J}$ *satisfies Assumption 1.1.*
2. *We have* $G = \overline{\langle \partial\mathfrak{I} \rangle}$.
3. *Assumptions 2.4–2.6 are fulfilled.*

Proof. By construction we have $\mathfrak{I} \subset \mathfrak{J}$, and $\mathfrak{J}$ is of the form (1.16). Moreover, Assumption 2.6 is fulfilled because of Assumption 5.1, and since $L(V)$ is finite dimensional. Inspecting the proof of [24, Prop. 5.1], it follows that Assumptions 1.1, 2.4 and 2.5 are fulfilled, too, and that $G = \overline{\langle \partial\mathfrak{I} \rangle}$. □

Consequently, we are in the framework of Section 2, and all results, which we have derived so far, apply. We define the subspace $K \subset \mathcal{D}(A)$ as

$$K := A^{-1}(\operatorname{ran}(S) + V). \tag{5.5}$$

5.4. **Proposition.** *The following statements are true:*

1. *We have* $\mathfrak{S} \subset K$.

(2) *For every direct sum decomposition* $\mathcal{D}(A) = \ker(A) \oplus F$ *with a closed subspace* $F \subset \mathcal{D}(A)$ *we have*

$$K = \ker(A) \oplus \Big(F \cap A^{-1}(\mathrm{ran}(S) + V) \Big).$$

(3) *If* $\ker(A)$ *is finite dimensional, then the subspace* K *is finite dimensional, too.*

Proof. Let $h \in \mathfrak{S}$ be arbitrary. Then, there exists $v \in V$ such that $\beta(h) = v$, and hence

$$Ah + S\sigma^2(h) = \beta(h) = v.$$

Therefore, we obtain

$$Ah = -S\sigma^2(h) + v \in \mathrm{ran}(S) + V,$$

which shows that $h \in K$, proving the first statement. The second statement is obvious. For the proof of the third statement, assume that $\ker(A)$ is finite dimensional. Note that $\mathrm{ran}(S)$ is finite dimensional, because $L(V)$ is finite dimensional. Therefore, we deduce that K is finite dimensional, too. □

5.5. **Remark.** *The assumption that* $\ker(A)$ *is finite dimensional is satisfied in many situations; for example:*

- *For the HJMM equation (see Section 6) we have* $A = d/dx$*.*
- *For the heat equation the generator is given by the Laplace operator* $A = \Delta$*.*

5.6. **Remark.** *If* $\ker(A)$ *is finite dimensional, then the singular set* $\mathfrak{S} \subset \mathcal{D}(A)$ *is closed and nowhere dense with respect to the graph norm* $\| \cdot \|_{\mathcal{D}(A)}$*. This is an immediate consequence of Remark 2.3 and Proposition 5.4.*

5.7. **Corollary.** *Suppose that* $\sigma(g) = 0$ *for all* $g \in G \cap \mathcal{D}(A)$*. Then, for every direct sum decomposition* $\mathcal{D}(A) = \ker(A) \oplus F$ *with a closed subspace* $F \subset \mathcal{D}(A)$ *we have the identity (4.5).*

Proof. Let $g \in G \cap \mathcal{D}(A)$ be arbitrary. Since $\sigma(g) = 0$, we have $\sigma^2(g) = 0$, which implies $\alpha(g) = S\sigma^2(g) = 0$. Therefore, applying Proposition 4.3 completes the proof. □

For the rest of this section, we consider the situation where the affine realization is generated by a subspace; that is, we have $C = \{0\}$ and $U = V$. Then we have

$$\partial \mathfrak{I} = \partial \mathfrak{J} = G \cap \mathcal{D}(A), \quad \mathfrak{I} = \mathfrak{J} = \mathcal{D}(A), \tag{5.6}$$

$$\partial \mathfrak{J} \cap \mathfrak{S} = G \cap \beta^{-1}(V), \quad \mathfrak{J} \cap \mathfrak{S} = \beta^{-1}(V). \tag{5.7}$$

5.8. **Proposition.** *Suppose that* $S\sigma^2(H), V \subset \mathrm{ran}(A)$*, and let* $\mathcal{D}(A) = \ker(A) \oplus F$ *be a direct sum decomposition with a closed subspace* $F \subset \mathcal{D}(A)$*. Then the following statements are true:*

(1) *We have*

$$\mathfrak{J} \cap \mathfrak{S} = \{ h \in \mathcal{D}(A) : h = f + w \textit{ with } f \in F \cap A^{-1}(-S\sigma^2(h)) \textit{ and } w \in A^{-1}(V) \}. \tag{5.8}$$

(2) *If* $\sigma \in V^n$ *is constant, then we have*

$$\mathfrak{J} \cap \mathfrak{S} = \big(F \cap A^{-1}(-S\sigma^2) \big) + A^{-1}(V).$$

(3) *If* $F \cap A^{-1}(-S\sigma^2(H)) \subset G$*, then we have*

$$\begin{aligned} \partial \mathfrak{J} \cap \mathfrak{S} = \{ g \in G \cap \mathcal{D}(A) : g = f + w \textit{ with } f \in F \cap A^{-1}(-S\sigma^2(g)) \\ \textit{and } w \in G \cap A^{-1}(V) \}. \end{aligned}$$

(4) *If $\sigma \in V^n$ is constant and $F \cap A^{-1}(-S\sigma^2) \subset G$, then we have*
$$\partial\mathfrak{J} \cap \mathfrak{S} = \big(F \cap A^{-1}(-S\sigma^2)\big) + \big(G \cap A^{-1}(V)\big).$$

Proof. Let $h \in \mathfrak{J} \cap \mathfrak{S}$ be arbitrary. By (5.7) we have $h \in \mathcal{D}(A)$ and $\beta(h) \in V$. Hence, there exists $v \in V$ such that $Ah + S\sigma^2(h) = v$, and hence
$$Ah = -S\sigma^2(h) + v.$$
Since $S\sigma^2(H), V \subset \operatorname{ran}(A)$, there exist $f, g \in F$ with $Af = -S\sigma^2(h)$ and $Ag = v$, which implies $f \in F \cap A^{-1}(-S\sigma^2(h))$ and
$$Ah = A(f+g).$$
Therefore, there exists $e \in \ker(A)$ such that $h = e + f + g$. Setting $w := e + g$, we have $w \in A^{-1}(V)$ and
$$h = f + (e+g) = f + w.$$
Conversely, let $h \in \mathcal{D}(A)$ be of the form $h = f+w$ with with $f \in F \cap A^{-1}(-S\sigma^2(h))$ and $w \in A^{-1}(V)$. Then we obtain
$$\beta(h) = Ah + S\sigma^2(h) = A(f+w) + S\sigma^2(h) = -S\sigma^2(h) + Aw + S\sigma^2(h) = Aw \in V.$$
Therefore, by (5.7) we obtain $h \in \mathfrak{J} \cap \mathfrak{S}$, proving the first statement. The second statement is an immediate consequence of the first statement. Moreover, by (5.7) we have
$$\partial\mathfrak{J} \cap \mathfrak{S} = G \cap (\mathfrak{J} \cap \mathfrak{S}).$$
Therefore, taking into account the new assumption $F \cap A^{-1}(-S\sigma^2(H)) \subset G$, the remaining two statements are immediate consequences. □

5.9. **Remark.** *Note that for each $h \in \mathfrak{J} \cap \mathfrak{S}$ the decomposition $h = f + w$ appearing in the right-hand side of (5.8) is uniquely determined.*

6. The HJMM equation

In this section, we briefly review the HJMM (Heath-Jarrow-Morton-Musiela) equation, which we will consider for the rest of this paper. The HJMM equation

$$\left\{ \begin{array}{rcl} dr_t & = & \big(\frac{d}{dx} r_t + \alpha_{\mathrm{HJM}}(r_t)\big)dt + \sigma(r_t)dW_t \\ r_0 & = & h_0 \end{array} \right. \tag{6.1}$$

is a SPDE which models the term structure of interest rates in a market of zero coupon bonds. It is a SPDE of the form (1.1) with $A = d/dx$ and the particular feature that the drift term $\alpha_{\mathrm{HJM}} : H \to H$ is given by the so-called HJM drift condition

$$\alpha_{\mathrm{HJM}}(h) = \sum_{k=1}^{n} \sigma_k(h)\Sigma_k(h), \quad h \in H, \tag{6.2}$$

where $\Sigma = (\Sigma_1, \ldots, \Sigma_n) : H \to H^n$ is defined as
$$\Sigma_k(h) := \int_0^{\bullet} \sigma_k(h)(\eta)d\eta \quad \text{for } h \in H \text{ and } k = 1, \ldots, n.$$
This ensures that the corresponding bond market is free of arbitrage opportunities. The state space H is the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that
$$\|h\|_H := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x)dx \right)^{1/2} < \infty,$$

where $w : \mathbb{R}_+ \to [1, \infty)$ is a nondecreasing C^1-function such that $w^{-1/3} \in \mathcal{L}^1(\mathbb{R}_+)$. For further details we refer to Sec. 5 in [24] and references therein. We define the subspace $F \subset \mathcal{D}(A)$ as

$$F := \{h \in \mathcal{D}(A) : h(0) = 0\}.$$

For each $\lambda \in H$ we agree to denote by Λ the primitive $\Lambda := \int_0^\bullet \lambda(\eta) d\eta$.

6.1. **Lemma.** *The following statements are true:*

(1) *We have* $\ker(A) = \langle \mathbb{1} \rangle$.
(2) *F is a closed subspace of* $\mathcal{D}(A)$.
(3) *We have the direct sum decomposition* $\mathcal{D}(A) = \ker(A) \oplus F$.
(4) *For each* $\lambda \in \operatorname{ran}(A)$ *we have*

$$F \cap A^{-1}(\{\lambda\}) = \{\Lambda\}.$$

Proof. We only need to prove the second statement. Since $\ell : H \to \mathbb{R}_+$ given by $\ell(h) = h(0)$ is a continuous linear functional, which follows from estimate (5.4) in [9], the representation

$$F = \mathcal{D}(A) \cap \{h \in H : h(0) = 0\}$$

shows that F is a closed subspace of $\mathcal{D}(A)$. □

6.2. **Assumption.** *We suppose that the following conditions are fulfilled:*

(1) *We have* $V \subset \mathcal{D}(A)$.
(2) *We have* $\sigma(H) \subset V^n$.
(3) *The mapping* $\sigma^2 : H \to L(V)$ *is Lipschitz continuous.*

6.3. **Lemma.** *If Assumption 6.2 is fulfilled, then Assumption 5.1 is fulfilled, too.*

Proof. This is a consequence of [24, Prop. 6.2]. □

Now, we suppose that Assumptions 5.2 and 6.2 are fulfilled, and we define the set of initial curves[1] $\mathfrak{I}$ according to the decomposition (1.3), where the boundary $\partial\mathfrak{I}$ is given by (5.4). Furthermore, we define the subset $\mathfrak{J} \subset H$ according to the decomposition (1.16) with boundary $\partial\mathfrak{J}$ given by (1.20).

For the rest of this section, we assume that $\operatorname{ran}(S), V \subset \operatorname{ran}(A)$, and that the volatility $\sigma : H \to H$ is of the form

$$\sigma(h) = \Phi(h)\lambda, \quad h \in H \tag{6.3}$$

with a continuous mapping $\Phi : H \to \mathbb{R}$ and a function $\lambda \in V$.

6.4. **Proposition.** *We have* $\mathfrak{S} \subset \langle \mathbb{1}, \Lambda^2 \rangle + (F \cap A^{-1}(V))$.

Proof. According to the proof of [24, Prop. 6.2], we have $\operatorname{ran}(S) = \langle \lambda\Lambda \rangle$. Therefore, denoting by $K \subset \mathcal{D}(A)$ the subspace defined in (5.5), by Proposition 5.4 we obtain

$$\mathfrak{S} \subset K = \ker(A) \oplus \Big(F \cap A^{-1}(\operatorname{ran}(S) + V) \Big) \subset \langle \mathbb{1}, \Lambda^2 \rangle + (F \cap A^{-1}(V)),$$

completing the proof. □

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[1]In the context of the HJMM equation, we agree to speak about initial curves instead of initial points.

7. Affine realizations for the HJMM equation generated by a subspace

In this section, we study affine dimensional realizations for the HJMM equation generated by a subspace, and will in particular focus on the one-dimensional case. Let the volatility $\sigma : H \to H$ be of the form (6.3) with a continuous mapping $\Phi : H \to \mathbb{R}$ and a function $\lambda \in \mathcal{D}(A^\infty)$. We assume that for each $h \in H$ the mapping

$$v \mapsto \Phi(h+v) : V \to \mathbb{R}$$

is constant, and that the subspace $U \subset H$ defined as

$$U := \langle A^k \lambda : k \in \mathbb{N}_0 \rangle$$

is finite dimensional. We set $C := \{0\}$ and $V := U$, and let $G \subset H$ be an arbitrary closed subspace such that we have the direct sum decomposition $H = G \oplus V$. Then Assumptions 5.2 and 6.2 are fulfilled, and the sets $\partial\mathfrak{I}$, $\mathfrak{I}$ and $\partial\mathfrak{J}$, $\mathfrak{J}$ are given by (5.6).

7.1. **Proposition.** *We suppose that* $\operatorname{ran}(S), V \subset \operatorname{ran}(A)$*. Then the following statements are true:*

(1) *We have*

$$\mathfrak{S} \subset \langle \mathbb{1}, \Lambda^2, \Lambda, \lambda, \lambda', \ldots, \lambda^{(d-2)} \rangle.$$

(2) *We have*

$$\mathfrak{J} \cap \mathfrak{S} = \left\{ h \in \mathcal{D}(A) : h = -\frac{\Phi(h)^2}{2}\Lambda^2 + w \text{ for some } w \in \langle \mathbb{1}, \Lambda, \lambda, \lambda', \ldots, \lambda^{(d-2)} \rangle \right\}.$$

(3) *If* $\Phi \equiv \rho$ *for some constant* $\rho \in \mathbb{R}$*, then we have*

$$\mathfrak{J} \cap \mathfrak{S} = -\frac{\rho^2}{2}\Lambda^2 + \langle \mathbb{1}, \Lambda, \lambda, \lambda', \ldots, \lambda^{(d-2)} \rangle.$$

Proof. This is an immediate consequence of Propositions 6.4 and 5.8. □

Now, we consider the situation $\dim U = 1$, which means that $U = V = \langle \lambda \rangle$. Here, we can distinguish two cases, which include the Hull-White extension of the Vasiček model and the Ho-Lee model.

7.2. **Example.** *We suppose that* $\lambda = e^{-\gamma \bullet}$ *for some constant* $\gamma \in (0,\infty)$*. Then the following statements are true:*

(1) *We have*

$$\mathfrak{S} \subset \langle \Lambda^2, \mathbb{1}, \lambda \rangle.$$

(2) *We have*

$$\mathfrak{J} \cap \mathfrak{S} = \left\{ h \in \mathcal{D}(A) : h = -\frac{\Phi(h)^2}{2}\Lambda^2 + w \text{ for some } w \in \langle \mathbb{1}, \lambda \rangle \right\}.$$

(3) *If* $\Phi \equiv \rho$ *for some constant* $\rho \in \mathbb{R}$*, then we have*

$$\mathfrak{J} \cap \mathfrak{S} = -\frac{\rho^2}{2}\Lambda^2 + \langle \mathbb{1}, \lambda \rangle.$$

7.3. **Example.** *Suppose that* $\lambda = \mathbb{1}$*. Then the following statements are true:*

(1) *We have*

$$\mathfrak{S} \subset \langle \mathrm{Id}^2, \mathbb{1}, \mathrm{Id} \rangle.$$

(2) *We have*

$$\mathfrak{J} \cap \mathfrak{S} = \left\{ h \in \mathcal{D}(A) : h = -\frac{\Phi(h)^2}{2}\mathrm{Id}^2 + w \text{ for some } w \in \langle \mathbb{1}, \mathrm{Id} \rangle \right\}.$$

(3) *If $\Phi \equiv \rho$ for some constant $\rho \in \mathbb{R}$, then we have*

$$\mathfrak{J} \cap \mathfrak{S} = -\frac{\rho^2}{2}\mathrm{Id}^2 + \langle \mathbb{1}, \mathrm{Id} \rangle.$$

7.4. **Remark.** *Note that for Example 7.3 we have to change to another space of forward curves, because* $\mathrm{Id}, \mathrm{Id}^2 \notin H$.

Now, we will have a closer look at Example 7.2. For what follows, we choose a continuous linear functional $\ell \in H^*$ such that $\ell(\lambda) = 1$ and $\ell(\Lambda^2) = 0$, and we set $G := \ker(\ell)$. Then we have the direct sum decomposition $H = G \oplus V$.

7.5. **Proposition.** *The following statements are true:*

(1) *We have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \left\{ h \in \mathcal{D}(A) : h = -\frac{\Phi(h)^2}{2}\Lambda^2 + w \text{ for some } w \in \langle \mathbb{1} - \ell(\mathbb{1})\lambda \rangle \right\}.$$

(2) *If $\Phi \equiv \rho$ for some constant $\rho \in \mathbb{R}$, then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = -\frac{\rho^2}{2}\Lambda^2 \oplus \langle \mathbb{1} - \ell(\mathbb{1})\lambda \rangle.$$

Proof. We have $A^{-1}(V) = \langle \mathbb{1}, \lambda \rangle$. Therefore, and since $\ell(\lambda) = 1$, we obtain

$$G \cap A^{-1}(V) = \ker(\ell) \cap \langle \mathbb{1}, \lambda \rangle = \langle \mathbb{1} - \ell(\mathbb{1})\lambda \rangle.$$

Moreover, since $\Lambda^2 \in G$, we have

$$F \cap A^{-1}(-S\sigma^2(H)) = \left\{ -\frac{\Phi(h)^2}{2}\Lambda^2 : h \in H \right\} \subset G.$$

Therefore, applying Proposition 5.8 completes the proof. □

7.6. **Proposition.** *Let $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ be arbitrary, denote by $h_0 = g_0 + v_0$ its decomposition according to (1.17), and let*

$$g_0 = a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda) \tag{7.1}$$

be the representation of g_0 with respect to the basis $(\Lambda^2, \mathbb{1} - \ell(\mathbb{1})\lambda)$. Then the strong solution r to the HJMM equation (6.1) with $r_0 = h_0$ is given by (1.14), where the V-valued time-homogeneous affine process X is the strong solution to the SDE

$$\begin{cases} dX_t &= \gamma(b\ell(\mathbb{1})\lambda - X_t)dt + \Phi(g_0)\lambda dW_t \\ X_0 &= v_0. \end{cases} \tag{7.2}$$

Proof. Let $v \in \langle \lambda \rangle$ be arbitrary. Since $\Phi(g_0 + \bullet)$ is constant, we have

$$\sigma(g_0 + v) = \Phi(g_0 + v)\lambda = \Phi(g_0)\lambda.$$

By Proposition 7.5, the constant $a \in \mathbb{R}$ appearing in (7.1) is given by

$$a = -\frac{\Phi(g_0)^2}{2}.$$

Therefore, and since $\Phi(g_0 + \bullet)$ is constant, we obtain

$$\begin{aligned} \beta(g_0 + v) &= A(g_0 + v) + \alpha_{\mathrm{HJM}}(g_0 + v) = \frac{d}{dx}(g_0 + v) + \Phi(g_0 + v)^2\lambda\Lambda \\ &= \frac{d}{dx}\Big(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda)\Big) + \frac{d}{dx}v + \Phi(g_0)^2\lambda\Lambda \\ &= 2a\lambda\Lambda + \gamma b\ell(\mathbb{1})\lambda - \gamma v + \Phi(g_0)^2\lambda\Lambda = \gamma(b\ell(\mathbb{1})\lambda - v). \end{aligned}$$

Consequently, applying Theorem 3.1 completes the proof. □

7.7. **Proposition.** *Let $h_0 \in \mathfrak{I} \cap \mathfrak{S}$ be arbitrary, and denote by*

$$h_0 = a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda) + c\lambda \tag{7.3}$$

its representation with respect to the basis $(\Lambda^2, \mathbb{1} - \ell(\mathbb{1})\lambda, \lambda)$. Then the strong solution r to the HJMM equation (6.1) with $r_0 = h_0$ is given by

$$r = a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda) + Y\lambda, \tag{7.4}$$

where the $\mathbb{R}$-valued time-homogeneous affine process $Y = \ell(r)$ is the strong solution to the SDE

$$\begin{cases} dY_t &= \gamma(b\ell(\mathbb{1}) - Y_t)dt + \Phi\big(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda)\big)dW_t \\ Y_0 &= c. \end{cases} \tag{7.5}$$

Proof. Let $y \in \mathbb{R}$ be arbitrary. Since $\ell(\lambda) = 1$, from the calculations in the proof of Proposition 7.6 we obtain

$$\begin{aligned} \ell\big(\sigma(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda) + y\lambda)\big) &= \ell\big(\Phi(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda))\lambda\big) \\ &= \Phi\big(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda)\big) \end{aligned}$$

as well as

$$\ell\big(\beta(a\Lambda^2 + b(\mathbb{1} - \ell(\mathbb{1})\lambda) + y\lambda)\big) = \gamma\ell(b\ell(\mathbb{1})\lambda - y\lambda) = \gamma(b\ell(\mathbb{1}) - y).$$

Therefore, applying Proposition 3.3 completes the proof. □

7.8. **Remark.** *Let us emphasize the following points:*

(1) *Since $G = \ker(\ell)$, the direct sum decomposition $H = G \oplus V$ depends on the choice of the linear functional ℓ.*
(2) *The expressions (7.1)–(7.5) depend on the value $\ell(\mathbb{1})$, and simplify in the case $\ell(\mathbb{1}) = 0$.*
(3) *We can choose the short rate evaluation $\ell \in H^*$ given by $\ell(h) = h(0)$, as it fulfills the conditions $\ell(\lambda) = 1$ and $\ell(\Lambda^2) = 0$. However, then the expressions (7.1)–(7.5) do not simplify, because $\ell(\mathbb{1}) = 1$.*

8. One-dimensional realizations for the HJMM equation generated by a cone

In this section, we study one-dimensional realizations for the HJMM equation generated by a cone. We assume that the volatility $\sigma : H \to H$ is of the form

$$\sigma(h) = \rho\sqrt{|\ell(h)|}\lambda, \quad h \in H \tag{8.1}$$

with a function $\lambda \in \mathcal{D}(A)$, a constant $\rho > 0$ and a continuous linear functional $\ell \in H^*$ satisfying $\ell(\lambda) = 1$. We assume that λ is a solution of the differential equation

$$\frac{d}{dx}\lambda + \rho^2\lambda\Lambda + \gamma\lambda = 0 \tag{8.2}$$

for some constant $\gamma \in \mathbb{R}$.

8.1. **Remark.** *The solution of the Riccati differential equation*

$$\frac{d}{dx}\Lambda + \frac{\rho^2}{2}\Lambda^2 + \gamma\Lambda = 1, \quad \Lambda(0) = 0$$

is given by

$$\Lambda(x) = \frac{2(\exp(x\sqrt{\gamma^2 + \rho^2}) - 1)}{(\sqrt{\gamma^2 + 2\rho^2} + \gamma)(\exp(x\sqrt{\gamma^2 + 2\rho^2}) - 1) + 2\sqrt{\gamma^2 + 2\rho^2}}, \quad x \in \mathbb{R}_+, \tag{8.3}$$

see, for example, [9, Sec. 7.4.1]. *Therefore, the function*

$$\lambda = 1 - \frac{\rho^2}{2}\Lambda^2 - \gamma\Lambda, \tag{8.4}$$

where Λ *is given by (8.3), is a solution to the differential equation (8.2). Figure 1 shows plots of the functions* Λ *and* λ *for the parameters* $\rho = 1$ *and* $\gamma = -1$.

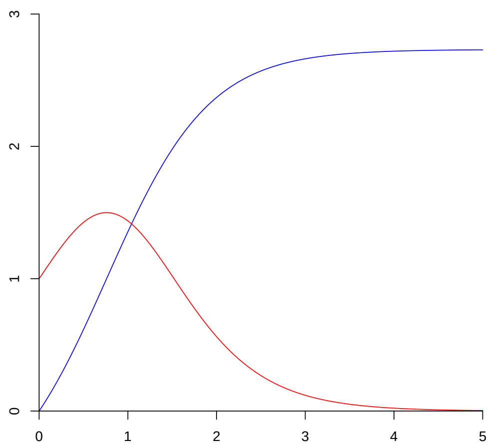

FIGURE 1. The functions Λ (blue) and λ (red) for $\rho = 1$ and $\gamma = -1$.

8.2. **Remark.** *If the functional* ℓ *is the short rate evaluation* $\ell(h) = h(0)$*, then the HJMM equation (6.1) with volatility (8.1) is the Hull-White extension of the Cox-Ingersoll-Ross model. We will treat this model at the end of this section.*

Let $\mathfrak{C} := \langle\lambda\rangle^+$, $U := \{0\}$ and $V := C := \langle\lambda\rangle$. Setting $G := \ker(\ell)$, we have the direct sum decomposition $H = G \oplus V$. Note that Assumptions 5.2 and 6.2 are fulfilled. We define the set of initial curves $\mathfrak{I}$ according to the decomposition (1.3), where the boundary $\partial\mathfrak{I}$ is given by (5.4). Furthermore, we define the subset $\mathfrak{J} \subset H$ according to the decomposition (1.16) with boundary $\partial\mathfrak{J}$ given by (1.20).

8.3. **Proposition.** *We have the identities*

$$\partial\mathfrak{I} = \{h \in \mathcal{D}(A) : \ell(h) = 0 \text{ and } \ell(h') > 0\}, \tag{8.5}$$

$$\mathfrak{I} = \{h \in \mathcal{D}(A) : \ell(h) \geq 0 \text{ and } \ell(h') + (\rho^2\ell(\lambda\Lambda) + \gamma)\ell(h) > 0\}, \tag{8.6}$$

$$\partial\mathfrak{J} = \{h \in \mathcal{D}(A) : \ell(h) = 0 \text{ and } \ell(h') \geq 0\}, \tag{8.7}$$

$$\mathfrak{J} = \{h \in \mathcal{D}(A) : \ell(h) \geq 0 \text{ and } \ell(h') + (\rho^2\ell(\lambda\Lambda) + \gamma)\ell(h) \geq 0\}. \tag{8.8}$$

Proof. This is a consequence of [24, Prop. 7.2] and its proof. □

8.4. **Corollary.** *We have* $\mathfrak{G} \subset \langle \mathbb{1}, \Lambda, \lambda\rangle$.

Proof. By Proposition 6.4 and identity (8.4) we have

$$\mathfrak{G} \subset \langle \mathbb{1}, \Lambda^2\rangle + (F \cap A^{-1}(V)) = \langle \mathbb{1}, \Lambda^2, \Lambda\rangle = \langle \mathbb{1}, \Lambda, \lambda\rangle,$$

finishing the proof. □

The following result shows that the intersection $\mathfrak{J} \cap \mathfrak{G}$ depends on the choice of the linear functional ℓ appearing in the volatility (8.1).

8.5. **Proposition.** *The following statements are true:*

(1) *If* $\Lambda, \mathbb{1} \in \ker(\ell)$*, then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \langle \mathbb{1} \rangle \oplus \langle \Lambda \rangle^+ \quad \text{and} \quad \mathfrak{J} \cap \mathfrak{S} = \langle \mathbb{1} \rangle \oplus \langle \Lambda, \lambda \rangle^+.$$

(2) *If* $\Lambda \in \ker(\ell)$ *and* $\mathbb{1} \notin \ker(\ell)$*, then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \langle \Lambda \rangle^+ \quad \text{and} \quad \mathfrak{J} \cap \mathfrak{S} = \langle \Lambda, \lambda \rangle^+.$$

(3) *If* $\Lambda \notin \ker(\ell)$ *and* $\mathbb{1} \in \ker(\ell)$*, then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \langle \mathbb{1} \rangle \quad \text{and} \quad \mathfrak{J} \cap \mathfrak{S} = \langle \mathbb{1} \rangle \oplus \langle \lambda \rangle^+.$$

(4) *If* $\Lambda, \mathbb{1} \notin \ker(\ell)$*, then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \left\langle \Lambda - \frac{\ell(\Lambda)}{\ell(\mathbb{1})}\mathbb{1} \right\rangle^+ \quad \text{and} \quad \mathfrak{J} \cap \mathfrak{S} = \left\langle \Lambda - \frac{\ell(\Lambda)}{\ell(\mathbb{1})}\mathbb{1}, \lambda \right\rangle^+.$$

(5) *In any case, we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} \subset \langle \mathbb{1} \rangle \oplus \langle \Lambda \rangle^+ \quad \text{and} \quad \mathfrak{J} \cap \mathfrak{S} \subset \langle \mathbb{1} \rangle \oplus \langle \Lambda, \lambda \rangle^+.$$

Proof. Since $G = \ker(\ell)$, by the representation (8.1) we have $\sigma(g) = 0$ for all $g \in G$. Therefore, by Corollary 5.7 we obtain

$$\partial\mathfrak{J} \cap \mathfrak{S} = G \cap \Big(\ker(A) \oplus \big(F \cap A^{-1}(\mathfrak{C})\big) \Big) = \ker(\ell) \cap (\langle \mathbb{1} \rangle \oplus \langle \Lambda \rangle^+),$$

completing the proof. □

8.6. **Remark.** *As the proof of Proposition 8.5 shows, we have the identity*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \ker(\ell) \cap (\langle \mathbb{1} \rangle \oplus \langle \Lambda \rangle^+).$$

Let $h \in \partial\mathfrak{J} \cap \mathfrak{S}$ *be arbitrary. Then we have* $\ell(h) = 0$ *and the representation* $h = a\mathbb{1} + b\Lambda$ *for some* $(a, b) \in \mathbb{R} \times \mathbb{R}_+$*. Therefore, noting that* $\ell(\lambda) = 1$*, we obtain*

$$\ell(h') = \ell(b\lambda) = b \geq 0,$$

which is in accordance with the representation (8.7) of $\partial\mathfrak{J}$ *from Proposition 8.3.*

8.7. **Proposition.** *Let* $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ *be arbitrary, and denote by* $h_0 = g_0 + v_0$ *its decomposition according to (1.17). Then the strong solution* r *to the HJMM equation (6.1) with* $r_0 = h_0$ *is given by (1.14), where the* $\mathfrak{C}$*-valued time-homogeneous affine process* X *is the strong solution to the SDE*

$$\begin{cases} dX_t &= (g_0' - \gamma \ell(X_t)\lambda)dt + \rho\sqrt{\ell(X_t)}\lambda dW_t \\ X_0 &= v_0. \end{cases}$$

Proof. Let $v \in \langle \lambda \rangle^+$ be arbitrary. Since $G = \ker(\ell)$, we obtain

$$\sigma(g_0 + v) = \rho\sqrt{|\ell(g_0 + v)|}\lambda = \rho\sqrt{\ell(v)}\lambda,$$

and by further taking into account Lemma 3.2 and equation (8.2), we obtain

$$\begin{aligned} \beta(g_0 + v) &= A(g_0 + v) + \alpha_{\text{HJM}}(g_0 + v) = \frac{d}{dx}(g_0 + v) + \rho^2 \ell(g_0 + v)\lambda\Lambda \\ &= g_0' + \ell(v)\frac{d}{dx}\lambda + \rho^2 \ell(v)\lambda\Lambda \\ &= g_0' - \ell(v)(\rho^2\lambda\Lambda + \gamma\lambda) + \rho^2\ell(v)\lambda\Lambda = g_0' - \gamma\ell(v)\lambda. \end{aligned}$$

Therefore, applying Theorem 3.1 completes the proof. □

8.8. **Proposition.** *Let $h_0 \in \mathfrak{I} \cap \mathfrak{S}$ be arbitrary, and denote by $(a,b,c) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ the vector such that*

$$h_0 = a\mathbb{1} + b\Lambda + c\lambda. \tag{8.9}$$

Then the strong solution r to the HJMM equation (6.1) with $r_0 = h_0$ is given by

$$r = a\mathbb{1} + b\Lambda + Y\lambda, \tag{8.10}$$

where the $\mathbb{R}_+$-valued time-homogeneous affine process $Y = \ell(r)$ is the strong solution to the SDE

$$\begin{cases} dY_t &= (b - \gamma Y_t)dt + \rho\sqrt{Y_t}dW_t \\ Y_0 &= c. \end{cases} \tag{8.11}$$

Proof. Let $y \in \mathbb{R}_+$ be arbitrary. Since $\ell(\lambda) = 1$, from the calculations in the proof of Proposition 8.7 we obtain

$$\ell(\sigma(a\mathbb{1} + b\Lambda + y\lambda)) = \ell(\rho\sqrt{\ell(y\lambda)}\lambda) = \rho\sqrt{y},$$

as well as

$$\ell(\beta(a\mathbb{1} + b\Lambda + y\lambda)) = \ell((a\mathbb{1} + b\Lambda)' - \gamma\ell(y\lambda)\lambda) = \ell(b\lambda - \gamma y\lambda) = b - \gamma y.$$

Therefore, applying Proposition 3.3 completes the proof. □

As pointed out in [12, Rem. 4.5], the Hull-White extension of the Cox-Ingersoll-Ross model is not positivity preserving. However, by choosing appropriate initial curves from the intersection $\mathfrak{I} \cap \mathfrak{S}$, we obtain nonnegative and even strictly positive forward curve evolutions.

8.9. **Proposition.** *Let $h_0 \in \mathfrak{I} \cap \mathfrak{S}$ be arbitrary, let $(a,b,c) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ be the vector such that we have (8.9), let r be the strong solution to the HJMM equation (6.1) with $r_0 = h_0$, and let Y be the strong solution to the SDE (8.11) with $Y_0 = c$. Then the following statements are true:*

1. *If $(a,b,c) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, then we have $r \geq 0$.*
2. *If $(a,b,c) \in (0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+$, then we have $r > 0$.*
3. *If $(a,b,c) \in \mathbb{R}_+ \times (0,\infty) \times \mathbb{R}_+$, then we have $r(x) > 0$ for all $x > 0$.*
4. *If $(a,b,c) \in \mathbb{R} \times [\rho^2/2,\infty) \times (0,\infty)$, then we have $Y > 0$.*
5. *If $(a,b,c) \in \mathbb{R}_+ \times [\rho^2/2,\infty) \times (0,\infty)$, then we have $r > 0$.*

Proof. According to Proposition 8.8 the strong solution r to the HJMM equation (6.1) with $r_0 = h_0$ is given by (8.10). Noting that $\lambda > 0$ and $\Lambda \geq 0$ with $\Lambda(x) > 0$ for all $x > 0$, the first three statements follow, and taking into account [17, Prop. 6.2.4.1], the two remaining statements follow as well. □

The Hull-White extension of the Cox-Ingersoll-Ross model is obtained by choosing the linear functional $\ell \in H^*$ as the evaluation at the short rate, that is $\ell(h) = h(0)$. Note that the condition $\ell(\lambda) = 1$ is fulfilled, because $\Lambda(0) = 0$ and we have the representation (8.4) of λ. For the Hull-White extension of the Cox-Ingersoll-Ross model, we obtain the following two results as corollaries.

8.10. **Corollary.** *We have $\mathfrak{I} \cap \mathfrak{S} = \langle \Lambda, \lambda \rangle^+$.*

Proof. Since $\Lambda \in \ker(\ell)$ and $\mathbb{1} \notin \ker(\ell)$, this is an immediate consequence of Proposition 8.5 □

8.11. **Corollary.** *Let $h_0 \in \langle \Lambda, \lambda \rangle^+$ be arbitrary, let $(b,c) \in \mathbb{R}_+ \times \mathbb{R}_+$ be the unique vector such that*

$$h_0 = b\Lambda + c\lambda,$$

and let r be the strong solution to the HJMM equation (6.1) with $r_0 = h_0$.

(1) *We have* $r \geq 0$.
(2) *If* $(b, c) \in (0, \infty) \times \mathbb{R}_+$, *then we have* $r(x) > 0$ *for all* $x > 0$.
(3) *If* $(b, c) \in [\rho^2/2, \infty) \times \mathbb{R}_+$, *then we have* $r > 0$.

Proof. This is an immediate consequence of Proposition 8.9 □

Note that Figure 1 provides an illustration of the two-dimensional cone $\langle \Lambda, \lambda \rangle^+$.

9. A two-dimensional example

In this section, we present a two-dimensional example. We assume that the volatility $\sigma : H \to H$ in the HJMM equation (6.1) is of the form

$$\sigma(h) = \rho \sqrt{|\ell_1(h)|} \lambda, \quad h \in H \tag{9.1}$$

with a constant $\rho > 0$, that the function $\lambda \in H$ is given by $\lambda(x) = e^{-\gamma x}$, $x \in \mathbb{R}_+$ for some constant $\gamma \in (0, \infty)$, and that the functional $\ell_1 \in H^*$ satisfies $\ell_1(\lambda) = 1$ and $\ell_1(\lambda^2) = 0$. Then, as pointed out in [24], the HJMM equation (6.1) cannot have an affine realization generated by some subspace with affine and admissible state processes, but it has such a realization generated by a two-dimensional cone, which is decomposed into a one-dimensional proper cone and a one-dimensional subspace. There exists $\ell_2 \in H^*$ such that $\ell = (\ell_1, \ell_2) \in L(H, \mathbb{R}^2)$ satisfies

$$\ell(\lambda) = e_1 \quad \text{and} \quad \ell(\lambda^2) = e_2, \tag{9.2}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ denote the unit vectors in $\mathbb{R}^2$.

We define $\mathfrak{C} := \langle \lambda \rangle^+$, $U := \langle \lambda^2 \rangle$ and $V := \langle \lambda, \lambda^2 \rangle$. Setting $G := \ker(\ell)$, we have the direct sum decomposition $H = G \oplus V$. Note that Assumptions 5.2 and 6.2 are fulfilled. We define the set of initial curves $\mathfrak{I}$ according to the decomposition (1.3), where the boundary $\partial \mathfrak{I}$ is given by (5.4). Furthermore, we define the subset $\mathfrak{J} \subset H$ according to the decomposition (1.16) with boundary $\partial \mathfrak{J}$ given by (1.20).

9.1. **Proposition.** *We have the identities*

$$\partial \mathfrak{I} = \{h \in \mathcal{D}(A) : \ell(h) = 0 \text{ and } \ell_1(h' + \gamma h) > 0\}, \tag{9.3}$$

$$\mathfrak{I} = \{h \in \mathcal{D}(A) : \ell_1(h) \geq 0 \text{ and } \ell_1(h' + \gamma h) > 0\}, \tag{9.4}$$

$$\partial \mathfrak{J} = \{h \in \mathcal{D}(A) : \ell(h) = 0 \text{ and } \ell_1(h' + \gamma h) \geq 0\}, \tag{9.5}$$

$$\mathfrak{J} = \{h \in \mathcal{D}(A) : \ell_1(h) \geq 0 \text{ and } \ell_1(h' + \gamma h) \geq 0\}. \tag{9.6}$$

Proof. This follows from [24, Prop. 7.6] and its proof. □

9.2. **Corollary.** *We have* $\mathfrak{S} \subset \langle \mathbb{1}, \lambda^2, \lambda \rangle$.

Proof. Noting that $\Lambda^2 \in \langle \mathbb{1}, \lambda, \lambda^2 \rangle$, by Proposition 6.4 we have

$$\mathfrak{S} \subset \langle \mathbb{1}, \Lambda^2 \rangle + (F \cap A^{-1}(V)) = \langle \mathbb{1}, \Lambda^2, \mathbb{1} - \lambda, \mathbb{1} - \lambda^2 \rangle = \langle \mathbb{1}, \Lambda^2, \lambda, \lambda^2 \rangle = \langle \mathbb{1}, \lambda^2, \lambda \rangle,$$

finishing the proof. □

9.3. **Proposition.** *The following statements are true:*

(1) *If* $\ell_1(\mathbb{1}) > 0$, *then we have*

$$\partial \mathfrak{I} \cap \mathfrak{S} = \langle \mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2 \rangle^+ \quad \text{and}$$
$$\mathfrak{I} \cap \mathfrak{S} = \{a\mathbb{1} + b\lambda + c\lambda^2 : a \in \mathbb{R}_+, b \geq -\ell_1(\mathbb{1})a \text{ and } c \in \mathbb{R}\}.$$

(2) *If* $\ell_1(\mathbb{1}) = 0$, *then we have*

$$\partial \mathfrak{I} \cap \mathfrak{S} = \langle \mathbb{1} - \ell_2(\mathbb{1})\lambda^2 \rangle \quad \text{and} \quad \mathfrak{I} \cap \mathfrak{S} = \langle \mathbb{1}, \lambda^2 \rangle \oplus \langle \lambda \rangle^+.$$

(3) *If* $\ell_1(\mathbb{1}) < 0$, *then we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \langle \ell_1(\mathbb{1})\lambda + \ell_2(\mathbb{1})\lambda^2 - \mathbb{1}\rangle^+ \quad and$$
$$\mathfrak{J} \cap \mathfrak{S} = \{a\mathbb{1} + b\lambda + c\lambda^2 : a \in \mathbb{R}_-, b \geq -\ell_1(\mathbb{1})a \text{ and } c \in \mathbb{R}\}.$$

(4) *In any case, we have*

$$\partial\mathfrak{J} \cap \mathfrak{S} \subset \langle \mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2\rangle \quad and$$
$$\mathfrak{J} \cap \mathfrak{S} \subset \langle \mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2\rangle \oplus \langle\lambda\rangle^+ \oplus \langle\lambda^2\rangle.$$

Proof. Since $G = \ker(\ell) \subset \ker(\ell_1)$, by the representation (9.1) we have $\sigma(g) = 0$ for all $g \in G$. Therefore, by Corollary 5.7 we obtain

$$\begin{aligned}\partial\mathfrak{J} \cap \mathfrak{S} &= G \cap \Big(\ker(A) \oplus \big(F \cap A^{-1}(\mathfrak{C} \oplus U)\big)\Big) = \ker(\ell) \cap (\langle \mathbb{1}, \mathbb{1} - \lambda^2\rangle \oplus \langle \mathbb{1} - \lambda\rangle^+) \\ &= \ker(\ell) \cap (\langle \mathbb{1}, \lambda^2\rangle \oplus \langle\lambda\rangle^-).\end{aligned}$$

Consequently, every $h \in \partial\mathfrak{J} \cap \mathfrak{S}$ is of the form

$$h = a\mathbb{1} + b\lambda + c\lambda^2 \tag{9.7}$$

with $a, b, c \in \mathbb{R}$ such that $b \leq 0$, and we have $\ell(h) = 0$. Moreover, by (9.2) we obtain

$$a\ell_1(\mathbb{1}) + b = 0, \tag{9.8}$$
$$a\ell_2(\mathbb{1}) + c = 0, \tag{9.9}$$

which completes the proof. □

9.4. **Remark.** *As the proof of Proposition 9.3 shows, we have the identity*

$$\partial\mathfrak{J} \cap \mathfrak{S} = \ker(\ell) \cap (\langle \mathbb{1}, \lambda^2\rangle \oplus \langle\lambda\rangle^-).$$

Let $h \in \partial\mathfrak{J} \cap \mathfrak{S}$ *be arbitrary. Then we have* $\ell(h) = 0$ *and the representation (9.7) for some* $(a, b, c) \in \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}$. *Therefore, noting (9.2) and (9.8), we obtain*

$$\ell_1(h' + \gamma h) = \ell_1(-b\gamma\lambda - 2c\gamma\lambda^2 + a\gamma\mathbb{1} + b\gamma\lambda + c\gamma\lambda^2) = a\gamma\ell_1(\mathbb{1}) = -\gamma b \geq 0,$$

which is in accordance with the representation (9.5) of $\partial\mathfrak{J}$ *from Proposition 9.1.*

9.5. **Proposition.** *Let* $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ *be arbitrary, denote by* $h_0 = g_0 + v_0$ *its decomposition according to (1.17), and let*

$$g_0 = a(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2)$$

be the representation of g_0 *with respect to the basis* $(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2)$. *Then the strong solution* r *to the HJMM equation (6.1) with* $r_0 = h_0$ *is given by (1.14), where the* $\mathfrak{C} \oplus U$*-valued time-homogeneous affine process* X *is the strong solution to the SDE*

$$\begin{cases} dX_t &= \Big[\big(a\gamma\ell_1(\mathbb{1}) + (\frac{\rho^2}{\gamma} - \gamma)\ell_1(X_t)\big)\lambda \\ &\quad +\big(2a\gamma\ell_2(\mathbb{1}) - \frac{\rho^2}{\gamma}\ell_1(X_t) - 2\gamma\ell_2(X_t)\big)\lambda^2\Big]dt \\ &\quad +\rho\sqrt{\ell_1(X_t)}\lambda dW_t \\ X_0 &= v_0. \end{cases}$$

Proof. Let $v \in \langle\lambda\rangle^+ \oplus \langle\lambda^2\rangle$ be arbitrary. Since $G = \ker(\ell)$, we obtain

$$\sigma(g_0 + v) = \rho\sqrt{|\ell_1(g_0 + v)|}\lambda = \rho\sqrt{\ell_1(v)}\lambda$$

as well as

$$\beta(g_0 + v) = A(g_0 + v) + \alpha_{\mathrm{HJM}}(g_0 + v) = \frac{d}{dx}(g_0 + v) + \rho^2\ell_1(g_0 + v)\lambda\Lambda.$$

Note the identities

$$\Lambda = \frac{1-\lambda}{\gamma} \quad \text{and} \quad g_0' = a\gamma\big(\ell_1(\mathbb{1})\lambda + 2\ell_2(\mathbb{1})\lambda^2\big).$$

Furthermore, by Lemma 3.2 we have $v = \ell_1(v)\lambda + \ell_2(v)\lambda^2$, and hence

$$v' = -\gamma\big(\ell_1(v)\lambda + 2\ell_2(v)\lambda^2\big).$$

Therefore, we obtain

$$\begin{aligned}\beta(g_0+v) &= a\gamma\big(\ell_1(\mathbb{1})\lambda + 2\ell_2(\mathbb{1})\lambda^2\big) - \gamma\big(\ell_1(v)\lambda + 2\ell_2(v)\lambda^2\big) + \frac{\rho^2}{\gamma}\ell_1(v)(\lambda-\lambda^2) \\ &= \bigg(a\gamma\ell_1(\mathbb{1}) - \gamma\ell_1(v) + \frac{\rho^2}{\gamma}\ell_1(v)\bigg)\lambda \\ &\quad + \bigg(2a\gamma\ell_2(\mathbb{1}) - 2\gamma\ell_2(v) - \frac{\rho^2}{\gamma}\ell_1(v)\bigg)\lambda^2,\end{aligned}$$

and hence, applying Theorem 3.1 completes the proof. □

9.6. **Proposition.** *Let $h_0 \in \mathfrak{J} \cap \mathfrak{S}$ be arbitrary, and denote by $(a,b,c) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ the vector such that*

$$h_0 = a(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2) + b\lambda + c\lambda^2.$$

Then the strong solution r to the HJMM equation (6.1) with $r_0 = h_0$ is given by

$$r = a(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2) + Y^1\lambda + Y^2\lambda^2,$$

where the $\mathbb{R}_+ \times \mathbb{R}$-valued time-homogeneous affine process $Y = \ell(r)$ is the strong solution to the SDE

(9.10)

$$\begin{cases} dY_t &= a\gamma \begin{pmatrix} \ell_1(\mathbb{1}) \\ 2\ell_2(\mathbb{1}) \end{pmatrix} + \begin{pmatrix} \frac{\rho^2}{\gamma} - \gamma & 0 \\ -\frac{\rho^2}{\gamma} & -2\gamma \end{pmatrix} \begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} dt + \begin{pmatrix} \rho\sqrt{Y_t^1} \\ 0 \end{pmatrix} dW_t \\ Y_0 &= \begin{pmatrix} b \\ c \end{pmatrix}. \end{cases}$$

Proof. Let $y \in \mathbb{R}_+ \times \mathbb{R}$ be arbitrary. Noting (9.2), from the calculations in the proof of Proposition 9.5 we obtain

$$\begin{aligned}\ell\big(\sigma(a(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2) + y^1\lambda + y^2\lambda^2)\big) &= \ell(\rho\sqrt{\ell_1(y^1\lambda + y^2\lambda^2)}\lambda) \\ &= \ell(\rho\sqrt{y^1}\lambda) = (\rho\sqrt{y^1}, 0)^\top\end{aligned}$$

as well as

$$\begin{aligned}&\ell\big(\beta(a(\mathbb{1} - \ell_1(\mathbb{1})\lambda - \ell_2(\mathbb{1})\lambda^2) + y^1\lambda + y^2\lambda^2)\big) \\ &= \bigg(a\gamma\ell_1(\mathbb{1}) - \gamma y^1 + \frac{\rho^2}{\gamma}y^1, 2a\gamma\ell_2(\mathbb{1}) - 2\gamma y^2 - \frac{\rho^2}{\gamma}y^1\bigg)^\top \\ &= a\gamma \begin{pmatrix} \ell_1(\mathbb{1}) \\ 2\ell_2(\mathbb{1}) \end{pmatrix} + \begin{pmatrix} \frac{\rho^2}{\gamma} - \gamma & 0 \\ -\frac{\rho^2}{\gamma} & -2\gamma \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}.\end{aligned}$$

Therefore, applying Proposition 3.3 completes the proof. □

9.7. **Remark.** *Noting that $\gamma > 0$, by Proposition 9.3 we always have $a\gamma\ell(\mathbb{1}) \geq 0$. Therefore, the drift of the SDE (9.10) is inward pointing, and the volatility is parallel at boundary points of the canonical state space $\mathbb{R}_+ \times \mathbb{R}$, which confirms that Y is a $\mathbb{R}_+ \times \mathbb{R}$-valued time-homogeneous affine process.*

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Leibniz Universität Hannover, Institut für Mathematische Stochastik, Welfengarten 1, 30167 Hannover, Germany

E-mail address: tappe@stochastik.uni-hannover.de