JUMP-DIFFUSIONS IN HILBERT SPACES: EXISTENCE, STABILITY AND NUMERICS

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Abstract. By means of an original approach, called "method of the moving frame", we establish existence, uniqueness and stability results for mild and weak solutions of stochastic partial differential equations (SPDEs) with path dependent coefficients driven by an infinite dimensional Wiener process and a compensated Poisson random measure. Our approach is based on a time-dependent coordinate transform, which reduces a wide class of SPDEs to a class of simpler SDE problems. We try to present the most general results, which we can obtain in our setting, within a self-contained framework to demonstrate our approach in all details. Also several numerical approaches to SPDEs in the spirit of this setting are presented.

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1. Introduction

Stochastic partial differential equations (SPDEs) are usually considered as stochastic perturbations of partial differential equations (PDEs). More precisely, let H be a Hilbert space and A the generator of a strongly continuous semigroup S on H, then

$$\frac{dr_t}{dt} = Ar_t + \alpha(r_t), \quad r_0 \in H$$

describes a (semi-linear) PDE on the Hilbert space of states H with linear generator A and (non-linear) term $\alpha : H \to H$. Solutions are usually defined in the mild or weak sense. A stochastic perturbation of this (semi-linear) PDE is given through a driving noise and (volatility) vector fields, for instance one can choose a one-dimensional Brownian motion W and $\sigma : H \to H$ and consider

$$dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t, \quad r_0 \in H.$$

Solution concepts, properties of solutions, manifold applications have been worked out in the most general cases, e.g., [9] in the case of Brownian noise or [27] in the case of Lévy noises.

We suggest in this article a new approach to SPDEs, which works for most of the SPDEs considered in the literature (namely those where the semigroup is pseudo-contractive). The advantages are three-fold: first one can consider most general noises with path-dependent coefficients and derive existence, uniqueness and

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stability results in an easy manner. Second the new approach easily leads to (numerical) approximation schemes for SPDEs, third the approach allows for rough path formulations (see [32]) and therefore for large deviation results, Freidlin-Wentzell type results, etc. In this article we shall mainly address existence, uniqueness and stability results for SPDEs with driving Poisson random measures and general path-dependent coefficients. An outline of the basic relation, namely short-time asymptotics, for high-order, weak or strong numerical schemes is presented, too.

In our point of view SPDEs are considered as time-dependent transformations of well-understood stochastic differential equations (SDEs). This is best described by a metaphor from physics: take the previous equation and assume $\dim H = 1$, $\alpha(r) = 0$ and $\sigma(r) = \sigma$ a constant, i.e. an Ornstein-Uhlenbeck process

$$dr_t = Ar_t dt + \sigma dW_t,\ r_0 \in \mathbb{R}$$

in dimension one describing the trajectory of a Brownian particle in a (damping) velocity field $x \mapsto Ax$. If we move our coordinate frame according to the vector field $x \mapsto Ax$ we observe a transformed movement of the particle, namely

$$df_t = \exp(-At)\sigma dW_t,\ f_0 = r_0,$$

which corresponds to a Brownian motion with time-dependent volatility, since space is scaled by a factor $\exp(-At)$ at time t and the speed of the movement of the coordinate frame makes the drift disappear. Loosely speaking, one "jumps on the moving frame", where the speed of the frame is chosen equal to the drift. In finite dimensions the advantage of this procedure is purely conceptual, since analytically the both equations can be equally well treated. If one imagines for a moment the same procedure for an SPDE the advantage is much more than conceptual, since the transformed equation, seen from the moving frame, is rather an SDE than an SPDE, as the non-continuous drift term disappears in the moving frame. More precisely, considering the variation of constants formula

$$r_t = S_t r_0 + \int_0^t S_{t-s}\alpha(r_s)ds + \int_0^t S_{t-s}\sigma(r_s)dW_s$$

we recognize the dynamics of the transformed SDE, namely the process $f_t = S_{-t}r_t$ satisfies

$$df_t = S_{-t}\alpha(S_t f_t)dt + S_{-t}\sigma(S_t f_t)dW_t,\ f_0 = r_0.$$

At this point it is clear that the drift term in infinite dimensions does usually not allow movements in negative time direction, which is crucial for the approach. This limitation can be overcome by the Szőkefalvi-Nagy theorem, which allows for group extensions of given (pseudo-contractive) semigroups of linear operators. We emphasize that we do not need the particular structure of this extension, which might be quite involved. The emphasis of this article is to provide a self-contained outline of this method in the realm of jump-diffusions with path-dependent coefficients, which has not been treated in the literature so far.

Therefore we suggest the following approach to SPDEs, which is the guideline through this article:

- consider the SDE obtained by transforming the SPDE with a time-dependent transformation $r \mapsto S_{-t}r$ (jump to the moving frame).
- solve the transformed SDE.
- transform the solution process by $r \mapsto S_t r$ in order to obtain a mild solution of the original SPDE (leave the moving frame).

In [1] and [26] existence, uniqueness and regular dependence on initial data are considered for SPDEs driven by a Wiener processes and Poisson random measures. The authors also apply the Szőkefalvi-Nagy theorem to prove certain inequalities, which are crucial for their considerations. In contrast our approach means that we reduce all these separate considerations to the analysis of *one* transformed SDE, which corresponds then – by means of the time-dependent transformation – to the solution of the given SPDE.

Our approach is based on the general jump-diffusion approach to stochastic partial differential equations as presented in [1] or [26]. In contrast, our vector fields can be path-dependent in a general sense, not only random as supposed in [26]. Applications of this setting can be found in recent work on volatility surfaces, where random dependence of the vector fields is not enough. We first do the obvious proofs for stochastic differential equations with values in (separable) Hilbert spaces. Then we show that by our transformation method ("jump to the moving frame") we can transfer those results to stochastic partial differential equations. In a completely similar way we could have taken the setting for stochastic differential equations in Ph. Protter's book [28], which is based on semi-martingales as driving processes and where we can literally transfer the respective theorems into the setting of stochastic partial differential equations. In particular all L^p-estimates – as extensively proved in [28] – can be transferred into the setting of stochastic partial differential equations.

The "moving frame approach" is a particular case of methods, where pull-backs with respect to flows are applied. Those methods have quite a long history in the theory of ODEs, PDEs and SDEs (pars pro toto we mention the Doss-Sussman method as described in [29] and the further material therein). In the realm of SPDEs the "pull-back" method has been successfully applied in [7] with respect to noise vector fields. See also a discussion in [6] where this point of view is applied again, but a pull-back with respect to the PDE part has not been applied yet.

We shall now provide a guideline for the remainder of the article. In Section 2 we define the fundamental concepts, notions and notations for stochastic integration with respect to Wiener processes and Poisson random measures. In Section 3 and 4 we provide for the sake of completeness existence and uniqueness results for Hilbert spaces valued SDEs and respective L^p-estimates. In Sections 5 and 6 we provide stability and regularity results for those SDEs. Section 7 we introduce all necessary solution concepts for (semi-linear) SPDEs. In Section 8 we apply our method of the moving frame to existence and uniqueness questions. Section 9 is devoted to the study of stability and regularity for SPDEs. Section 10 and Section 11 describe Markovian SPDE problems and several high order numerical schemes for SPDEs in this case. Again for the sake of completeness we provide a stochastic Fubini theorem with respect to compensated Poisson random measures in Appendix A.

2. Stochastic integration in Hilbert spaces

In this section, we shall outline the notion of stochastic integrals with respect to an infinite dimensional Wiener process and with respect to a compensated Poisson random measure. The construction of the stochastic integral with respect to a Brownian motion follows [9, Sec. 4.2]. The construction of the stochastic integral with respect to a Poisson measure is similar and can be found in [30] or [21, Sec. 2].

2.1. **Setting and Definitions.** From now on, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Furthermore, let H denote a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and associated norm $\| \cdot \|_H$. If there is no ambiguity, we shall simply write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

In the sequel, $\mathcal{P}$ denotes the predictable σ-algebra on $\mathbb{R}_+$ and $\mathcal{P}_T$ denotes the predictable σ-algebra on $[0, T]$ for an arbitrary $T \in \mathbb{R}_+$. We denote by λ the Lebesgue measure on $\mathbb{R}$.

For an arbitrary $p \geq 1$ and a finite time horizon $T \in \mathbb{R}_+$ we define

$$L_T^p(\lambda; H) := L^p(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; H)$$

and let $L^p(\lambda; H)$ be the space of all predictable process $\Phi : \Omega \times \mathbb{R}_+ \to H$ such that for each $T \in \mathbb{R}_+$ the restriction of Φ to $\Omega \times [0,T]$ belongs to $L_T^p(\lambda; H)$. Furthermore, $\mathcal{L}_{\text{loc}}^p(\lambda; H)$ denotes the space of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to H$ such that

$$\mathbb{P}\bigg(\int_0^T \|\Phi_t\|^p dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

Clearly, for each $\Phi \in \mathcal{L}_{\text{loc}}^p(\lambda; H)$ the path-by-path Stieltjes integral $\int_0^t \Phi_s ds$ exists.

Let $M_T^2(H)$ be the space of all square-integrable càdlàg martingales $M : \Omega \times [0,T] \to H$, where indistinguishable processes are identified. Endowed with the inner product

$$(M,N) \mapsto \mathbb{E}[\langle M_T, N_T \rangle],$$

the space $M_T^2(H)$ is a Hilbert space. The space $M_T^{2,c}(H)$, consisting of all continuous elements from $M_T^2(H)$, is a closed subspace of $M_T^2(H)$, which is a consequence of Doob's martingale inequality [9, Thm. 3.8].

2.2. **Stochastic Integration with respect to Wiener processes.** Let U be another separable Hilbert space and $Q \in L(U)$ be a compact, self-adjoint, strictly positive linear operator. Then there exist an orthonormal basis $\{e_j\}$ of U and a bounded sequence λ_j of strictly positive real numbers such that

$$Qu = \sum_j \lambda_j \langle u, e_j \rangle e_j, \quad u \in U$$

namely, the λ_j are the eigenvalues of Q, and each e_j is an eigenvector corresponding to λ_j, see, e.g., [33, Thm. VI.3.2].

The space $U_0 := Q^{\frac{1}{2}}(U)$, equipped with inner product $\langle u, v \rangle_{U_0} := \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_U$, is another separable Hilbert space and $\{\sqrt{\lambda_j} e_j\}$ is an orthonormal basis.

Let W be a Q-Wiener process [9, p. 86,87]. We assume that $\operatorname{tr}(Q) = \sum_j \lambda_j < \infty$. Otherwise, which is the case if W is a cylindrical Wiener process, there always exists a separable Hilbert space $U_1 \supset U$ on which W has a realization as a finite trace class Wiener process, see [9, Chap. 4.3].

We denote by $L_2^0 := L_2(U_0, H)$ the space of Hilbert-Schmidt operators from U_0 into H, which, endowed with the Hilbert-Schmidt norm

$$\|\Phi\|_{L_2^0} := \sqrt{\sum_j \lambda_j \|\Phi e_j\|^2}, \quad \Phi \in L_2^0$$

itself is a separable Hilbert space.

Following [9, Chap. 4.2], we define the stochastic integral $\int_0^t \Phi_s dW_s$ as an isometry, extending the obvious isometry on simple predictable processes, from $L_T^2(W; L_2^0)$ to $M_T^{2,c}(H)$, where

$$L_T^2(W; L_2^0) := L^2(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; L_2^0).$$

In particular, we obtain the *Itô-isometry*

$$\mathbb{E}\Bigg[\bigg\| \int_0^t \Phi_s dW_s \bigg\|^2 \Bigg] = \mathbb{E}\bigg[\int_0^t \|\Phi_s\|_{L_2^0}^2 ds \bigg], \quad t \in [0,T] \tag{2.1}$$

for all $\Phi \in L_T^2(W; L_2^0)$. In a straightforward manner, we extend the stochastic integral to the space $L^2(W; L_2^0)$ of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to L_2^0$ such that the restriction of Φ to $\Omega \times [0,T]$ belongs to $L_T^2(W; L_2^0)$ for all $T \in \mathbb{R}_+$, and, furthermore, to the space $\mathcal{L}^2_{\mathrm{loc}}(W; L^0_2)$ consisting of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \to L^0_2$ such that

$$\mathbb{P}\bigg(\int_0^T \|\Phi_t\|^2_{L^0_2} dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

The integral process is unique up to indistinguishability.

There is an alternative view on the stochastic integral, which we shall use in this text. According to [9, Prop. 4.1], the sequence of stochastic processes $\{\beta^j\}$ defined as $\beta^j := \frac{1}{\sqrt{\lambda_j}} \langle W, e_j \rangle$ is a sequence of real-valued independent $(\mathcal{F}_t)$-Brownian motions and we have the expansion

$$W = \sum_j \sqrt{\lambda_j} \beta^j e_j,$$

where the series is convergent in the space $M^2(U)$ of U-valued square-integrable martingales. Let $\Phi \in \mathcal{L}^2_{\mathrm{loc}}(W; L^0_2)$ be arbitrary. For each j we set $\Phi^j := \sqrt{\lambda_j} \Phi e_j$. Then we have

$$\int_0^t \Phi_s dW_s = \sum_j \int_0^t \Phi^j_s d\beta^j_s, \quad t \in \mathbb{R}_+$$

where the convergence is uniformly on compact time intervals in probability, see [9, Thm. 4.3].

2.3. **Stochastic Integration with respect to Poisson random measures.** Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [11, 15]). We remark that every Polish space with its Borel σ-field is a Blackwell space.

Now let μ be a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [18, Def. II.1.20]. Then its compensator is of the form $dt \otimes F(dx)$, where F is a σ-finite measure on $(E, \mathcal{E})$.

We define the Itô-integral $\int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$ as an isometry, which extends the obvious isometry on simple predictable processes, from $L^2_T(\mu; H)$ to $M^2_T(H)$, where

$$L^2_T(\mu; H) := L^2(\Omega \times [0,T] \times E, \mathcal{P}_T \otimes \mathcal{E}, \mathbb{P} \otimes \lambda \otimes F; H). \tag{2.2}$$

In particular, for each $\Phi \in L^2_T(\mu; H)$ we obtain the *Itô-isometry*

$$\mathbb{E}\Bigg[\bigg\| \int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds) \bigg\|^2 \Bigg] = \mathbb{E}\bigg[\int_0^t \int_E \|\Phi(s,x)\|^2 F(dx)ds \bigg] \tag{2.3}$$

for all $t \in [0,T]$. In a straightforward manner, we extend the stochastic integral to the space $L^2(\mu; H)$ of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \times E \to H$ such that the restriction of Φ to $\Omega \times [0,T] \times E$ belongs to $L^2_T(\mu; H)$ for all $T \in \mathbb{R}_+$, and, furthermore, to the space $\mathcal{L}^2_{\mathrm{loc}}(\mu; H)$ consisting of all predictable processes $\Phi : \Omega \times \mathbb{R}_+ \times E \to H$ such that

$$\mathbb{P}\bigg(\int_0^T \int_E \|\Phi(t,x)\|^2 F(dx)dt < \infty \bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

The integral process is unique up to indistinguishability.

Such a construction of the stochastic integral can, e.g., be found in [2, Sec. 4] for the finite dimensional case and in [30], [21, Sec. 2] for the infinite dimensional case.

2.4. **Path properties of stochastic integrals.** It is apparent that for every $\Phi \in \mathcal{L}^p_{\mathrm{loc}}(\lambda; H)$, where $p \geq 1$, the path-by-path Stieltjes integral $\int_0^\bullet \Phi_s ds$ has continuous sample paths.

As outlined in Section 2.2, we have first defined the stochastic integral $\int_0^t \Phi_s dW_s$ as an isometry from $L^2_T(W; L^0_2)$ to $M^{2,c}_T(H)$, the space of all square-integrable continuous martingales, and then extended it by localization. Therefore, for each $\Phi \in \mathcal{L}^2_{\mathrm{loc}}(W; L^0_2)$, the trajectories of the integral process $\int_0^\bullet \Phi_s dW_s$ are continuous.

Similarly, the stochastic integral $\int_0^t \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$, outlined in Section 2.3, is, in the first step, defined as an isometry from $L^2_T(\mu; H)$ to $M^2_T(H)$, the space of all square-integrable càdlàg martingales, and then extended by localization. Hence, for each $\Phi \in \mathcal{L}^2_{\mathrm{loc}}(\mu; H)$ the integral process $\int_0^\bullet \int_E \Phi(s,x)(\mu(ds,dx) - F(dx)ds)$ has càdlàg sample paths.

2.5. **Independence of the driving terms.** We remark that the Wiener process W and the Poisson random measure μ are independent, which we will actually only need in Section 11.

The asserted independence is provided by using the semimartingale theory from Jacod and Shiryaev [18]. Indeed, for a continuous local martingale M and a purely discontinuous local martingale N, which are both assumed to be processes with independent increments and both considered with respect to the same filtration, the semimartingale $X = (M, N)$ is again a process with independent increments, because its semimartingale characteristics (see [18, Def. II.2.6]), which we can easily compute from those of M and N, are also deterministic. Here we need the fact that $(M, N) = (M, 0) + (0, N)$ is a decomposition into a continuous and purely discontinuous local martingale. Computing the characteristic functions of M, N and X by means of [18, Thm. II.4.15] yields the desired independence.

3. Existence and uniqueness of solutions for stochastic differential equations

Since we shall show that – in case of pseudo-contractive strongly continuous semigroups – it is equivalent to consider SPDEs on the one hand or time-dependent SDEs on the other hand, we need the basic results for time-dependent SDEs with possibly infinite dimensional state space at hand. In this section we prove existence and uniqueness results for stochastic differential equations (SDEs) on a possibly infinite dimensional state space. The results are fairly standard, but we provide them in order to keep our presentation self-contained and to introduce certain notation which we shall need in the further sections.

For an interval $I \subset \mathbb{R}_+$ we define the space $C(I; H) := C(I; L^2(\Omega; H))$ of all continuous functions from I into $L^2(\Omega; H)$. If the interval I is compact, then $C(I; H)$ is a Banach space with respect to the norm

$$\|r\|_I := \sup_{t \in I} \|r_t\|_{L^2(\Omega;H)} = \sqrt{\sup_{t \in I} \mathbb{E}[\|r_t\|^2]}.$$

Note that $C(I; H)$ is a space consisting of continuous curves of equivalence classes of random variables. For each element $r \in C(I; H)$ we can associate an H-valued, mean-square continuous process $\tilde{r} = (\tilde{r}_t)_{t \in I}$, which is unique up to a version.

Let $C_{\mathrm{ad}}(I; H)$ be the subspace consisting of all adapted curves from $C(I; H)$. Note that, by the completeness of the filtration $(\mathcal{F}_t)_{t \geq 0}$, adaptedness of a curve $r \in C(I; H)$ is independent of the choice of the representative. If the interval I is compact, then the subspace $C_{\mathrm{ad}}(I; H)$ is closed with respect to the norm $\| \cdot \|_I$.

We shall also consider the spaces $\mathcal{C}(I; H)$ and $\mathcal{C}_{\mathrm{ad}}(I; H)$ of all mean-square continuous and of all adapted, mean-square continuous processes $r \in C(I; \mathcal{L}^2(\Omega; H))$.

Note that for each $r \in \mathcal{C}(I;H)$ the equivalence class $[r]$ belongs to $C(I;H)$, and if $r \in \mathcal{C}_{\mathrm{ad}}(I;H)$, then we have $[r] \in C_{\mathrm{ad}}(I;H)$.

If no confusion concerning the Hilbert space H is possible, we shall use the abbreviations $C(I)$, $C_{\mathrm{ad}}(I)$, $\mathcal{C}(I)$ and $\mathcal{C}_{\mathrm{ad}}(I)$ for $C(I;H)$, $C_{\mathrm{ad}}(I;H)$, $\mathcal{C}(I;H)$ and $\mathcal{C}_{\mathrm{ad}}(I;H)$.

We denote by $H_{\mathcal{P}}$ resp. $H_{\mathcal{P}\otimes\mathcal{E}}$ the space of all predictable processes $r : \Omega \times \mathbb{R}_+ \to H$ resp. $r : \Omega \times \mathbb{R}_+ \times E \to H$.

We shall now deal with stochastic differential equations of the kind

$$\left\{ \begin{array}{rcl} dr_t & = & \alpha(r)_t dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r|_{[0,t_0]} & = & h, \end{array} \right. \tag{3.1}$$

where $\alpha : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$. Fix $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$.

3.1. **Definition.** *A process $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ is called a* solution *for (3.1) if we have $r|_{[0,t_0]} = h$, $\alpha([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^1_{\mathrm{loc}}(\lambda;H)$, $\sigma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(W;L_2^0)$, $\gamma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(\mu;H)$ and almost surely*

$$\begin{aligned} r_t = h_{t_0} &+ \int_{t_0}^t \alpha([r])_s ds + \int_{t_0}^t \sigma([r])_s dW_s \\ &+ \int_{t_0}^t \int_E \gamma([r])(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq t_0. \end{aligned} \tag{3.2}$$

As pointed out in Section 2, the stochastic integrals at the right-hand side of (3.2) are only determined up to indistinguishability. Therefore, uniqueness *of solutions for (3.1) is meant up to indistinguishability on the interval $[t_0,\infty)$, that is, for two solutions $r, \tilde{r} \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ we have $\mathbb{P}(\bigcap_{t \geq t_0}\{r_t = \tilde{r}_t\}) = 1$.*

3.2. **Remark.** *Note that in this definition time-dependence of the vector fields is naturally included into the setting. Also observe that for $t_0 = 0$ we have $C_{\mathrm{ad}}[0,t_0] = L^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$ and $\mathcal{C}_{\mathrm{ad}}[0,t_0] = \mathcal{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$.*

The following standard assumptions are crucial for existence and uniqueness:

3.3. **Assumption.** *We assume that for all $T \in \mathbb{R}_+$ and all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$ with $r^1|_{[0,T]} = r^2|_{[0,T]}$ we have*

$$\begin{aligned} \alpha(r^1)|_{[0,T]} &= \alpha(r^2)|_{[0,T]}, \\ \sigma(r^1)|_{[0,T]} &= \sigma(r^2)|_{[0,T]}, \\ \gamma(r^1)|_{[0,T]\times E} &= \gamma(r^2)|_{[0,T]\times E}. \end{aligned}$$

3.4. **Assumption.** *Denoting by $\mathbf{0} \in C_{\mathrm{ad}}(\mathbb{R}_+)$ the zero process, we assume that*

$$t \mapsto \mathbb{E}[\|\alpha(\mathbf{0})_t\|^2] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \tag{3.3}$$

$$t \mapsto \mathbb{E}[\|\sigma(\mathbf{0})_t\|^2_{L_2^0}] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \tag{3.4}$$

$$t \mapsto \mathbb{E}\left[\int_E \|\gamma(\mathbf{0})(t,x)\|^2 F(dx)\right] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+). \tag{3.5}$$

3.5. **Assumption.** *We assume there is a function*

$$L \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+) \tag{3.6}$$

such that for all $t \in \mathbb{R}_+$ *we have*

$$\mathbb{E}[\|\alpha(r^1)_t - \alpha(r^2)_t\|^2] \leq L(t)^2 \|r^1 - r^2\|^2_{[0,t]}, \tag{3.7}$$

$$\mathbb{E}[\|\sigma(r^1)_t - \sigma(r^2)_t\|^2_{L_2^0}] \leq L(t)^2 \|r^1 - r^2\|^2_{[0,t]}, \tag{3.8}$$

$$\mathbb{E}\bigg[\int_E \|\gamma(r^1)(t,x) - \gamma(r^2)(t,x)\|^2 F(dx)\bigg] \leq L(t)^2 \|r^1 - r^2\|^2_{[0,t]} \tag{3.9}$$

for all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$.

3.6. **Remark.** *For* $p \geq 1$ *the space* $\mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}_+)$ *denotes the space of all measurable functions* $f : \mathbb{R}_+ \to \mathbb{R}$ *such that the restriction* $f|_{[0,T]}$ *belongs to* $\mathcal{L}^p[0,T]$ *for every* $T \in \mathbb{R}_+$. *Note that (3.3), (3.4), (3.5) and (3.6) are in particular satisfied if the respective functions are bounded on compact intervals.*

3.7. **Remark.** *Note that Assumptions 3.3, 3.4, 3.5 are satisfied for a wide class of SDEs (and thus – by the method of the moving frame – SPDEs) with path-dependent coefficients. As an example, we will consider equations with characteristic coefficients depending on the randomness* ω, *the time* t *and finitely many states from the path on the interval* $[0,t]$, *see Corollary 10.3 below, which together with Remark 10.10 (see also Remark 8.6) generalizes* [26, Thm. 2.4]. *We also emphasize that the Lipschitz function* L *only needs to be locally square-integrable.*

3.8. **Remark.** *In the book of Ph. Protter* [28] *stochastic differential equations driven by semimartingales are studied. The characteristic coefficients are mappings* $F : \mathbb{D} \to \mathbb{D}$, *where* $\mathbb{D}$ *denotes the space of adapted càdlàg processes. In* [28, Thm. V.7] *they are assumed to be* functional Lipschitz, *i.e. for any* $X, Y \in \mathbb{D}$ *we have*

$$F(X)^{\tau-} = F(Y)^{\tau-} \quad \text{for any stopping time } \tau \text{ with } X^{\tau-} = Y^{\tau-} \tag{3.10}$$

and almost surely

$$\|F(X)_t - F(Y)_t\| \leq K_t \sup_{s \in [0,t]} \|X_s - Y_s\| \quad \text{for each } t \geq 0, \tag{3.11}$$

where $K = (K_t)_{t \geq 0}$ *is an increasing (finite) process. By localization, Protter* [28] *assumes that* K *is uniformly bounded by some finite constant* $k > 0$, *see* [28, Lemmas V.1, V.2]. *Taking expectation in (3.11) then yields*

$$\mathbb{E}[\|F(X)_t - F(Y)_t\|^2] \leq k^2 \mathbb{E}\bigg[\sup_{s \in [0,t]} \|X_s - Y_s\|^2\bigg] = k^2 \|X - Y\|^2_{S^2[0,t]}, \tag{3.12}$$

and existence and uniqueness is proven by a fixed point argument on the space S^2. *We, in contrast, will apply a fixed point argument on the space* $C_{\mathrm{ad}}(\mathbb{R}_+)$, *and show the existence of a càdlàg version afterwards, see Theorem 3.11 below. Note that Assumption 3.3 corresponds to (3.10) and Assumption 3.5 corresponds to (3.12). Hence, our assumptions can be regarded as an analogue to the functional Lipschitz property in* [28].

3.9. **Lemma.** *For each* $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ *the functions*

$$t \mapsto \mathbb{E}\bigg[\int_0^t \|\alpha(r)_s\|^2 ds\bigg], \tag{3.13}$$

$$t \mapsto \mathbb{E}\bigg[\int_0^t \|\sigma(r)_s\|^2_{L_2^0} ds\bigg], \tag{3.14}$$

$$t \mapsto \mathbb{E}\bigg[\int_0^t \int_E \|\gamma(r)(s,x)\|^2 F(dx) ds\bigg] \tag{3.15}$$

are well-defined and continuous on $\mathbb{R}_+$.

Proof. Let $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$ be arbitrary. Using the Lipschitz conditions (3.7), (3.8), (3.9) we obtain

$$\mathbb{E}\bigg[\int_0^t \|\alpha(r)_s\|^2 ds \bigg] \leq 2 \int_0^t L(s)^2 \|r\|_{[0,s]}^2 ds + 2 \int_0^t \mathbb{E}[\|\alpha(\mathbf{0})_s\|^2] ds,$$
$$\mathbb{E}\bigg[\int_0^t \|\sigma(r)_s\|_{L_2^0}^2 ds \bigg] \leq 2 \int_0^t L(s)^2 \|r\|_{[0,s]}^2 ds + 2 \int_0^t \mathbb{E}[\|\sigma(\mathbf{0})_s\|_{L_2^0}^2] ds,$$
$$\mathbb{E}\bigg[\int_0^t \int_E \|\gamma(r)(s,x)\|^2 F(dx) ds \bigg]$$
$$\leq 2 \int_0^t L(s)^2 \|r\|_{[0,s]}^2 ds + 2 \int_0^t \mathbb{E}\bigg[\int_E \|\gamma(\mathbf{0})(s,x)\|^2 F(dx) \bigg] ds.$$

Note that, by (3.6), we have

$$\int_0^t L(s)^2 \|r\|_{[0,s]}^2 ds \leq \|r\|_{[0,t]}^2 \int_0^t L(s)^2 ds < \infty.$$

Together with (3.3), (3.4), (3.5) we deduce that the functions in (3.13), (3.14), (3.15) are well-defined. The continuity follows from Lebesgue's theorem. □

According to Lemma 3.9, for all $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ we have $\alpha(r) \in L^2(\lambda; H)$, $\sigma(r) \in L^2(W; L_2^0)$ and $\gamma(r) \in L^2(\mu; H)$. This ensures that the following stochastic integrals in (3.16) are well-defined.

For any $T \in \mathbb{R}_+$ and $r \in C_{\mathrm{ad}}[0,T]$ we define

$$\alpha(r) := \alpha(\tilde{r})|_{[0,T]},$$
$$\sigma(r) := \sigma(\tilde{r})|_{[0,T]},$$
$$\gamma(r) := \gamma(\tilde{r})|_{[0,T]\times E},$$

where we have chosen $\tilde{r} \in C_{\mathrm{ad}}(\mathbb{R}_+)$ such that $r = \tilde{r}|_{[0,T]}$. Such an element $\tilde{r}$ always exists. Take, for example, the constant continuation $\tilde{r}_t := r_T$ for $t \geq T$. Notice also that this definition is independent of the choice of $\tilde{r}$ by virtue of Assumption 3.3.

Let us fix $t_0 \in \mathbb{R}_+$, $T \geq t_0$, $h \in C_{\mathrm{ad}}[0,t_0]$ and $r \in C_{\mathrm{ad}}[0,T]$. We define $\Lambda_h(r)$ by $\Lambda_h(r)|_{[0,t_0]} := h$ and

$$\begin{aligned} \Lambda_h(r)_t &:= h_{t_0} + \int_{t_0}^t \alpha(r)_s ds + \int_{t_0}^t \sigma(r)_s dW_s \\ &\quad + \int_{t_0}^t \int_E \gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \in [t_0,T]. \end{aligned} \tag{3.16}$$

By Hölder's inequality, the Itô-isometries (2.1), (2.3) and Lemma 3.9, the process $\Lambda_h(r)$ is mean-square continuous. By taking the respective equivalence classes, this induces a mapping $\Lambda_h : C_{\mathrm{ad}}[0,T] \to C_{\mathrm{ad}}[0,T]$.

In an analogous fashion, we define a mapping $\Lambda_h : C_{\mathrm{ad}}(\mathbb{R}_+) \to C_{\mathrm{ad}}(\mathbb{R}_+)$.

Now we fix $T_1, T_2 \in \mathbb{R}_+$ with $T_1 \leq T_2$ and $r^1 \in C_{\mathrm{ad}}[0,T_1]$. For $r^2 \in C_{\mathrm{ad}}[T_1,T_2]$ we have

$$(r^1, r^2) := \big((r_s^1)_{s\in[0,T_1]}, (r_s^2 + r_{T_1}^1 - r_{T_1}^2)_{s\in(T_1,T_2]} \big) \in C_{\mathrm{ad}}[0,T_2].$$

Hence, we can define

$$\begin{aligned} \Gamma_{r^1}(r^2)_t &:= r_{T_1}^1 + \int_{T_1}^t \alpha(r^1,r^2)_s ds + \int_{T_1}^t \sigma(r^1,r^2)_s dW_s \\ &\quad + \int_{T_1}^t \gamma(r^1,r^2)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \in [T_1,T_2]. \end{aligned}$$

By Hölder's inequality, the Itô-isometries (2.1), (2.3) and Lemma 3.9, the process $\Gamma_{r^1}(r^2)$ is mean-square continuous. By taking the respective equivalence classes, this induces a mapping $\Gamma_{r^1} : C_{\mathrm{ad}}[T_1, T_2] \to C_{\mathrm{ad}}[T_1, T_2]$.

3.10. **Lemma.** *Let $t_0 \in \mathbb{R}_+$ be arbitrary. There exists a sequence $t_0 = T_0 < T_1 < T_2 < \dots$ with $T_n \to \infty$ such that for all $n \in \mathbb{N}_0$ and all $h \in C_{\mathrm{ad}}[0, T_n]$ the map Γ_h is a contraction on $C_{\mathrm{ad}}[T_n, T_{n+1}]$.*

Proof. We choose an arbitrary $\epsilon \in (0,1)$. Let $\delta > 0$ be such that

$$f(t) \leq 25, \quad t \in [0, \delta] \tag{3.17}$$

where $f(t) := 12(t+2)$. By (3.6) and Lebesgue's theorem, the map $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(t) = \int_0^t L(s)^2 ds$ is continuous. Since g is uniformly continuous on compact intervals of $\mathbb{R}_+$, there exists a sequence $t_0 = T_0 < T_1 < T_2 < \dots$ with $\sup_{n \in \mathbb{N}_0} |T_{n+1} - T_n| \leq \delta$ and $T_n \to \infty$ such that

$$|g(T_n) - g(T_{n+1})| \leq \frac{\epsilon^2}{25} \quad \text{for all } n \in \mathbb{N}_0. \tag{3.18}$$

Let $n \in \mathbb{N}_0$ and $h \in C_{\mathrm{ad}}[0, T_n]$ be arbitrary. We fix $r^1, r^2 \in C_{\mathrm{ad}}[T_n, T_{n+1}]$ and $t \in [T_n, T_{n+1}]$. By using Hölder's inequality and (3.7) we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg\| \int_{T_n}^t (\alpha(h, r^1)_s - \alpha(h, r^2)_s) ds \bigg\|^2\Bigg] \\
&\leq (t - T_n) \int_{T_n}^t L(s)^2 \| r^1 + h_{T_n} - r^1_{T_n} - (r^2 + h_{T_n} - r^2_{T_n}) \|^2_{[T_n, s]} ds \\
&\leq 4(t - T_n) \bigg(\int_{T_n}^t L(s)^2 ds \bigg) \| r^1 - r^2 \|^2_{[T_n, T_{n+1}]}.
\end{aligned}$$

The Itô-isometry (2.1) and (3.8) yield

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg\| \int_{T_n}^t (\sigma(h, r^1)_s - \sigma(h, r^2)_s) dW_s \bigg\|^2\Bigg] \\
&\leq \int_{T_n}^t L(s)^2 \| r^1 + h_{T_n} - r^1_{T_n} - (r^2 + h_{T_n} - r^2_{T_n}) \|^2_{[T_n, s]} ds \\
&\leq 4 \bigg(\int_{T_n}^t L(s)^2 ds \bigg) \| r^1 - r^2 \|^2_{[T_n, T_{n+1}]},
\end{aligned}$$

and the Itô-isometry (2.3) and (3.9) give us an analogous estimate for the jump part. Thus, we obtain for all $t \in [T_n, T_{n+1}]$ the estimate

$$\begin{aligned}
\mathbb{E}[\| \Gamma_h(r^1)_t - \Gamma_h(r^2)_t \|^2] &\leq 12(t - T_n + 2) \bigg(\int_{T_n}^t L(s)^2 ds \bigg) \| r^1 - r^2 \|^2_{[T_n, T_{n+1}]} \\
&= f(t - T_n)(g(t) - g(T_n)) \| r^1 - r^2 \|^2_{[T_n, T_{n+1}]},
\end{aligned}$$

which implies, by taking into account (3.17) and (3.18),

$$\| \Gamma_h(r^1) - \Gamma_h(r^2) \|_{[T_n, T_{n+1}]} \leq \epsilon \| r^1 - r^2 \|_{[T_n, T_{n+1}]},$$

proving that Γ_h is a contraction on $C_{\mathrm{ad}}[T_n, T_{n+1}]$. □

3.11. **Theorem.** *Suppose that Assumptions 3.3, 3.4, 3.5 are fulfilled. Then, for each $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0, t_0]$ there exists a unique solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (3.1) with càdlàg paths on $[t_0, \infty)$, and it satisfies*

$$\mathbb{E}\bigg[\sup_{t \in [t_0, T]} \| r_t \|^2 \bigg] < \infty \quad \textit{for all } T \geq t_0. \tag{3.19}$$

Proof. Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ be arbitrary. We identify h with its equivalence class and fix a sequence $(T_n)_{n\in\mathbb{N}}$ as in Lemma 3.10. By induction we shall prove that for each $n \in \mathbb{N}_0$ the fixed point equation

$$r^n = \Lambda_h(r^n), \quad r^n \in C_{\mathrm{ad}}[0,T_n] \tag{3.20}$$

has a unique solution. For $n=0$ the unique solution for (3.20) is given by $r^0 = h$. We proceed with the induction step $n \to n+1$. By the Banach fixed point theorem there exists a unique solution for

$$\tilde{r}^{n+1} = \Gamma_{r^n}(\tilde{r}^{n+1}), \quad \tilde{r}^{n+1} \in C_{\mathrm{ad}}[T_n, T_{n+1}]. \tag{3.21}$$

The process $r^{n+1} := ((r^n)_{t\in[0,T_n]}, (\tilde{r}^{n+1})_{t\in(T_n,T_{n+1}]})$ belongs to $C_{\mathrm{ad}}[0,T_{n+1}]$, because $r^n_{T_n} = \tilde{r}^{n+1}_{T_n}$ by (3.21), and, by taking into account Assumption 3.3, it is the unique solution for

$$r^{n+1} = \Lambda_h(r^{n+1}), \quad r^{n+1} \in C_{\mathrm{ad}}[0,T_{n+1}].$$

Since $T_n \to \infty$, there exists, by noting Assumption 3.3 again, a unique solution $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ for the fixed point equation

$$r = \Lambda_h(r), \quad r \in C_{\mathrm{ad}}(\mathbb{R}_+). \tag{3.22}$$

The right-hand side of (3.22) consists of the sum of stochastic integrals. Therefore, there exists a representative $\tilde{r} \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ of $\Lambda_h(r)$ with càdlàg paths on $[t_0,\infty)$, see Section 2.4. Equation (3.22) yields, up to indistinguishability,

$$\begin{aligned}\tilde{r}_t = h_{t_0} &+ \int_{t_0}^t \alpha([\tilde{r}])_s ds + \int_{t_0}^t \sigma([\tilde{r}])_s dW_s \\ &+ \int_{t_0}^t \int_E \gamma([\tilde{r}])(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq t_0.\end{aligned}$$

Since any two representatives of r, which are càdlàg on $[t_0,\infty)$, are indistinguishable on $[t_0,\infty)$, this shows that $\tilde{r}$ is the unique solution for (3.1). Relation (3.19) is established by Hölder's inequality, Doob's martingale inequality [9, Thm. 3.8], the Itô-isometries (2.1), (2.3) and Lemma 3.9. □

3.12. **Remark.** *The idea to work on the space $C_{\mathrm{ad}}(\mathbb{R}_+)$ already appears in the proof of* [14, Thm. 4.1]*, which deals with infinite dimensional stochastic differential equations driven by Wiener processes.*

4. L^p-ESTIMATES

In order to carry L^p-theory from SDEs with possibly infinite dimensional state space to SPDEs we provide the relevant results for SDEs here. For the SDEs of Section 3 the full theory of L^p-estimates for solutions of stochastic differential equations holds true.

Let $p \geq 2$ be arbitrary. In this section, for any interval $I \subset \mathbb{R}_+$ we consider the space $C(I) := C(I; L^p(\Omega;H))$ of all continuous functions from I into $L^p(\Omega;H)$. If the interval I is compact, we equip $C(I)$ with the norm

$$\|r\|_I := \sup_{t\in I} \|r_t\|_{L^p(\Omega;H)} = \left(\sup_{t\in I} \mathbb{E}[\|r_t\|^p] \right)^{\frac{1}{p}}.$$

We replace Assumptions 3.4 and 3.5 by the following stronger assumptions.

4.1. **Assumption.** *Denoting by $\mathbf{0} \in C_{\mathrm{ad}}(\mathbb{R}_+)$ the zero process, we assume that*

$$
\begin{aligned}
t &\mapsto \mathbb{E}[\|\alpha(\mathbf{0})_t\|^p] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \mathbb{E}[\|\sigma(\mathbf{0})_t\|^p_{L_2^0}] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \mathbb{E}\bigg[\int_E \|\gamma(\mathbf{0})(t,x)\|^p F(dx) \bigg] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+).
\end{aligned}
$$

4.2. **Assumption.** *We assume there is a function*

$$
L \in \mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}_+)
$$

such that for all $t \in \mathbb{R}_+$ we have

$$
\mathbb{E}[\|\alpha(r^1)_t - \alpha(r^2)_t\|^p] \leq L(t)^p \|r^1 - r^2\|^p_{[0,t]}, \tag{4.1}
$$

$$
\mathbb{E}[\|\sigma(r^1)_t - \sigma(r^2)_t\|^p_{L_2^0}] \leq L(t)^p \|r^1 - r^2\|^p_{[0,t]}, \tag{4.2}
$$

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_E \|\gamma(r^1)(t,x) - \gamma(r^2)(t,x)\|^p F(dx) \bigg] \\
&+\mathbb{E}\bigg[\bigg(\int_E \|\gamma(r^1)(t,x) - \gamma(r^2)(t,x)\|^2 F(dx) \bigg)^{\frac{p}{2}} \bigg] \leq L(t)^p \|r^1 - r^2\|^p_{[0,t]}
\end{aligned} \tag{4.3}
$$

for all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$.

4.3. **Theorem.** *Suppose that Assumptions 3.3, 4.1, 4.2 are fulfilled. Then, for each $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ there exists a unique solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (3.1) with càdlàg paths on $[t_0, \infty)$, and it satisfies*

$$
\mathbb{E}\bigg[\sup_{t \in [t_0,T]} \|r_t\|^p \bigg] < \infty \quad \text{for all } T \geq t_0.
$$

The proof is established by applying the reasonings from the previous section directly. We do not go into detail here, but indicate how we apply the Banach fixed point theorem in this situation, which relies on Burkholder-Davis-Gundy and Bichteler-Jacod type arguments.

By using Hölder's inequality and (4.1) we obtain

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg\| \int_{T_n}^t (\alpha(h,r^1)_s - \alpha(h,r^2)_s) ds \bigg\|^p \bigg] \\
&\leq (t - T_n)^{p-1} \mathbb{E}\bigg[\int_{T_n}^t \|\alpha(h,r^1)_s - \alpha(h,r^2)_s\|^p ds \bigg] \\
&\leq 2^p (t - T_n)^{p-1} \bigg(\int_0^t L(s)^p ds \bigg) \|r^1 - r^2\|^p_{[T_n, T_{n+1}]}.
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality, Hölder's inequality and (4.2) we have

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg\| \int_{T_n}^t (\sigma(h,r^1)_s - \sigma(h,r^2)_s) dW_s \bigg\|^p \bigg] \\
&\leq C_p \mathbb{E}\bigg[\bigg(\int_{T_n}^t \|\sigma(h,r^1)_s - \sigma(h,r^2)_s\|^2_{L_2^0} ds \bigg)^{\frac{p}{2}} \bigg] \\
&\leq C_p (t - T_n)^{\frac{p}{2}-1} \int_{T_n}^t \mathbb{E}[\|\sigma(h,r^1)_s - \sigma(h,r^2)_s\|^p_{L_2^0}] ds \\
&\leq 2^p C_p (t - T_n)^{\frac{p}{2}-1} \bigg(\int_{T_n}^t L(s)^p ds \bigg) \|r^1 - r^2\|^p_{[T_n, T_{n+1}]},
\end{aligned}
$$

with a constant $C_p > 0$. By means of the Bichteler-Jacod inequality (see [26, Lemma 3.1]) and (4.3) we get

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg\| \int_{T_n}^t \int_E (\gamma(h,r^1)(s,x) - \gamma(h,r^2)(s,x))(\mu(ds,dx) - F(dx)ds) \bigg\|^p\bigg] \\
&\leq N\mathbb{E}\bigg[\int_{T_n}^t \int_E \|\gamma(h,r^1)(s,x) - \gamma(h,r^2)(s,x)\|^p F(dx)ds\bigg] \\
&\quad + N\mathbb{E}\bigg[\int_{T_n}^t \bigg(\int_E \|\gamma(h,r^1)(s,x) - \gamma(h,r^2)(s,x)\|^2 F(dx)\bigg)^{\frac{p}{2}} ds\bigg] \\
&\leq 2^{p+1} N \bigg(\int_{T_n}^t L(s)^p ds\bigg) \|r^1 - r^2\|^p_{[T_n,T_{n+1}]}
\end{aligned}$$

with a constant $N = N(p,t) > 0$. Proceeding as in the proof of Lemma 3.10, we obtain, after choosing an appropriate sequence $(T_n)_{n\in\mathbb{N}}$, that the fixed point mappings Γ_h for $h \in C_{\mathrm{ad}}[0,T_n]$ are contractions on $C_{\mathrm{ad}}[T_n,T_{n+1}]$.

5. Stability of stochastic differential equations

We shall now deal with stability of stochastic differential equations of the kind (3.1). Again these are standard results which we do only give for the sake of completeness. Using the method of the moving frame, we will transfer the results to stochastic partial differential equations in Section 9.

As in Section 3, we assume that $\alpha : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$ fulfill Assumptions 3.3, 3.4, 3.5. Furthermore, let, for each $n \in \mathbb{N}$, $\alpha_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$ be given. We make the following additional assumptions.

5.1. **Assumption.** *We assume that for all $T \in \mathbb{R}_+$ and all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$ with $r^1|_{[0,T]} = r^2|_{[0,T]}$ we have*

$$\begin{aligned}
\alpha_n(r^1)|_{[0,T]} &= \alpha_n(r^2)|_{[0,T]}, \quad n \in \mathbb{N} \\
\sigma_n(r^1)|_{[0,T]} &= \sigma_n(r^2)|_{[0,T]}, \quad n \in \mathbb{N} \\
\gamma_n(r^1)|_{[0,T]\times E} &= \gamma_n(r^2)|_{[0,T]\times E}, \quad n \in \mathbb{N}.
\end{aligned}$$

5.2. **Assumption.** *Denoting by $\mathbf{0} \in C_{\mathrm{ad}}(\mathbb{R}_+)$ the zero process, we assume that*

$$\begin{aligned}
t \mapsto \mathbb{E}[\|\alpha_n(\mathbf{0})_t\|^2] &\in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \quad n \in \mathbb{N} \\
t \mapsto \mathbb{E}[\|\sigma_n(\mathbf{0})_t\|^2_{L_2^0}] &\in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \quad n \in \mathbb{N} \\
t \mapsto \mathbb{E}\bigg[\int_E \|\gamma_n(\mathbf{0})(t,x)\|^2 F(dx)\bigg] &\in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \quad n \in \mathbb{N}.
\end{aligned}$$

5.3. **Assumption.** *We assume that for all $t \in \mathbb{R}_+$ we have*

$$\mathbb{E}[\|\alpha_n(r^1)_t - \alpha_n(r^2)_t\|^2] \leq L(t)^2 \|r^1 - r^2\|_t^2, \tag{5.1}$$

$$\mathbb{E}[\|\sigma_n(r^1)_t - \sigma_n(r^2)_t\|^2_{L_2^0}] \leq L(t)^2 \|r^1 - r^2\|_t^2, \tag{5.2}$$

$$\mathbb{E}\bigg[\int_E \|\gamma_n(r^1)(t,x) - \gamma_n(r^2)(t,x)\|^2 F(dx)\bigg] \leq L(t)^2 \|r^1 - r^2\|_t^2 \tag{5.3}$$

for all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$ and $n \in \mathbb{N}$, where $L \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+)$ denotes the function from Assumption 3.5

5.4. **Remark.** *Notice the slight difference of the previous Assumption 5.3 to Assumption 3.5 for each α_n, σ_n and γ_n, namely, that the function L does not depend on $n \in \mathbb{N}$.*

Furthermore, let $t_0 \in \mathbb{R}_+$, $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ and for each $n \in \mathbb{N}$ let $h^n \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ and $B_n \in \mathcal{E}$ be given.

According to Theorem 3.11, there exists a unique solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (3.1) with $r|_{[0,t_0]} = h$ with càdlàg paths on $[t_0,\infty)$ satisfying (3.19), and for each $n \in \mathbb{N}$ there exists a unique solution $r^n \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for

$$\begin{cases} dr_t^n &= \alpha_n(r^n)_t dt + \sigma_n(r^n)_t dW_t + \int_{B_n} \gamma_n(r^n)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r^n|_{[0,t_0]} &= h^n, \end{cases}$$

with càdlàg paths on $[t_0,\infty)$ satisfying $\mathbb{E}[\sup_{t\in[t_0,T]} \|r_t^n\|^2] < \infty$ for all $T \geq t_0$.

We also make the following assumption, in which $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ denotes the solution for (3.1) with $r|_{[0,t_0]} = h$.

5.5. **Assumption.** *We assume that $B_n \uparrow E$ and*

$$\begin{aligned} &\alpha_n([r])\mathbb{1}_{[t_0,\infty)} \to \alpha([r])\mathbb{1}_{[t_0,\infty)} \quad \textit{in } L^2(\lambda;H), \\ &\sigma_n([r])\mathbb{1}_{[t_0,\infty)} \to \sigma([r])\mathbb{1}_{[t_0,\infty)} \quad \textit{in } L^2(W;L_2^0), \\ &\gamma_n([r])\mathbb{1}_{[t_0,\infty)} \to \gamma([r])\mathbb{1}_{[t_0,\infty)} \quad \textit{in } L^2(\mu;H). \end{aligned}$$

Notice that, by Assumption 5.5, for all $T \geq t_0$ we have

(5.4)
$$\begin{aligned} C_n(T,r) := \Bigg(&\mathbb{E}\bigg[\int_{t_0}^T \|\alpha([r])_s - \alpha_n([r])_s\|^2 ds\bigg] + \mathbb{E}\bigg[\int_{t_0}^T \|\sigma([r])_s - \sigma_n([r])_s\|_{L_2^0}^2 ds\bigg] \\ &+ \mathbb{E}\bigg[\int_{t_0}^T \int_E \|\gamma([r])(s,x) - \gamma_n([r])(s,x)\|^2 F(dx)ds\bigg] \\ &+ \mathbb{E}\bigg[\int_{t_0}^T \int_{E\setminus B_n} \|\gamma([r])(s,x)\|^2 F(dx)ds\bigg]\Bigg)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For a compact interval $I \subset \mathbb{R}_+$ we shall also consider the norm

$$\|r\|_{S^2(I)} := \sqrt{\mathbb{E}\bigg[\sup_{t\in I} \|r_t\|^2\bigg]}.$$

By Theorem 3.11, for any $T \geq t_0$ we have $\|r\|_{S^2[t_0,T]} < \infty$, where $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ denotes the solution for (3.1) with $r|_{[0,t_0]} = h$.

5.6. **Proposition.** *Suppose that Assumptions 3.3, 3.4, 3.5, 5.1, 5.2, 5.3 and 5.5 are fulfilled. Then, there exist maps $K_1, K_2 : \mathbb{R}_+ \to \mathbb{R}_+$, only depending on the Lipschitz function L, such that the following statements are valid:*

(1) *If $h^n \to h$ in $C_{\mathrm{ad}}[0,t_0]$, then for each $T \geq t_0$ we have the estimate*

$$\sup_{t\in[0,T]} \mathbb{E}[\|r_t - r_t^n\|^2] \leq K_1\big(\|h - h^n\|_{[0,t_0]}^2 + C_n^2\big) \to 0 \quad \textit{for } n \to \infty, \tag{5.5}$$

where $K_1 = K_1(T)$ and $C_n = C_n(T,r)$ is defined in (5.4).

(2) *If even $h^n \to h$ in $S^2[0,t_0]$, then for each $T \geq t_0$ we have the estimate*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r_t - r_t^n\|^2\bigg] \leq K_2\big(\|h - h^n\|_{S^2[0,t_0]}^2 + C_n^2\big) \to 0 \quad \textit{for } n \to \infty, \tag{5.6}$$

where $K_2 = K_2(T)$ and $C_n = C_n(T,r)$ is defined in (5.4).

Proof. Let $T \geq t_0$ and $n \in \mathbb{N}$ be arbitrary. By Hölder's inequality, the Itô-isometries (2.1), (2.3) and the Lipschitz conditions (5.1), (5.2), (5.3) we obtain, by writing

$$\begin{aligned}
&\int_{t_0}^{t} \int_{E} \gamma([r])(s,x)(\mu(ds,dx) - F(dx)ds) \\
&\quad - \int_{t_0}^{t} \int_{B_n} \gamma_n([r^n])(s,x)(\mu(ds,dx) - F(dx)ds) \\
&= \int_{t_0}^{t} \int_{B_n} (\gamma([r])(s,x) - \gamma_n([r])(s,x))(\mu(ds,dx) - F(dx)ds) \\
&\quad + \int_{t_0}^{t} \int_{E \setminus B_n} \gamma([r])(s,x)(\mu(ds,dx) - F(dx)ds) \\
&\quad + \int_{t_0}^{t} \int_{B_n} (\gamma_n([r])(s,x) - \gamma_n([r^n])(s,x)(\mu(ds,dx) - F(dx)ds),
\end{aligned}$$

for all $t \in [t_0, T]$ the estimate

$$\begin{aligned}
&\|r - r^n\|_{[0,t]}^2 = \sup_{s \in [0,t]} \mathbb{E}[\|r_s - r_s^n\|^2] \leq 8\big(\|h - h^n\|_{[0,t_0]}^2 + ((t - t_0) \vee 1)C_n(t,r)^2\big) \\
&\quad + 8 \sup_{s \in [t_0,t]} \mathbb{E}\left[\left\| \int_{t_0}^{s} (\alpha_n([r])_v - \alpha_n([r^n])_v) dv \right\|^2\right] \\
&\quad + 8 \sup_{s \in [t_0,t]} \mathbb{E}\left[\left\| \int_{t_0}^{s} (\sigma_n([r])_v - \sigma_n([r^n])_v) dW_v \right\|^2\right] \\
&\quad + 8 \sup_{s \in [t_0,t]} \mathbb{E}\left[\left\| \int_{t_0}^{s} \int_{B_n} (\gamma_n([r])(v,x) - \gamma_n([r^n])(v,x))(\mu(dv,dx) - F(dx)dv) \right\|^2\right] \\
&\leq 8\big(\|h - h^n\|_{[0,t_0]}^2 + ((T - t_0) \vee 1)C_n(T,r)^2\big) \\
&\quad + 8(T - t_0 + 2) \int_{t_0}^{t} L(s)^2 \|r - r^n\|_{[0,s]}^2 ds.
\end{aligned}$$

Applying the Gronwall Lemma gives us

$$\begin{aligned}
&\sup_{s \in [0,t]} \mathbb{E}[\|r_s - r_s^n\|^2] = \|r - r^n\|_{[0,t]}^2 \\
&\leq 8\big(\|h - h^n\|_{[0,t_0]}^2 + ((T - t_0) \vee 1)C_n(T,r)^2\big) e^{8(T - t_0 + 2) \int_{t_0}^{t} L(s)^2 ds}
\end{aligned}$$

for all $t \in [t_0, T]$, implying (5.5). Analogously, by also taking into account Doob's martingale inequality [9, Thm. 3.8], we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \in [0,T]} \|r_t - r_t^n\|^2\right] \leq 8\big(\|h - h^n\|_{S^2[0,t_0]}^2 + ((T - t_0) \vee 4)C_n(T,r)^2\big) \\
&\quad + 8\mathbb{E}\left[\sup_{t \in [t_0,T]} \left\| \int_{t_0}^{t} (\alpha_n([r])_s - \alpha_n([r^n])_s) ds \right\|^2\right] \\
&\quad + 8\mathbb{E}\left[\sup_{t \in [t_0,T]} \left\| \int_{t_0}^{t} (\sigma_n([r])_s - \sigma_n([r^n])_s) dW_s \right\|^2\right] \\
&\quad + 8\mathbb{E}\left[\sup_{t \in [t_0,T]} \left\| \int_{t_0}^{t} \int_{B_n} (\gamma_n([r])(s,x) - \gamma_n([r^n])(s,x))(\mu(ds,dx) - F(dx)ds) \right\|^2\right] \\
&\leq 8\big(\|h - h^n\|_{S^2[0,t_0]}^2 + ((T - t_0) \vee 4)C_n(T,r)^2\big) \\
&\quad + 8(T - t_0 + 8)\left(\int_{t_0}^{T} L(s)^2 ds\right) \|r - r^n\|_{[0,T]}^2.
\end{aligned}$$

Noting that $\|h - h^n\|_{[0,t_0]} \leq \|h - h^n\|_{S^2[0,t_0]}$, inserting (5.5) shows (5.6). $\square$

5.7. **Remark.** *Fix a finite time $T \geq t_0$ and denote for $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ by r^h the unique solution for (3.1) with $r|_{[0,t_0]} = h$, which has càdlàg paths on $[t_0,\infty)$. Restricting it to the interval $[0,T]$, estimates (5.5), (5.6) show that the solution map $h \mapsto r^h$ is Lipschitz continuous with a constant $L = L(T) > 0$, if considered as a map $C_{\mathrm{ad}}[0,t_0] \to C_{\mathrm{ad}}[0,T]$ or as a map $S^2[0,t_0] \to S^2[0,T]$. In particular, there exists a constant $C = C(T) > 0$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[\|r_t^h\|^2] \leq C\Big(1 + \sup_{t\in[0,t_0]} \mathbb{E}[\|h_t\|^2]\Big), \quad h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$$

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r_t^h\|^2\bigg] \leq C\bigg(1 + \mathbb{E}\bigg[\sup_{t\in[0,t_0]} \|h_t\|^2\bigg]\bigg), \quad h \in \mathcal{S}^2[0,t_0].$$

Notice further for $t_0 = 0$ the coincidence $\mathcal{C}_{\mathrm{ad}}[0,t_0] = \mathcal{S}^2[0,t_0] = \mathcal{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$.

5.8. **Remark.** *Using Burkholder-Davis-Gundy and Bichteler-Jacod type arguments as in the previous section, we can, in an analogous fashion, derive the L^p-version of the stability result above.*

6. Regular dependence on initial data for stochastic differential equations

In this section, we study regular dependence on initial data for SDEs. Some related ideas can be found in [25]. By the method of the moving frame, which we present in Section 8, we can transfer the upcoming results to SPDEs.

We understand the question of regular dependence on initial data as a conclusion of the stability results of Section 5. We consider

$$(6.1) \quad \begin{cases} dr_t &= \alpha(r)_t dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r|_{[0,t_0]} &= h, \end{cases}$$

under Assumptions 3.3, 3.4, 3.5, such that we can conclude the existence and uniqueness of solutions for $t_0 \in \mathbb{R}_+$ and $h \in C_{\mathrm{ad}}[0,t_0]$. Motivated by ideas from convenient analysis, see [22], we fix a curve of initial data $\epsilon \mapsto c(\epsilon) \in C_{\mathrm{ad}}[0,t_0]$, which is differentiable for all ϵ with derivative $c'(\epsilon) \in C_{\mathrm{ad}}[0,t_0]$. We consider the following system of equations,

$$(6.2) \quad \begin{cases} dr_t^\epsilon &= \alpha(r^\epsilon)_t dt + \sigma(r^\epsilon)_t dW_t + \int_E \gamma(r^\epsilon)(t,x)(\mu(dt,dx) - F(dx)dt), \\ r^\epsilon|_{[0,t_0]} &= c(\epsilon), \\ d\frac{r_t^\epsilon - r_t^0}{\epsilon} &= \frac{\alpha(r^\epsilon)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(r^\epsilon)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ &\quad + \int_E \frac{\gamma(r^\epsilon)(t,x) - \gamma(r^0)(t,x)}{\epsilon}(\mu(dt,dx) - F(dx)dt), \\ \frac{r^\epsilon - r^0}{\epsilon}|_{[0,t_0]} &= \frac{c(\epsilon) - c(0)}{\epsilon}, \end{cases}$$

for $\epsilon \neq 0$, where r^0 denotes the solution for (6.1) with $h = c(0)$. We can consider those equations indeed as two SDEs in our sense. More precisely let

$$(6.3) \quad \begin{cases} dr_t^\epsilon &= \alpha(r^\epsilon)_t dt + \sigma(r^\epsilon)_t dW_t + \int_E \gamma(r^\epsilon)(t,x)(\mu(dt,dx) - F(dx)dt), \\ r^\epsilon|_{[0,t_0]} &= c(\epsilon), \\ d\Delta_t^\epsilon &= \frac{\alpha(\epsilon\Delta^\epsilon + r^0)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(\epsilon\Delta^\epsilon + r^0)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ &\quad + \int_E \frac{\gamma(\epsilon\Delta^\epsilon + r^0)(t,x) - \gamma(r^0)(t,x)}{\epsilon}(\mu(dt,dx) - F(dx)dt), \\ \Delta^\epsilon|_{[0,t_0]} &= \frac{c(\epsilon) - c(0)}{\epsilon}, \end{cases}$$

for $\epsilon \neq 0$, then this system of equations can be seen as two stochastic differential equations. We can readily check that the Assumptions 3.3, 3.4, 3.5 are true for the second SDE in (6.3) for every $\epsilon \neq 0$. Its solution is given by

$$\Delta_t^\epsilon = \frac{r_t^\epsilon - r_t^0}{\epsilon}, \quad t \geq 0. \tag{6.4}$$

We assume now that the maps α, σ and γ admit directional derivatives in all directions of $C_{\mathrm{ad}}(\mathbb{R}_+)$. We denote those directional derivatives at the point $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ into direction $v \in C_{\mathrm{ad}}(\mathbb{R}_+)$ by $D\alpha(r) \bullet v$, $D\sigma(r) \bullet v$ and $D\gamma(r) \bullet v$. By D, we always mean the Fréchet derivative.

6.1. **Assumption.** *We define the first variation process $J(r) \bullet w$ in direction w, where $w \in C_{\mathrm{ad}}[0, t_0]$, to be the unique solution of the SDE*

$$\left\{ \begin{array}{rcl} d(J(r) \bullet w)_t & = & \big(D\alpha(r) \bullet (J(r) \bullet w)\big)_t dt + \big(D\sigma(r) \bullet (J(r) \bullet w)\big)_t dW_t + \\ & & + \int_E \big(D\gamma(r) \bullet (J(r) \bullet w)\big)(t,x)(\mu(dt,dx) - F(dx)dt), \\ (J(r) \bullet w)|_{[0,t_0]} & = & w, \end{array} \right. \tag{6.5}$$

where r solves equation (6.1). *We assume that Assumptions 3.3, 3.4, 3.5 are true for equation* (6.5). *We assume furthermore that*

$$\frac{\alpha(\epsilon(J(r) \bullet w) + r) - \alpha(r)}{\epsilon} \mathbb{1}_{[t_0,\infty)} \to D\alpha(r) \bullet (J(r) \bullet w) \mathbb{1}_{[t_0,\infty)},$$
$$\frac{\sigma(\epsilon(J(r) \bullet w) + r) - \sigma(r)}{\epsilon} \mathbb{1}_{[t_0,\infty)} \to D\sigma(r) \bullet (J(r) \bullet w) \mathbb{1}_{[t_0,\infty)},$$
$$\frac{\gamma(\epsilon(J(r) \bullet w) + r) - \gamma(r)}{\epsilon} \mathbb{1}_{[t_0,\infty)} \to D\gamma(r) \bullet (J(r) \bullet w) \mathbb{1}_{[t_0,\infty)}$$

as $\epsilon \to 0$ in the respective spaces $L^2(\lambda; H)$, $L^2(W; L_2^0)$ and $L^2(\mu; H)$. The process r denotes the solution of equation (6.1) *and $J(r) \bullet w$ denotes the solution of the first variation equation* (6.5).

6.2. **Proposition.** *Suppose that Assumptions 5.1, 5.2, 5.3 for equation*

$$\left\{ \begin{array}{rcl} d\Delta_t^\epsilon & = & \frac{\alpha(\epsilon\Delta^\epsilon + r^0)_t - \alpha(r^0)_t}{\epsilon} dt + \frac{\sigma(\epsilon\Delta^\epsilon + r^0)_t - \sigma(r^0)_t}{\epsilon} dW_t + \\ & & + \int_E \frac{\gamma(\epsilon\Delta^\epsilon + r^0)(t,x) - \gamma(r^0)(t,x)}{\epsilon} (\mu(dt,dx) - F(dx)dt), \\ \Delta^\epsilon|_{[0,t_0]} & = & \frac{c(\epsilon) - c(0)}{\epsilon}, \end{array} \right. \tag{6.6}$$

are valid in the obvious sense for $\epsilon \neq 0$ in a neighborhood of 0, and assume that Assumption 6.1 is fulfilled for $w = c'(0)$ for a chosen curve of initial values $\epsilon \mapsto c(\epsilon)$. Then, for each $T \geq t_0$ we have the estimate

$$\sup_{t \in [0,T]} \mathbb{E}[\|(J(r) \bullet w)_t - \Delta_t^\epsilon\|^2] \leq K_1 \left(\left\| c'(0) - \frac{c(\epsilon) - c(0)}{\epsilon} \right\|_{[0,t_0]}^2 + C_\epsilon^2 \right) \to 0 \tag{6.7}$$

for $\epsilon \to 0$, and if $\epsilon \mapsto c(\epsilon)$ is even a curve in $S^2[0, t_0]$, then for each $T \geq t_0$ we have the estimate

$$\mathbb{E}\Bigg[\sup_{t \in [0,T]} \|(J(r) \bullet w)_t - \Delta_t^\epsilon\|^2 \Bigg] \leq K_2 \left(\left\| c'(0) - \frac{c(\epsilon) - c(0)}{\epsilon} \right\|_{S^2[0,t_0]}^2 + C_\epsilon^2 \right) \to 0 \tag{6.8}$$

for $\epsilon \to 0$. In particular, the map $w \mapsto J(r) \bullet w$ is linear and continuously depending on w in the sense that for every $T \geq t_0$ we have

$$\sup_{t \in [0,T]} \mathbb{E}[\|(J(r) \bullet w)_t - (J(r) \bullet w^n)_t\|^2] \leq K_1 \|w - w^n\|_{[0,t_0]}^2 \to 0 \tag{6.9}$$

for variation of the initial value $w^n \to w \in C_{\text{ad}}[0,t_0]$, and if $\epsilon \mapsto c(\epsilon)$ is even a curve in $S^2[0,t_0]$, then for every $T \geq t_0$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|(J(r) \bullet w)_t - (J(r) \bullet w^n)_t\|^2 \right] \leq K_2 \|w - w^n\|^2_{S^2[0,t_0]} \to 0 \tag{6.10}$$

for variation of the initial value $w^n \to w \in S^2[0,t_0]$.

6.3. **Remark.** *The notion C_ϵ is defined corresponding to (5.4) and K_1, K_2 according to Proposition 5.6.*

Proof. The assertion is a corollary of Proposition 5.6. Assumption 6.1 corresponds precisely to Assumption 5.5, which is needed for the proof of Proposition 5.6. Note that the "continuous" parameter ϵ replaces the index n, which does not cause any problems, since we do not speak about almost sure convergence results here. □

6.4. **Remark.** *Fix a finite time $T \geq t_0$ and a curve of initial data $\epsilon \mapsto c(\epsilon) \in C_{\text{ad}}[0,t_0]$ or $\epsilon \mapsto c(\epsilon) \in S^2[0,t_0]$. Then, we can consider the curve of solution processes $\epsilon \mapsto r^\epsilon \in C_{\text{ad}}[0,T]$ or $\epsilon \mapsto r^\epsilon \in S^2[0,T]$, respectively. By Remark 5.7 we already know that the solution map $\epsilon \mapsto r^\epsilon$ is continuous. Now, estimates (6.7), (6.8) show that, subject to our previous assumptions, $\epsilon \mapsto r^\epsilon$ is also differentiable with derivative $\epsilon \mapsto J(r) \bullet c'(\epsilon)$. Moreover, regarding the variation $w \mapsto J(r) \bullet w$ of the initial value as a linear map $C_{\text{ad}}[0,t_0] \to C_{\text{ad}}[0,T]$ or as a linear map $S^2[0,t_0] \to S^2[0,T]$, estimates (6.9), (6.10) show its continuity.*

Considering the construction for all possible curves of initial values c we can define the first (and possibly higher) variation processes in a coherent way for all variations of the initial values and also for variations of the process up to time t by shifting $\mathcal{F}_t$ to $\mathcal{F}_0$. Properties of this variation process can be established by considering the equation, which follows right from Proposition 6.2,

$$r^\epsilon - r^0 = \int_0^\epsilon J(r) \bullet c'(\eta)\, d\eta \tag{6.11}$$

and which reveals the true meaning of the first variation process.

7. Solution concepts for stochastic partial differential equations

When dealing with SPDEs there are several solution concepts, which we will discuss in this section. The main difficulty is that solutions of SPDEs usually leave the realm of semi-martingales and one therefore has to modify the usual semi-martingale decomposition. The method of the moving frame, which will be presented in the next section, is a new approach how to handle this problem.

In this section, we review the well-known concepts of strong, weak and mild solutions and show, how they are related. The proofs from [9] (or [27]) can be transferred to the present situation, whence we keep this section rather short. The decisive tool in order to prove Lemma 7.7 is an appropriate Stochastic Fubini Theorem with respect to Poisson measures, which we provide in Appendix A.

Now let $(S_t)_{t\geq 0}$ be a C_0-semigroup on the separable Hilbert space H with infinitesimal generator $A : \mathcal{D}(A) \subset H \to H$. We denote by $A^* : \mathcal{D}(A^*) \subset H \to H$ the adjoint operator of A. Recall that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense in H, see, e.g., [33, Satz VII.4.6, p. 351].

In this section, we are interested in stochastic partial differential equations of the form

$$\begin{cases} dr_t &= (Ar_t + \alpha(r)_t)dt + \sigma(r)_t dW_t + \int_E \gamma(r)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r|_{[0,t_0]} &= h \end{cases} \tag{7.1}$$

where $\alpha : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$. Fix $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0, t_0]$.

7.1. **Definition.** *A process $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ is called a* strong solution *for (7.1) if we have $r|_{[0,t_0]} = h$, $\mathbb{P}(r_t \in \mathcal{D}(A)) = 1$, $t \geq t_0$, the relations $A(r\mathbb{1}_{[t_0,\infty)}) + \alpha([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^1_{\mathrm{loc}}(\lambda; H)$, $\sigma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(W; L_2^0)$, $\gamma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(\mu; H)$ and almost surely*

$$
\begin{aligned}
r_t = h_{t_0} &+ \int_{t_0}^t (Ar_s + \alpha([r])_s) ds + \int_{t_0}^t \sigma([r])_s dW_s \\
&+ \int_{t_0}^t \int_E \gamma([r])(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq t_0.
\end{aligned} \tag{7.2}
$$

7.2. **Definition.** *A process $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ is called a* weak solution *for (7.1) if $r|_{[0,t_0]} = h$, $\alpha([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^1_{\mathrm{loc}}(\lambda; H)$, $\sigma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(W; L_2^0)$, $\gamma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(\mu; H)$ and for all $\zeta \in \mathcal{D}(A^*)$ we have almost surely*

$$
\begin{aligned}
\langle \zeta, r_t \rangle = \langle \zeta, h_{t_0} \rangle &+ \int_{t_0}^t (\langle A^*\zeta, r_s \rangle + \langle \zeta, \alpha([r])_s \rangle) ds + \int_{t_0}^t \langle \zeta, \sigma([r])_s \rangle dW_s \\
&+ \int_{t_0}^t \int_E \langle \zeta, \gamma([r])(s,x) \rangle (\mu(ds,dx) - F(dx)ds), \quad t \geq t_0.
\end{aligned} \tag{7.3}
$$

7.3. **Definition.** *A process $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ is called a* mild solution *for (7.1) if $r|_{[0,t_0]} = h$, $\alpha([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^1_{\mathrm{loc}}(\lambda; H)$, $\sigma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(W; L_2^0)$, $\gamma([r])\mathbb{1}_{[t_0,\infty)} \in \mathcal{L}^2_{\mathrm{loc}}(\mu; H)$ and we have almost surely*

$$
\begin{aligned}
r_t = S_{t-t_0} h_{t_0} &+ \int_{t_0}^t S_{t-s}\alpha([r])_s ds + \int_{t_0}^t S_{t-s}\sigma([r])_s dW_s \\
&+ \int_{t_0}^t \int_E S_{t-s}\gamma([r])(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq t_0.
\end{aligned} \tag{7.4}
$$

7.4. **Remark.** *For all the three just defined solution concepts* uniqueness *of solutions for (7.1) is, as in Definition 3.1, meant up to indistinguishability on the interval $[t_0, \infty)$.*

7.5. **Lemma.** *Let $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ be a strong solution for (7.1). Then, r is also a weak solution for (7.1).*

Proof. For all $\zeta \in \mathcal{D}(A^*)$ we have

$$
\begin{aligned}
\langle \zeta, r_t \rangle = \langle \zeta, h_{t_0} \rangle &+ \int_{t_0}^t \langle \zeta, Ar_s + \alpha([r])_s \rangle ds + \int_{t_0}^t \langle \zeta, \sigma([r])_s \rangle dW_s \\
&+ \int_{t_0}^t \int_E \langle \zeta, \gamma([r])(s,x) \rangle (\mu(ds,dx) - F(dx)ds), \quad t \geq t_0
\end{aligned}
$$

implying that r is also a weak solution for (7.1), because $\langle \zeta, Ah \rangle = \langle A^*\zeta, h \rangle$ for all $h \in \mathcal{D}(A)$. □

7.6. **Lemma.** *Let $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ be a weak solution for (7.1). Then, r is also a mild solution for (7.1).*

Proof. Let $T \geq t_0$ be arbitrary. As in the proof of [27, Thm. 9.15] we show that

$$
\begin{aligned}
\langle g(t), r_t \rangle &= \langle g(t_0), h_{t_0} \rangle + \int_{t_0}^t \Big(\langle g'(s) + A^* g(s), r_s \rangle + \langle g(s), \alpha([r])_s \rangle \Big) ds \\
&+ \int_{t_0}^t \langle g(s), \sigma([r])_s \rangle dW_s + \int_{t_0}^t \int_E \langle g(s), \gamma([r])(s,x) \rangle (\mu(ds,dx) - F(dx)ds)
\end{aligned}
$$

for all $g \in C^1([t_0,T];\mathcal{D}(A^*))$ and $t \in [t_0,T]$. For an arbitrary $t \geq t_0$ and an arbitrary $\zeta \in \mathcal{D}(A^*)$ we apply this identity to $g(s) := S_{t-s}^* \zeta$, $s \in [t_0,t]$, which yields that the process r is also a mild solution for (7.1). □

7.7. **Lemma.** *Let $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ be a mild solution for (7.1) such that $\sigma([r])\mathbb{1}_{[t_0,\infty)} \in L^2(W;L_2^0)$ and $\gamma([r])\mathbb{1}_{[t_0,\infty)} \in L^2(\mu;H)$. Then, r is also a weak solution for (7.1).*

Proof. We proceed as in the proof of [27, Thm. 9.15]. The change of order of integration for the stochastic integrals with respect to the compensated Poisson random measure is valid by the Stochastic Fubini Theorem A.2 provided in Appendix A. □

8. Existence and uniqueness of mild and weak solutions for stochastic partial differential equations

In this section we introduce the method of the moving frame, which has been announced in the introduction. Loosely speaking we apply a time-dependent coordinate transformation to the SPDE such that "from the point of view of the moving frame" the SPDE looks like an SDE with appropriately transformed coefficients. The method is in contrast to the point of view, that an SPDE is a PDE together with a non-linear stochastic perturbation. Here we consider an SPDE rather as a time-transformed SDE, where the time transform contains the respective PDE aspect.

We apply this method for an "easy" proof of existence and uniqueness in this general setting. The key argument, which allows to apply the method, is the Szőkefalvi-Nagy theorem, which has been brought to our attention by [17]. We emphasize that in our article we do not need a particular representation of the Hilbert space involved in the Szőkefalvi-Nagy theorem (see the subsequent remark). The Szőkefalvi-Nagy theorem is a "ladder", which allows us to "climb" towards several new assertions, but which is not necessary to understand the statements of those assertions.

During this section, we impose the following assumption.

8.1. **Assumption.** *There exist another separable Hilbert space $\mathcal{H}$, a C_0-group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ and continuous linear operators $\ell \in L(H,\mathcal{H})$, $\pi \in L(\mathcal{H},H)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{U_t} & \mathcal{H} \\ \uparrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \pi} \\ H & \xrightarrow{S_t} & H \end{array}$$

commutes for every $t \in \mathbb{R}_+$, that is

$$\pi U_t \ell = S_t \quad \text{for all } t \in \mathbb{R}_+. \tag{8.1}$$

In particular, we see that $\pi\ell = \mathrm{Id}$.

8.2. **Remark.** *In the spirit of* [31]*, the group $(U_t)_{t\in\mathbb{R}}$ is a* dilation *of the semigroup $(S_t)_{t\geq 0}$.*

8.3. **Remark.** *Assumption 8.1 is not only frequently fulfilled, which seems surprising at a first view, but it is also possible to describe the respective Hilbert space $\mathcal{H}$ more precisely. Take for instance a self-adjoint strongly continuous semigroup of contractions S on the complex Hilbert space H, then – as a part of the Szőkefalvi-Nagy theorem – the map $t \mapsto S_{|t|}$, where the semigroup is extended by $S_{-t} := S_t$ for $t \geq 0$, is a strongly continuous, positive definite map, i.e. for all $\psi_1,\ldots,\psi_n \in H$ and all real times $t_1,\ldots,t_n$ the matrix $\big(\langle S_{|t_i-t_j|}\psi_i,\psi_j\rangle\big)$ is positive definite. A positive definite map with values in bounded linear operators can be considered as characteristic function of a vector-valued measure η taking values in positive operators on H.*

One can define the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \eta; H)$, i.e. the space of square-integrable H-valued measurable maps f, such that the integral

$$\int_{\mathbb{R}} \langle f(x), \eta(dx) f(x) \rangle < \infty$$

is finite. H can be embedded via the constant maps $f(x) \equiv h$ for $h \in H$ and $x \in \mathbb{R}$ and the semigroup U is defined via

$$U_t f(x) = \exp(itx) f(x)$$

for $t, x \in \mathbb{R}$. Consequently, more precise analysis of the respective generator of S on $\mathcal{H}$ can be performed. Details of the previous considerations and impacts on SPDEs will be presented elsewhere.

According to Proposition 8.7 below, Assumption 8.1 is in particular satisfied if the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive.

8.4. **Definition.** *The C_0-semigroup $(S_t)_{t \geq 0}$ is called* pseudo-contractive *if there exists $\omega \in \mathbb{R}$ such that*

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0. \tag{8.2}$$

8.5. **Remark.** *Sometimes in the literature, e.g., see* [26]*, the notion quasi-contractive is used instead of pseudo-contractive.*

8.6. **Remark.** *By the theorem of Lumer-Phillips, a densely defined operator A generates a pseudo-contractive semigroup $(S_t)_{t \geq 0}$ with growth estimate (8.2) for some $\omega \geq 0$ if and only if A is ω-m-dissipative, that is, $A - \omega$ is dissipative, which means*

$$\langle Ah, h \rangle \leq \omega \|h\|^2 \quad \text{for all } h \in \mathcal{D}(A), \tag{8.3}$$

and there exists $\lambda > 0$ such that $\lambda + \omega - A$ is surjective. For example, consider the Hilbert space $H = L^2(0, \infty)$ and the Laplace operator $A = \Delta$ defined by $\Delta h = h''$ on the Sobolev space $\mathcal{D}(\Delta) = H_0^1(0, \infty) \cap W^2(0, \infty)$. Then, Δ is densely defined, because $C_0^\infty(0, \infty)$ is dense in $L^2(0, \infty)$. Let us check the dissipativity of Δ. For $h \in H_0^1(0, \infty) \cap W^2(0, \infty)$ choose a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_0^\infty(0, \infty)$ with $\|h - \varphi_n\|_{H_0^1} \to 0$. By integration by parts, we have

$$\langle h'', h \rangle_{L^2} = \lim_{n \to \infty} \langle h'', \varphi_n \rangle_{L^2} = - \lim_{n \to \infty} \langle h', \varphi_n' \rangle_{L^2} = -\langle h', h' \rangle_{L^2} \leq 0,$$

showing (8.3) with $\omega = 0$. For $\lambda > 0$ and $f \in L^2(0, \infty)$ there exists a unique solution $h \in H_0^1(0, \infty) \cap W^2(0, \infty)$ of the second order differential equation

$$\lambda h - \Delta h = f,$$

see [23, Thm. 8.2.7]*. Hence, $\lambda - \Delta$ is surjective.*

For every C_0-semigroup $(S_t)_{t \geq 0}$ there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S_t\| \leq M e^{\omega t}, \quad t \geq 0 \tag{8.4}$$

see, e.g., [33, Lemma VII.4.2]. Hence, in other words, the semigroup $(S_t)_{t \geq 0}$ is contractive if we can choose $M = 1$ and $\omega = 0$ in (8.4), and it is pseudo-contractive, if we can choose $M = 1$ in (8.4).

Every C_0-semigroup is not far from being pseudo-contractive. Indeed, for an arbitrary $s > 0$, we have, by (8.4), the estimate

$$\|S_t\| \leq e^{\omega(s) t}, \quad t \geq s$$

where we have set $\omega(s) := \frac{\ln M}{s} + \omega$. Nevertheless, there are C_0-semigroups, which are not pseudo-contractive. For a counter example, we choose, following [12, Ex.

I.5.7.iii], the Hilbert space $H := L^2(\mathbb{R})$ and the shift semigroup $(S_t)_{t \geq 0}$ with jump, defined as

$$S_t h(x) := \begin{cases} 2h(x+t), & x \in [-t, 0] \\ h(x+t), & \text{otherwise} \end{cases}$$

for $h \in H$. Then $(S_t)_{t \geq 0}$ is a C_0-semigroup on H with $\|S_t\| = 2$ for all $t > 0$, because $\|S_t \mathbb{1}_{[0,t]}\| = 2 \|\mathbb{1}_{[0,t]}\|$.

However, many semigroups of practical relevance are pseudo-contractive, and then the following result shows that Assumption 8.1 is satisfied.

8.7. **Proposition.** *Assume the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive. Then there exist another separable Hilbert space $\mathcal{H}$ and a C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ such that (8.1) is satisfied, where $\ell \in L(H, \mathcal{H})$ is an isometric embedding and $\pi := \ell^* \in L(\mathcal{H}, H)$ is the orthogonal projection from $\mathcal{H}$ into H.*

Proof. Since the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive, there exists $\omega \geq 0$ such that (8.2) is satisfied. Hence, the C_0-semigroup $(T_t)_{t \geq 0}$ defined as $T_t := e^{-\omega t} S_t$, $t \in \mathbb{R}_+$ is contractive. By the Szőkefalvi-Nagy theorem on unitary dilations (see e.g. [31, Thm. I.8.1], or [10, Sec. 7.2]), there exist another separable Hilbert space $\mathcal{H}$ and a unitary C_0-group $(V_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ such that

$$\pi V_t \ell = T_t \quad \text{for all } t \in \mathbb{R}_+,$$

where $\ell \in L(H, \mathcal{H})$ is an isometric embedding and the adjoint operator $\pi := \ell^* \in L(\mathcal{H}, H)$ is the orthogonal projection from $\mathcal{H}$ into H. Defining the C_0-group $(U_t)_{t \in \mathbb{R}}$ as $U_t := e^{\omega t} V_t$, $t \in \mathbb{R}$ completes the proof. □

We suppose from now on Assumption 8.1. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U_t\| \leq M e^{\omega |t|}, \quad t \in \mathbb{R} \tag{8.5}$$

see [12, p. 79]. Now let $\alpha : C_{\text{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma : C_{\text{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\text{ad}}(\mathbb{R}_+) \to H_{\mathcal{P} \otimes \mathcal{E}}$ be given. We suppose that Assumptions 3.3, 3.4, 3.5 are satisfied.

In order to solve the stochastic partial differential equation (7.1), we consider the $\mathcal{H}$-valued stochastic differential equation

$$\left\{ \begin{array}{rcl} dR_t & = & \tilde{\alpha}(R)_t dt + \tilde{\sigma}(R)_t dW_t + \int_E \tilde{\gamma}(R)(t,x)(\mu(dt,dx) - F(dx)dt) \\ R|_{[0,t_0]} & = & h, \end{array} \right. \tag{8.6}$$

where $t_0 \in \mathbb{R}_+$ and $h \in C_{\text{ad}}([0,t_0]; \mathcal{H})$, and where $\tilde{\alpha} : C_{\text{ad}}(\mathbb{R}_+; \mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma} : C_{\text{ad}}(\mathbb{R}_+; \mathcal{H}) \to L_2(U_0, \mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma} : C_{\text{ad}}(\mathbb{R}_+; \mathcal{H}) \to \mathcal{H}_{\mathcal{P} \otimes \mathcal{E}}$ are defined as

$$\tilde{\alpha}(R)_t := U_{-t}^{t_0} \ell \alpha(\pi U^{t_0} R)_t, \tag{8.7}$$

$$\tilde{\sigma}(R)_t := U_{-t}^{t_0} \ell \sigma(\pi U^{t_0} R)_t, \tag{8.8}$$

$$\tilde{\gamma}(R)(t,x) := U_{-t}^{t_0} \ell \gamma(\pi U^{t_0} R)(t,x). \tag{8.9}$$

In the above definitions, we have used the notation

$$U_t^{t_0} := \begin{cases} U_{t-t_0}, & t \geq t_0 \\ \text{Id}, & t \in (-t_0, t_0) \\ U_{t_0+t}, & t \leq -t_0 \end{cases}$$

and $\pi U^{t_0} R \in C_{\text{ad}}(\mathbb{R}_+; H)$ denotes the process $(\pi U^{t_0} R)_t := \pi U_t^{t_0} R_t$, $t \geq 0$. Note that $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$ indeed map into the respective spaces of predictable processes, because

$(t,h) \mapsto U_t h$ is continuous on $\mathbb{R} \times \mathcal{H}$, see, e.g., [33, Lemma VII.4.3]. By (8.5), they also fulfill Assumptions 3.3, 3.4, 3.5, where the function L is replaced by

$$L(t) \rightsquigarrow \|\ell\|\big(\mathbb{1}_{[0,t_0)} + M^2 e^{2\omega(t-t_0)} \mathbb{1}_{[t_0,\infty)}\big)\|\pi\| L(t), \quad t \geq 0 \tag{8.10}$$

According to Theorem 3.11, for each $h \in \mathcal{C}_{\mathrm{ad}}([0,t_0];\mathcal{H})$ there exists a unique solution $R \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H})$ for (8.6) with càdlàg paths on $[t_0,\infty)$, and it satisfies

$$\mathbb{E}\bigg[\sup_{t\in[t_0,T]} \|R_t\|^2\bigg] < \infty \quad \text{for all } T \geq t_0. \tag{8.11}$$

8.8. **Theorem.** *Suppose that Assumptions 3.3, 3.4, 3.5 and 8.1 are fulfilled. Then, for each $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ there exists a unique mild and weak solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (7.1) with càdlàg paths on $[t_0,\infty)$, and it satisfies (3.19). The solution is given by $r = \pi U^{t_0} R$, where $R \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H})$ denotes the solution for (8.6) with $R|_{[0,t_0]} = \ell h$.*

Proof. Let $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ be arbitrary. The H-valued process $r := \pi U^{t_0} R$ belongs to $\mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$, it satisfies (3.19) by virtue of (8.11), and it has also càdlàg paths on $[t_0,\infty)$, because $(t,h) \mapsto U_t h$ is continuous on $\mathbb{R}_+ \times \mathcal{H}$, see, e.g., [33, Lemma VII.4.3]. Using (8.1) we obtain $r|_{[0,t_0]} = \pi \ell h = h$ and almost surely

$$\begin{aligned}
r_t &= (\pi U^{t_0} R)_t = \pi U_{t-t_0} R_t \\
&= \pi U_{t-t_0}\bigg(\ell h_{t_0} + \int_{t_0}^t U_{t_0-s}\ell\alpha(\pi U^{t_0}R)_s ds + \int_{t_0}^t U_{t_0-s}\ell\sigma(\pi U^{t_0}R)_s dW_s \\
&\quad + \int_{t_0}^t \int_E U_{t_0-s}\ell\gamma(\pi U^{t_0}R)(s,x)(\mu(ds,dx) - F(dx)ds)\bigg) \\
&= S_{t-t_0} h_{t_0} + \int_{t_0}^t S_{t-s}\alpha(r)_s ds + \int_{t_0}^t S_{t-s}\sigma(r)_s dW_s \\
&\quad + \int_{t_0}^t \int_E S_{t-s}\gamma(r)(s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq t_0
\end{aligned}$$

showing that r is a mild solution for (7.1). By virtue of Lemma 3.9 we have $\sigma(r) \in L^2(W;L_2^0)$ and $\gamma(r) \in L^2(\mu;H)$. Applying Lemma 7.7 proves that r is also a weak solution for (7.1).

For two mild solutions $r, \tilde{r} \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ of (7.1), which are càdlàg on $[t_0,\infty)$, and an arbitrary $T \geq t_0$, by using Hölder's inequality, the Itô-isometries (2.1), (2.3) and the Lipschitz conditions (3.7), (3.8), (3.9), the inequality

$$\begin{aligned}
\|r - \tilde{r}\|_{[t_0,t]}^2 &= \sup_{s\in[t_0,t]} \mathbb{E}[\|r_s - \tilde{r}_s\|^2] \\
&\leq 3M^2 e^{2\omega(T-t_0)}(T - t_0 + 2) \int_{t_0}^t L(v)^2 \|r - \tilde{r}\|_{[t_0,v]}^2 dv, \quad t \in [t_0,T]
\end{aligned}$$

is valid, where $M \geq 1$ and $\omega \in \mathbb{R}$ stem from (8.5). Using the Gronwall Lemma and the hypothesis that r and $\tilde{r}$ are càdlàg on $[t_0,\infty)$, we conclude that r and $\tilde{r}$ are indistinguishable on $[t_0,\infty)$. Taking into account Lemma 7.6, this proves the desired uniqueness of mild and weak solutions for (7.1). □

8.9. **Remarks.**

(1) *The idea to use the Szőkefalvi-Nagy theorem on unitary dilations in order to overcome the difficulties arising from stochastic convolutions, is due to E. Hausenblas and J. Seidler, see* [17] *and* [16].
(2) *Imposing Assumptions 3.3, 4.1, 4.2 and 8.1 we obtain the L^p-version of Theorem 8.8.*

8.10. **Remark.** *Another interpretation of Theorem 8.8 is the following: it is well known that generic mild (or weak) solutions of SPDEs* (7.1) *are not Hilbert space valued semi-martingales due to lack of regularity in time of the finite variation part. However, our method shows that we can decompose every mild (or weak) solution as $r_t = \pi U_{t-t_0} R_t$, $t \geq t_0$ where R is a semi-martingale, U a strongly continuous group and π the orthogonal projection due to Assumption 8.1.*

9. Stability and regularity of stochastic partial differential equations

We shall now deal with stability and regularity of stochastic partial differential equations of the kind (7.1). Stability and regularity results for SPDEs can also be found in [1] and [26]. Here, we can easily transfer the results on stability from Section 5 and on regularity from Section 6 to SPDEs by the method of the moving frame. For stability results, we provide the details in this section.

As in Section 8, we suppose Assumption 8.1 and that $\alpha : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$ fulfill Assumptions 3.3, 3.4, 3.5. For each $n \in \mathbb{N}$, let $\alpha_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\sigma_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\gamma_n : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}\otimes\mathcal{E}}$ be such that Assumptions 5.1, 5.2, 5.3 are fulfilled. Furthermore, let $t_0 \in \mathbb{R}_+$, $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ and for each $n \in \mathbb{N}$ let $h^n \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ and $B_n \in \mathcal{E}$ be given.

According to Theorem 8.8, there exists a unique solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (7.1) with $r|_{[0,t_0]} = h$ with càdlàg paths on $[t_0,\infty)$ satisfying (3.19), and for each $n \in \mathbb{N}$ there exists a unique solution $r^n \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for

$$\begin{cases} dr_t^n &= (Ar_t^n + \alpha(r^n)_t)dt + \sigma(r^n)_t dW_t \\ & \quad + \int_{B_n} \gamma(r^n)(t,x)(\mu(dt,dx) - F(dx)dt) \\ r^n|_{[0,t_0]} &= h^n \end{cases}$$

with càdlàg paths on $[t_0,\infty)$ satisfying $\mathbb{E}[\sup_{t\in[t_0,T]} \|r_t^n\|^2] < \infty$ for all $T \geq t_0$. We suppose that Assumption 5.5, in which $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ denotes the mild and weak solution for (7.1), holds true.

9.1. **Proposition.** *Suppose that Assumptions 3.3, 3.4, 3.5, 5.1, 5.2, 5.3, 5.5 and 8.1 are fulfilled. Then, there exist maps $K_1, K_2 : \mathbb{R}_+ \to \mathbb{R}_+$, only depending on the Lipschitz function L, such that the following statements are valid:*

(1) *If $h^n \to h$ in $C_{\mathrm{ad}}[0,t_0]$, then for each $T \geq t_0$ we have the estimate*

$$\sup_{t\in[0,T]} \mathbb{E}[\|r_t - r_t^n\|^2] \leq K_1\big(\|h - h^n\|_{[0,t_0]}^2 + C_n^2\big) \to 0 \quad \text{for } n \to \infty, \tag{9.1}$$

where $K_1 = K_1(T)$ and $C_n = C_n(T,r)$ is defined in (5.4).

(2) *If even $h^n \to h$ in $S^2[0,t_0]$, then for each $T \geq t_0$ we have the estimate*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r_t - r_t^n\|^2\bigg] \leq K_2\big(\|h - h^n\|_{S^2[0,t_0]}^2 + C_n^2\big) \to 0 \quad \text{for } n \to \infty, \tag{9.2}$$

where $K_2 = K_2(T)$ and $C_n = C_n(T,r)$ is defined in (5.4).

Proof. We define $\tilde{\alpha} : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma} : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to L_2(U_0,\mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma} : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to \mathcal{H}_{\mathcal{P}\otimes\mathcal{E}}$ by (8.7), (8.8), (8.9). Moreover, for each $n \in \mathbb{N}$, we define $\tilde{\alpha}_n : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to \mathcal{H}_{\mathcal{P}}$, $\tilde{\sigma}_n : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to L_2(U_0,\mathcal{H})_{\mathcal{P}}$ and $\tilde{\gamma}_n : C_{\mathrm{ad}}(\mathbb{R}_+;\mathcal{H}) \to \mathcal{H}_{\mathcal{P}\otimes\mathcal{E}}$ as

$$\tilde{\alpha}_n(R)_t := U_{-t}^{t_0} \ell \alpha_n(\pi U^{t_0} R)_t,$$
$$\tilde{\sigma}_n(R)_t := U_{-t}^{t_0} \ell \sigma_n(\pi U^{t_0} R)_t,$$
$$\tilde{\gamma}_n(R)(t,x) := U_{-t}^{t_0} \ell \gamma_n(\pi U^{t_0} R)(t,x).$$

According to Theorem 8.8, we have $r = \pi U^{t_0} R$, where $R \in \mathcal{C}_{\rm ad}(\mathbb{R}_+; \mathcal{H})$ denotes the solution for (8.6) with $R|_{[0,t_0]} = \ell h$, and for each $n \in \mathbb{N}$ we have $r^n = \pi U^{t_0} R^n$, where $R^n \in \mathcal{C}_{\rm ad}(\mathbb{R}_+; \mathcal{H})$ denotes the solution for

$$\left\{ \begin{array}{rcl} dR_t^n & = & \tilde{\alpha}(R^n)_t dt + \tilde{\sigma}(R^n)_t dW_t + \int_{B_n} \tilde{\gamma}(R^n)(t,x)(\mu(dt,dx) - F(dx)dt) \\ R^n|_{[0,t_0]} & = & \ell h^n. \end{array} \right.$$

By (8.5), the coefficients $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$ and $\tilde{\alpha}_n, \tilde{\sigma}_n, \tilde{\gamma}_n$, $n \in \mathbb{N}$ fulfill Assumptions 3.3, 3.4, 3.5, 5.1, 5.2, 5.3, where the function L is replaced by (8.10). Moreover, by (8.5), for each $n \in \mathbb{N}$ we have

(9.3)

$$\begin{aligned} \tilde{C}_n(T,R) := \bigg(& \mathbb{E}\bigg[\int_{t_0}^T \| \tilde{\alpha}([R])_s - \tilde{\alpha}_n([R])_s \|^2 ds \bigg] \\ & + \mathbb{E}\bigg[\int_{t_0}^T \| \tilde{\sigma}([R])_s - \tilde{\sigma}_n([R])_s \|_{L_2(U_0,\mathcal{H})}^2 ds \bigg] \\ & + \mathbb{E}\bigg[\int_{t_0}^t \int_E \| \tilde{\gamma}([R])(s,x) - \tilde{\gamma}_n([R])(s,x) \|^2 F(dx) ds \bigg] \\ & + \mathbb{E}\bigg[\int_{t_0}^T \int_{E \setminus B_n} \| \tilde{\gamma}([R])(s,x) \|^2 F(dx) ds \bigg] \bigg)^{\frac{1}{2}} \leq \|\ell\| M e^{\omega(T-t_0)} C_n(T,r) \end{aligned}$$

for all $T \geq t_0$. In particular, Assumption 5.5 is fulfilled for $\tilde{\alpha}([R])$, $\tilde{\sigma}([R])$, $\tilde{\gamma}([R])$ and $\tilde{\alpha}_n([R])$, $\tilde{\sigma}_n([R])$, $\tilde{\gamma}_n([R])$, $n \in \mathbb{N}$. If $h^n \to h$ in $C_{\rm ad}[0,t_0]$, then by applying Proposition 5.6 and noting (8.5) and (9.3), for each $T \geq t_0$ we obtain the estimate

$$\begin{aligned} & \sup_{t \in [0,T]} \mathbb{E}[\|r_t - r_t^n\|^2] \leq \|\pi\|^2 M^2 e^{2\omega(T-t_0)} \sup_{t \in [0,T]} \mathbb{E}[\|R_t - R_t^n\|^2] \\ & \leq \|\pi\|^2 M^2 e^{2\omega(T-t_0)} K_1(T) \big(\|h - h^n\|_{[0,t_0]}^2 + \tilde{C}_n(T,R)^2 \big) \\ & \leq \|\pi\|^2 M^2 e^{2\omega(T-t_0)} K_1(T) \big(\|h - h^n\|_{[0,t_0]}^2 + \|\ell\|^2 M^2 e^{2\omega(T-t_0)} C_n(T,r)^2 \big) \to 0 \end{aligned}$$

for $n \to \infty$, where the map $K_1 : \mathbb{R}_+ \to \mathbb{R}_+$ stems from Proposition 5.6, showing (9.1). Analogously, if $h^n \to h$ in $S^2[0,t_0]$, we get (9.2). □

By Proposition 9.1, the statement of Remark 5.7 concerning the Lipschitz continuity of the solution map $h \mapsto r^h$ is also valid for SPDEs.

Analogously to stability results also the results on regularity can be transferred to SPDEs by the method of the moving frame. The arguments of Section 6 can be transferred literally. The same arguments hold true for L^p-estimates.

10. Stochastic partial differential equations with state dependent coefficients

In this section, we deal with stochastic partial differential equations with state dependent coefficients, which may depend on the randomness ω, the time t and finitely many states of the path of the solution. As we shall see, this is a special case of the framework from Section 8.

Let $K \in \mathbb{N}$ and $0 \leq \delta_1 < \ldots < \delta_K \leq 1$ be given. Moreover, let $\alpha : \Omega \times \mathbb{R}_+ \times H^K \to H$, $\sigma : \Omega \times \mathbb{R}_+ \times H^K \to L_2^0$ be $\mathcal{P} \otimes \mathcal{B}(H^K)$-measurable and $\gamma : \Omega \times \mathbb{R}_+ \times H^K \times E \to H$ be $\mathcal{P} \otimes \mathcal{B}(H^K) \otimes \mathcal{E}$-measurable.

10.1. **Assumption.** *Denoting by $\mathbf{0} \in H^K$ the zero vector, we assume that*

$$
\begin{aligned}
t &\mapsto \mathbb{E}[\|\alpha(t,\mathbf{0})\|^2] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \mathbb{E}[\|\sigma(t,\mathbf{0})\|^2_{L_2^0}] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \mathbb{E}\bigg[\int_E \|\gamma(t,\mathbf{0},x)\|^2 F(dx) \bigg] \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+).
\end{aligned}
$$

10.2. **Assumption.** *We assume there is a function $L \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that almost surely*

$$
\begin{aligned}
\|\alpha(t,h_1) - \alpha(t,h_2)\| &\leq L(t) \sum_{i=1}^K \|h_1^i - h_2^i\|, \\
\|\sigma(t,h_1) - \sigma(t,h_2)\|_{L_2^0} &\leq L(t) \sum_{i=1}^K \|h_1^i - h_2^i\|, \\
\bigg(\int_E \|\gamma(t,h_1,x) - \gamma(t,h_2,x)\|^2 F(dx) \bigg)^{\frac{1}{2}} &\leq L(t) \sum_{i=1}^K \|h_1^i - h_2^i\|
\end{aligned}
$$

for all $t \in \mathbb{R}_+$ and all $h_1, h_2 \in H^K$.

10.3. **Corollary.** *Suppose that Assumptions 8.1 and 10.1, 10.2 are fulfilled. Then, for each $t_0 \in \mathbb{R}_+$ and $h \in \mathcal{C}_{\mathrm{ad}}[0,t_0]$ there exists a unique mild and weak solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for*

$$
\left\{
\begin{array}{rcl}
dr_t &=& (Ar_t + \alpha(t, r_{\delta_1 t}, \dots, r_{\delta_K t}))dt + \sigma(t, r_{\delta_1 t}, \dots, r_{\delta_K t})dW_t \\
&& + \int_E \gamma(t, r_{\delta_1 t-}, \dots, r_{\delta_K t-}, x)(\mu(dt,dx) - F(dx)dt) \\
r|_{[0,t_0]} &=& h
\end{array}
\right. \tag{10.1}
$$

with càdlàg paths on $[t_0,\infty)$, and it satisfies (3.19).

Proof. For every $r \in C_{\mathrm{ad}}(\mathbb{R}_+)$ let ${}^p r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ be a predictable representative of r, which, due to [9, Prop. 3.6.ii], always exists. Now, we define the maps $\tilde{\alpha} : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P}}$, $\tilde{\sigma} : C_{\mathrm{ad}}(\mathbb{R}_+) \to (L_2^0)_{\mathcal{P}}$ and $\tilde{\gamma} : C_{\mathrm{ad}}(\mathbb{R}_+) \to H_{\mathcal{P} \otimes \mathcal{E}}$ by

$$
\begin{aligned}
\tilde{\alpha}(r)_t(\omega) &:= \alpha(\omega, t, {}^p r_{\delta_1 t}(\omega), \dots, {}^p r_{\delta_K t}(\omega)), \quad (\omega,t) \in \Omega \times \mathbb{R}_+ \\
\tilde{\sigma}(r)_t(\omega) &:= \sigma(\omega, t, {}^p r_{\delta_1 t}(\omega), \dots, {}^p r_{\delta_K t}(\omega)), \quad (\omega,t) \in \Omega \times \mathbb{R}_+ \\
\tilde{\gamma}(r)(t,x)(\omega) &:= \gamma(\omega, t, {}^p r_{\delta_1 t}(\omega), \dots, {}^p r_{\delta_K t}(\omega), x), \quad (\omega,t,x) \in \Omega \times \mathbb{R}_+ \times E.
\end{aligned}
$$

Note that for a predictable process $(r_t)_{t\geq 0}$ and an arbitrary $0 \leq \delta \leq 1$ the process $(r_{\delta t})_{t \geq 0}$ is predictable, too. Hence, $\tilde{\alpha}$, $\tilde{\sigma}$, $\tilde{\gamma}$ indeed map into the respective spaces of predictable processes, because α, σ are $\mathcal{P} \otimes \mathcal{B}(H^K)$-measurable and γ is $\mathcal{P} \otimes \mathcal{B}(H^K) \otimes \mathcal{E}$-measurable. Assumption 3.3 holds true by the definition of $\tilde{\alpha}$, $\tilde{\sigma}$, $\tilde{\gamma}$ and Assumption 3.4 is satisfied by Assumption 10.1. Using Assumption 10.2, for all $t \in \mathbb{R}_+$ and all $r^1, r^2 \in C_{\mathrm{ad}}(\mathbb{R}_+)$ we obtain

$$
\begin{aligned}
&\mathbb{E}[\|\alpha(r^1)_t - \alpha(r^2)_t\|^2] = \mathbb{E}[\|\alpha(t, r^1_{\delta_1 t}, \dots, r^1_{\delta_K t}) - \alpha(t, r^2_{\delta_1 t}, \dots, r^2_{\delta_K t})\|^2] \\
&\leq L(t)^2 \mathbb{E}\Bigg[\bigg(\sum_{i=1}^K \|r^1_{\delta_i t} - r^2_{\delta_i t}\| \bigg)^2 \Bigg] \leq K L(t)^2 \sum_{i=1}^K \mathbb{E}[\|r^1_{\delta_i t} - r^2_{\delta_i t}\|^2] \\
&\leq K^2 L(t)^2 \max_{i=1,\dots,K} \mathbb{E}[\|r^1_{\delta_i t} - r^2_{\delta_i t}\|^2] \leq K^2 L(t)^2 \|r^1 - r^2\|^2_{[0,t]}.
\end{aligned}
$$

An analogous argumentation for σ and γ proves that Assumption 3.5 is fulfilled. Applying Theorem 8.8, there exists a unique mild and weak solution $r \in \mathcal{C}_{\mathrm{ad}}(\mathbb{R}_+)$ for (7.1) with càdlàg paths on $[t_0,\infty)$ satisfying (3.19). For every $T \geq t_0$ we have, by using Assumption 10.2, and since each path of r has only countably many jumps on the interval $[t_0, T]$,

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{t_0}^T \int_E \| \gamma(t, {}^p r_{\delta_1 t}, \ldots, {}^p r_{\delta_K t}, x) - \gamma(t, r_{\delta_1 t-}, \ldots, r_{\delta_K t-}, x) \|^2 F(dx) dt \bigg] \\
&\leq \int_{t_0}^T \mathbb{E}\Bigg[L(t)^2 \bigg(\sum_{i=1}^K \| {}^p r_{\delta_i t} - r_{\delta_i t-} \| \bigg)^2 \Bigg] dt \leq K \sum_{i=1}^K \int_{t_0}^T \mathbb{E}\big[L(t)^2 \| r_{\delta_i t} - r_{\delta_i t-} \|^2 \big] dt \\
&= K \sum_{i=1}^K \mathbb{E}\bigg[\int_{t_0}^T L(t)^2 \| \Delta r_{\delta_i t} \|^2 dt \bigg] = 0.
\end{aligned}$$

Therefore, $\gamma(t, {}^p r_{\delta_1 t}, \ldots, {}^p r_{\delta_K t}, x) \mathbb{1}_{[t_0,\infty)}$ and $\gamma(t, r_{\delta_1 t-}, \ldots, r_{\delta_K t-}, x) \mathbb{1}_{[t_0,\infty)}$ coincide in the space $L^2(\mu; H)$. Consequently, the process r is also the unique mild and weak solution for (10.1). □

As a particular case, we now turn to the Markovian framework. Let $\alpha : \mathbb{R}_+ \times H \to H$, $\sigma : \mathbb{R}_+ \times H \to L_2^0$ and $\gamma : \mathbb{R}_+ \times H \times E \to H$ be measurable.

10.4. **Assumption.** *We assume that*

$$\begin{aligned}
t &\mapsto \| \alpha(t, 0) \| \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \| \sigma(t, 0) \|_{L_2^0} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+), \\
t &\mapsto \int_E \| \gamma(t, 0, x) \|^2 F(dx) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+).
\end{aligned}$$

10.5. **Assumption.** *We assume there is a function $L \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$\begin{aligned}
\| \alpha(t, h_1) - \alpha(t, h_2) \| &\leq L(t) \| h_1 - h_2 \|, \\
\| \sigma(t, h_1) - \sigma(t, h_2) \|_{L_2^0} &\leq L(t) \| h_1 - h_2 \|, \\
\bigg(\int_E \| \gamma(t, h_1, x) - \gamma(t, h_2, x) \|^2 F(dx) \bigg)^{\frac{1}{2}} &\leq L(t) \| h_1 - h_2 \|
\end{aligned}$$

for all $t \in \mathbb{R}_+$ and all $h_1, h_2 \in H$.

10.6. **Corollary.** *Suppose that Assumptions 8.1 and 10.4, 10.5 are fulfilled. Then, for each $h_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ there exists a unique càdlàg, adapted, mean-square continuous mild and weak solution $(r_t)_{t \geq 0}$ for*

$$\begin{cases} dr_t &= (A r_t + \alpha(t, r_t)) dt + \sigma(t, r_t) dW_t + \int_E \gamma(t, r_{t-}, x) (\mu(dt, dx) - F(dx) dt) \\ r_0 &= h_0, \end{cases}$$

and it satisfies (3.19).

Proof. The assertion follows from Corollary 10.3 with $K = 1$, $\delta_1 = 1$ and $t_0 = 0$. □

We close this section with the time-homogeneous case. Let $\alpha : H \to H$, $\sigma : H \to L_2^0$ and $\gamma : H \times E \to H$ be measurable.

10.7. **Assumption.** *We assume $\int_E \| \gamma(0, x) \|^2 F(dx) < \infty$.*

10.8. **Assumption.** *We assume that there is a constant $L \geq 0$ such that*

$$\begin{aligned}
\| \alpha(h_1) - \alpha(h_2) \| &\leq L \| h_1 - h_2 \|, \\
\| \sigma(h_1) - \sigma(h_2) \|_{L_2^0} &\leq L \| h_1 - h_2 \|, \\
\bigg(\int_E \| \gamma(h_1, x) - \gamma(h_2, x) \|^2 F(dx) \bigg)^{\frac{1}{2}} &\leq L \| h_1 - h_2 \|
\end{aligned}$$

for all $h_1, h_2 \in H$.

10.9. **Corollary.** *Suppose that Assumptions 8.1 and 10.7, 10.8 are fulfilled. Then, for each $h_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ there exists a unique càdlàg, adapted, mean-square continuous mild and weak solution $(r_t)_{t \geq 0}$ for*

$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0, \end{cases}$$

and it satisfies (3.19).

Proof. The statement is an immediate consequence of Corollary 10.6. □

10.10. **Remark.** *An analogous reasoning also provides the corresponding L^p-versions of Corollaries 10.3, 10.6 and 10.9.*

10.11. **Remark.** *The time-inhomogeneous case can be considered by an extension of the state space from H to $\mathbb{R} \times H$. However, one has to pay attention at the boundary points of the interval $[0,T]$, where the vector fields have to be extended to the whole real line. Nevertheless we shall consider in the setting of our numerical applications the time-homogeneous case as the most characteristic one for all further applications.*

11. High-order (explicit-implicit) numerical schemes for stochastic partial differential equations with weak convergence order

We sketch in this last section high-order explicit-implicit numerical schemes for stochastic partial differential equations with state-dependent coefficients as introduced in Section 10. In this section (and only here) we will actually use that the Wiener process and the Poisson random measure are independent, see Section 2.5. By the stability results from Section 9 we can reduce the problem to simpler driving signals, namely a finitely active Poisson random measure and to a finite number of driving Wiener processes. We apply the results of [5] for the time-dependent SDE, which – due to the "method of the moving frame" – can be transferred to the general SPDE case. Our main focus here is to work out so-called cubature schemes, extended by finite activity jump parts, for time-dependent SDEs and therefore – by the method of the moving frame – for SPDEs. This also allows for high order numerical approximation schemes. Notice that cubature schemes are very adapted to SPDEs, since every local step can – in contrast to Taylor schemes – preserve the regularity of the states. Additionally a simple complexity analysis in the case of SPDEs yields that having a small amount p of local high order steps is cheaper than having a large amount p of local low order steps. The reason is that every local step means practically to solve a PDE numerically.

For this purpose we apply the respective notions from Section 7, 8 and 10 in order to formulate our conditions on the vector fields. Having the stability results of Section 9 in mind we do assume finite activity of the Poisson random measure, i.e. $F(E) < \infty$ and a finite dimensional Wiener process. Notice that this also allows for a statement on the rate of convergence to the original equations with possibly infinitely active jumps and infinitely many Brownian motions.

We consider here SPDEs of the type

(11.1)
$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &\in H \end{cases}$$

Let $T > 0$ denote a time-horizon. As in Section 10 we introduce measurable maps $\alpha : H \to H$, $\sigma : H \to L_2^0$ and $\gamma : H \times E \to H$ and define maps $\tilde{\alpha} : [0,T] \times \mathcal{H} \to \mathcal{H}$,

$\tilde{\sigma} : [0,T] \times \mathcal{H} \to \mathcal{H}$ and $\tilde{\gamma} : [0,T] \times \mathcal{H} \times E \to \mathcal{H}$ defined as

$$\begin{aligned} \tilde{\alpha}(t,R) &:= U_{-t}\ell\alpha(\pi U_t R), \quad t \in [0,T] \\ \tilde{\sigma}(t,R) &:= U_{-t}\ell\sigma(\pi U_t R), \quad t \in [0,T] \\ \tilde{\gamma}(t,R,x) &:= U_{-t}\ell\gamma(\pi U_t R, x), \quad (t,x) \in [0,T] \times E. \end{aligned}$$

11.1. **Assumption.** *Fix $m \geq 2$ (a degree of accuracy for the high order scheme) and $T > 0$. We assume that for the vector fields $\tilde{\alpha}$, $\tilde{\sigma}$, $\tilde{\gamma}$ there is a constant $M > 0$ such that for every radius $C_1 > 0$ we have*

$$\begin{aligned} \sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C_1} \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\alpha}(t,h)\| &\leq MC_1^{k_1}, \\ \sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C_1} \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\sigma}(t,h)\|_{L_2^0} &\leq MC_1^{k_1}, \\ \sup_{t\in[0,T],\, h\in\mathcal{H}, \|h\|\leq C_1} \int_E \|\partial_t^{k_1}\partial_h^{k_2}\tilde{\gamma}(t,h,x)\|^2 F(dx) &\leq MC_1^{k_1}, \end{aligned}$$

holds true for all $k_1 + k_2 \leq m+1$. In words, the growth of derivatives of the time-dependent vector fields up to order $m+1$ is polynomial in the radius of order k_1, when k_1 denotes the order of the time derivative.

11.2. **Example.** *Typical examples for vector fields satisfying Assumption 11.1 are those of functional form (as applied in* [13]*), i.e., choose a smooth map with all derivatives bounded $\phi : \mathbb{R}^n \to D(\mathcal{A}^\infty)$, where $\mathcal{A}$ denotes the infinitesimal generator of U, and choose $\xi_1, \cdots, \xi_n \in D((\mathcal{A}^*)^\infty)$, then*

$$\tilde{\sigma}(t,h) = U_{-t}\ell\phi(\langle \xi_1, \pi U_t h\rangle, \cdots, \langle \xi_n, \pi U_t h\rangle) \tag{11.2}$$

for $h \in \mathcal{H}$ satisfies Assumption 11.1 for every $m \geq 2$.

11.3. **Remark.** *As outlined in* [32] *the previous assumptions imply* Lip$(m+1)$*-conditions on the ball with radius C_1 for the corresponding time-dependent vector fields $\tilde{\alpha}, \tilde{\sigma}, \tilde{\gamma}$. This has an important meaning for the extension of our theory towards rough paths, see* [32].

11.4. **Remark.** *The Assumption 11.1 leads to global existence and uniqueness of the corresponding time-dependent stochastic differential equations and the corresponding SPDEs* (11.1)*. Due to the finite activity of the jump process the conditions also lead to existence of moments of any order of the solution process.*

We do first assume $\gamma = 0$, such that we find ourselves in a pure diffusion case. Furthermore we have assumed that the driving Wiener noise is finite dimensional, in other words we can write the stochastic partial differential equation in the moving frame and on the extended phase space $\mathbb{R} \times \mathcal{H}$:

$$dR_t = \tilde{\alpha}(s,R_t)dt + \sum_{i=1}^d \tilde{\sigma}_i(s,R_t)dW_t^i,\ ds = dt, \tag{11.3}$$

$$R_0 = r_0,\ s_0 = 0. \tag{11.4}$$

Let us fix $m \geq 2$. Notice that the Assumption 11.1 implies Assumptions 3.4 and 3.5, in particular the vector fields are $(m+1)$-times differentiable in all variables. This allows us to state the standard result on short-time asymptotic for the stochastic differential equation (11.3).

We now apply the notations for iterated stochastic integrals, i.e., the abbreviation

$$W_t^{(i_1,\dots,i_k)} = \int_{0\leq t_1\leq\cdots\leq t_k\leq t} \circ dW_{t_1}^{i_1} \cdots \circ dW_{t_k}^{i_k}$$

is short for iterated Stratonovich integrals, where we apply $\circ dW_t^0 = dt$. Notice also the degree mapping

$$\deg(i_1, \dots, i_k) = k + \operatorname{card}\{j \mid i_j = 0\},$$

which counts any appearance of 0 in the multiindex $(i_1, \dots, i_k)$ twice, since $\circ dW_t^0 = dt$ comes with twice the order of short time asymptotics than a Brownian motion. This is also the reason for the particular structure of the Assumptions 11.1. Recall also that any vector field σ can be interpreted as a first order differential operator on test functions f by

$$(\sigma f)(x) = Df(x) \cdot \sigma(x), \quad x \in H,$$

which will be applied extensively in the sequel.

11.5. **Theorem.** *Let $g : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ be a smooth function with all derivatives bounded. Then we have the following asymptotic formula,*

$$g(t, R_t) = \sum_{\deg(i_1,\dots,i_k) \le m} (\beta_{i_1} \cdots \beta_{i_k} g)(0, R_0) W_t^{(i_1,\dots,i_k)} + R_m(t, g, R_0), \quad R_0 \in \mathcal{H}, \tag{11.5}$$

with

$$\sqrt{E(R_m(t,g,R_0)^2)} \le C t^{\frac{m+1}{2}} \max_{m < \deg(i_1,\dots,i_k) \le m+2} \sup_{0 \le s \le t} \sqrt{E(|\beta_{i_1} \cdots \beta_{i_k} g(R_s, s)|^2)} \le \tag{11.6}$$

$$\le \tilde{M} \sup_{2k_1 \le m+1} \sup_{0 \le s \le t} E(||R_s||^{k_1}) < \infty, \tag{11.7}$$

where $\tilde{M}$ is a constant derived from Assumption 11.1.

Proof. The proof is a direct consequence of the results of [5, Prop. 3.1], where one additionally observes the necessary degrees of differentiability which are needed for the result. Notice in particular that the remainder term stays bounded due to the conditions of Assumption 11.1, in particular due to polynomial growth of the derivatives of at most order $m+1$ and the existence of moments up to order $m+1$ of the solution process. $\square$

11.6. **Example.** *We formulate the short-time asymptotic formula in the case $m = 2$ by taking the definitions of the vector fields $\beta_0, \dots, \beta_d$ and a smooth test function $g : \mathcal{H} \to \mathbb{R}$, which does not depend on the additional (time-)state s, then*

$$\begin{aligned}
g(R_t) =& g(R_0) + \beta_0 g(R_0) t + \sum_{i=1}^d \beta_i g(R_0) W_t^i + \\
& \sum_{i,j=1}^d \beta_i \beta_j g(R_0) W_t^{(i,j)} + \mathcal{O}(t^{\frac{3}{2}}) \\
=& g(R_0) + Dg(R_0) \bullet \alpha(R_0) t + \sum_{i=1}^d Dg(R_0) \bullet \sigma_i(R_0) W_t^i + \\
& + \sum_{i=1}^d Dg(R_0) \bullet (D\sigma_i(R_0) \bullet \sigma_i(R_0)) \frac{(W_t^i)^2 - t}{2} + \\
& + \sum_{i \ne j = 1}^d Dg(R_0) \bullet (D\sigma_i(R_0) \bullet \sigma_j(R_0)) W_t^{(i,j)} + \mathcal{O}(t^{\frac{3}{2}})
\end{aligned}$$

for $t \geq 0$, $R_0 \in \mathcal{H}$. Heading for a strong Euler-Maruyama-scheme the previous formula yields – formally evaluated for $g = id$ – the first iteration step from $0 \to t$

$$R_0 \mapsto R_0 + \alpha(R_0)t + \sum_{i=1}^{d} \sigma_i(R_0)W_t^i.$$

For the next step in the iteration we need the asymptotic expansion at time t and therefore also the vector fields $\tilde{\alpha}, \tilde{\sigma}$ appear at time t, namely

$$R_t \mapsto R_t + \tilde{\alpha}(t, R_t)t + \sum_{i=1}^{d} \tilde{\sigma}_i(t, R_t)W_t^i.$$

However, when one transfer the iteration of these two steps via πU_{2t} to H the described cancellation happens and one obtains the two-fold iteration of the time-homogeneous scheme

$$r_0 \mapsto S_t r_0 + S_t\alpha(r_0)t + \sum_{i=1}^{d} S_t\sigma_i(r_0)W_t^i.$$

Notice that this scheme is implicit in the linear PDE-part and explicit in the stochastic components and the non-linear drift component. Notice also that the weak convergence order 1 is obtained if the Assumptions 11.1 for $m = 2$ are satisfied for smooth test functions with all derivatives bounded.

As explained in the literature, for instance in [5] or [19], we can derive high-order schemes (strong or weak) from the given short time-asymptotic expansion. The weak order of convergence – given a short-time asymptotics of order $t^{\frac{m+1}{2}}$ – is then $\frac{m-1}{2}$. Therefore we obtain high-order Taylor schemes for the time-dependent system (11.3). However, even though possible, those Taylor schemes are usually not interesting – except for the case $m = 2$ – since one has to work at each step with time derivatives of the vector fields, which corresponds to working with the infinitesimal generator of the semigroup.

11.7. **Example.** *Consider a vector field $\tilde{\sigma}$ of functional form* (11.2)*, then apparently the time-derivative of the vector field, which appears in the stochastic Taylor expansion for $m \geq 3$, has the formula*

$$\frac{\partial}{\partial t}\tilde{\sigma}(t, R) = D\sigma(\langle \xi_1, \pi U_t\rangle, \cdots, \langle \xi_n, \pi U_t\rangle) \bullet (\langle \pi\mathcal{A}^*\xi_1, U_t h\rangle, \cdots, \langle \pi\mathcal{A}^*\xi_n, U_t h\rangle),$$

which contains the infinitesimal generator and which is linearly growing in h.

We present here a method to circumvent the problem that in each local step the infinitesimal generator appears, namely the cubature method: its implementation and structure work in the case of Hilbert space valued SDEs of type (11.3) in precisely the same way as in the finite dimensional case (see for instance [5] for details), since we do not have to deal with the unbounded infinitesimal generator. Convergence of global order $\frac{m-1}{2}$ follows from Assumption 11.1 on any bounded set. On the other hand, each local time step does not contain derivatives of the vector fields in question, and preserves therefore the regularity of the state vector. We need one analytical preparation for this, namely the following lemma which tells that – under Assumption 11.1 – we can suppose that on each bounded set there is a Lip$(m + 1)$-extension of the vector fields on the whole extended phase space.

11.8. **Lemma.** *Define vector fields β_i on the extended phase space $[0,T]\times\mathcal{H}$ by the following formulas:*

$$\beta_0(s,R)=\big(1,\tilde\alpha(s,R)-\frac{1}{2}\sum_{i=1}^d D\tilde\sigma_i(s,R)\bullet\tilde\sigma_i(s,R)\big), \tag{11.8}$$

$$\beta_i(s,R)=\big(0,\tilde\sigma_i(s,R)\big), \tag{11.9}$$

for $i=0,\dots,d$. Then for each C_1 we find vector fields $\beta_0^{C_1},\dots,\beta_d^{C_1}$ which conincide with the previous vector fields on the ball with radius C_1 but are $\mathrm{Lip}(m+1)$ *on the whole extended phase space $\mathbb{R}\times\mathcal{H}$.*

Proof. This is due to the fact that on balls with radius C_1+1 even the vector fields β_i satisfy a $\mathrm{Lip}(m+1)$ condition. Multiplying with a bump function being equal to 1 on the ball of radius C_1 and vanishing outside a ball of radius C_1+1 yields the result. □

In the rest of the section we develop the necessary terminology for cubature methods: Theorem 11.5 shows that iterated Stratonovich integrals play the same role as polynomials play in deterministic Taylor expansion. Consequently, it is natural to use them in order to define cubature formulas. Let $C_{bv}([0,t];\mathbb{R}^d)$ denote the space of continuous paths of bounded variation taking values in $\mathbb{R}^d$. As for the Brownian motion, we append a component $\omega^0(t)=t$ for any $\omega\in C_{bv}([0,t];\mathbb{R}^d)$. Furthermore, we establish the following convention: whenever r_t is the solution to some stochastic differential equation driven by Brownian motions W, whether on a finite or infinite dimensional space, and $\omega\in C_{bv}([0,t];\mathbb{R}^d)$, we denote by $r_t(\omega)$ the solution of the deterministic differential equation given by formally replacing all occurrences of "$\circ dW_s^i$" with "$d\omega^i(s)$" (with the same initial values). Note that it is necessary that the SDE for r is formulated in the Stratonovich sense (recall that the Stratonovich formulation does not necessarily make sense). With the following simple lemma we see that time dependent coordinate transforms commute with the procedure of replacing Brownian motions by deterministic trajectories.

11.9. **Lemma.** *Let $\omega:[0,T]\to\mathbb{R}^d$ be a continuous curve with finite total variation. Then the time-dependent ordinary differential equation*

$$dR_t(\omega)=(\tilde\alpha(t,R_t(\omega))-\frac{1}{2}\sum_{i=1}^d D\tilde\sigma_i(t,R_t(\omega))\bullet\tilde\sigma_i(t,R_t(\omega)))dt+ \tag{11.10}$$

$$+\sum_{i=1}^d\tilde\sigma_i(t,R_t(\omega))d\omega^i(t),\ R_0=r_0, \tag{11.11}$$

driven by ω instead of the finite dimensional Wiener process W, has a strong solution, which transfers via $r_t(\omega)=\pi U_tR_t(\omega)$ to a mild solution of

$$dr_t(\omega)=(Ar_t(\omega)+\alpha(r_t(\omega))-\frac{1}{2}\sum_{i=1}^d D\sigma_i(r_t(\omega))\bullet\sigma_i(r_t(\omega)))dt+\sum_{i=1}^d\sigma_i(r_t(\omega))d\omega^i(t). \tag{11.12}$$

Having in mind that one replaces Brownian motion W by a finite set of deterministic curves appearing with certain probabilities, we have to keep track of necessary moment conditions for (high-order) weak convergence, which is done in the following definition:

11.10. **Definition.** *Fix $t>0$ and $m\geq 1$. Positive weights $\lambda_1,\dots,\lambda_N$ summing up to 1 and paths $\omega_1,\dots,\omega_N\in C_{bv}([0,t];\mathbb{R}^d)$ form a* cubature formula on Wiener space *of degree* m *if for all multi-indices* $(i_1,\dots,i_k)\in\mathcal{A}$ *with* $\deg(i_1,\dots,i_k)\le m$, $k\in\mathbb{N}$, *we have that*

$$E(W_t^{(i_1,\dots,i_k)})=\sum_{l=1}^{N}\lambda_l W_t^{(i_1,\dots,i_k)}(\omega_l),$$

where we used the convention in line with the previous one, namely

$$W_t^{(i_1,\dots,i_k)}(\omega)=\int_{0\le t_1\le\dots\le t_k\le t} d\omega^{i_1}(t_1)\cdots d\omega^{i_k}(t_k).$$

Lyons and Victoir [20] show the existence of cubature formulas on Wiener space for any d and size $N\le\#\{I\in\mathcal{A}|\deg(I)\le m\}$ by applying Chakalov's theorem on cubature formulas and Chow's theorem for nilpotent Lie groups. Moreover, due to the scaling properties of Brownian motion (and its iterated Stratonovich integrals), i.e.

$$W_t^{(i_1,\dots,i_k)}\overset{\text{law}}{=}\sqrt{t}^{\deg(i_1,\dots,i_k)}W_1^{(i_1,\dots,i_k)},$$

it is sufficient to construct cubature paths for $t=1$.

11.11. **Assumption.** *Once and for all, we fix one cubature formula* $\widetilde{\omega}_1,\dots,\widetilde{\omega}_N$ *with weights* $\lambda_1,\dots,\lambda_N$ *of degree* $m\ge 2$ *on the interval* $[0,1]$. *Without loss of generality we assume that* $\widetilde{\omega}_i(0)=0$. *By abuse of notation, for any* $t>0$, *we will denote* $\omega_l(s)=\sqrt{t}\widetilde{\omega}_l(s/t)$, $s\in[0,t]$, $l=1,\dots,N$, *which yields a cubature formula for* $[0,t]$ *with the same weights* $\lambda_1,\dots,\lambda_N$.

11.12. **Example.** *For* $d=1$ *Brownian motions, a cubature formula on Wiener space of degree* $m=3$ *is given by* $N=2$ *paths*

$$\omega_1(s)=-\frac{s}{\sqrt{t}},\ \omega_2(s)=\frac{s}{\sqrt{t}}$$

for fixed time horizon t. *The corresponding weights are given by* $\lambda_1=\lambda_2=\frac{1}{2}$.

When we deal with $\mathrm{Lip}(m+1)$ vector fields on extended phase space we can write down – by means of the finitely many cubature trajectories – a local scheme. Notice that we have to replace the original vector fields $\beta_0,\dots,\beta_d$ by globally $\mathrm{Lip}(m+1)$ vector fields $\beta_0^{C_1},\dots,\beta_d^{C_1}$ on some large ball of radius C_1, see Lemma 11.8. The respective solutions of the SDEs are denoted by R^{C_1}. Combining then the stochastic Taylor expansion, the deterministic Taylor expansion for solutions of ODEs driven by ω_i for a cubature formula on Wiener space one obtains a one-step scheme for weak approximation of equations of type (11.3) precisely the same way as in [20]. Indeed, we get

$$\begin{aligned}(11.13)\quad \sup_{r_0\in\mathcal{H}}&|E(g(t,R_t^{C_1}))-\textstyle\sum_{l=1}^{N}\lambda_l g(t,R_t^{C_1}(\omega_l))|\\ &\le Ct^{\frac{m+1}{2}}\max_{\substack{(i_1,\dots,i_k)\in\mathcal{A}\\ m<\deg(i_1,\dots,i_k)\le m+2}}\sup_{r\in\mathcal{H}}|\beta_{i_1}^{C_1}\cdots\beta_{i_k}^{C_1}g(t,r)|,\end{aligned}$$

for $0<t<1$ and some test function g with all derivatives bounded.

11.13. **Remark.** *Due to the a priori bounds on the moments of the solution process* R_t, *we can estimate the probability for* R *to leave a ball of radius* C_1 *and we can therefore control "a priori" the error of replacing the vector fields* $\beta_0,\dots,\beta_d$ *by globally* $\mathrm{Lip}(m+1)$ *vector fields* $\beta_0^{C_1},\dots,\beta_d^{C_1}$ *on some large ball of radius* C_1.

For the global method (in fact an iteration due to the Markov property), divide the interval $[0,T]$ into p subintervals according to the partition $0=t_0<t_1<\dots<t_p=T$. For a multi-index $(l_1,\dots,l_p)\in\{1,\dots,N\}^p$ consider the path $\omega_{l_1,\dots,l_p}$

defined by concatenating the paths $\omega_{l_1}, \dots, \omega_{l_p}$, i.e. $\omega_{l_1,\dots,l_p}(t) = \omega_{l_1}(t)$ for $t \in]0, t_1]$ and

$$\omega_{l_1,\dots,l_p}(t) = \omega_{l_1,\dots,l_p}(t_{r-1}) + \omega_{l_r}(t - t_{r-1})$$

for r such that $t \in]t_{r-1}, t_r]$, where ω_{l_r} is scaled to be a cubature path on the interval $[0, t_r - t_{r-1}]$.

11.14. **Proposition.** *Fix $T > 0$, $m \in \mathbb{N}$, $C_1 > 0$, a cubature formula of degree m as in Definition 11.10 and a partition of $[0,T]$ as above. For every test function g there is a constant D independent of the partition such that*

$$\sup_{r \in \mathcal{H}} \Big| E(g(t, R_T^{C_1})) - \sum_{(l_1,\dots,l_p) \in \{1,\dots,N\}^p} \lambda_{l_1} \cdots \lambda_{l_p} g(t, R_T^{C_1}(\omega_{l_1,\dots,l_p})) \Big| \leq DT \max_{r=1,\dots,p} (t_r - t_{r-1})^{(m-1)/2}.$$

Additionally $R_T^{C_1}(\omega_{l_1,\dots,l_p})$, due to [24]*, we can allow a local error of order $\frac{m+1}{2}$ along each ω_{l_j}.*

Due to Lemma 11.9 we can transfer the previous result including the rate of convergence on the original space. Notice that the projection of the equations with vector fields $\beta_0^{C_1}, \dots, \beta_d^{C_1}$ only coincide on some bounded set of the original Hilbert space H with the original equation, which is, however, for numerical purposes sufficient. The transfer works so well due to the linearity of the semigroup and the projection.

11.15. **Remark.** *The same techniques as in* [5] *for the inclusion of finite activity jump processes also work in this setting. We do not outline this aspect here, since our main purpose was to show that high-order weak approximation schemes exist in the realm of SPDEs under fairly general assumptions on vector fields and test functions.*

We can summarize the method as follows:

- Choose a degree of accuracy $m \geq 2$ and a set of cubature paths $\omega_1, \dots, \omega_N$.
- Choose trajectories $\omega_{l_1,\dots,l_p}$ by means of a MC-procedure.
- Calculate numerically, with error of order $\frac{m+1}{2}$, the solution of the PDE obtained by "evaluating" the SPDE (11.1) along ω_{l_j}.
- Apply the main result to obtain a high order convergence scheme of order $\frac{m-1}{2}$.

11.16. **Remark.** *The advantage of high-order schemes becomes visible when the calculation of each local step is expensive: in this case a small number p is a true advantage.*

Appendix A. Stochastic Fubini theorem with respect to Poisson measures

In this appendix, we provide a stochastic Fubini theorem with respect to compensated Poisson random measures, see Theorem A.2, which we require for the proof of Lemma 7.7.

We could not find a proof in the literature. In the appendix of [8], it is merely mentioned that it can be provided the same way as in [28], where stochastic integrals with respect to semimartingales are considered. The stochastic Fubini theorem [3, Thm. 5], which is used in the proof of [21, Prop. 5.3], only deals with finite measure spaces.

We start with an auxiliary result.

A.1. **Lemma.** *Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$ be two σ-finite measure spaces. We define the product space*

$$(\Omega, \mathcal{F}, \mu) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2).$$

For each $\Phi \in L^2(\Omega, \mathcal{F}, \mu)$ there exists a sequence

$$(\Phi_n)_{n\in\mathbb{N}} \subset \operatorname{span}\{\mathbb{1}_{A_1}\mathbb{1}_{A_2} : A_i \in \mathcal{F}_i \text{ with } \mu_i(A_i) < \infty,\ i = 1, 2\} \tag{A.1}$$

such that $\Phi_n \to \Phi$ in $L^2(\Omega, \mathcal{F}, \mu)$.

Proof. Let $\Phi \in L^2(\Omega, \mathcal{F}, \mu)$ be arbitrary. We decompose $\Phi = \Phi^+ - \Phi^-$ into its positive and negative part. There are sequences $(\Phi_n^+)_{n\in\mathbb{N}}$, $(\Phi_n^-)_{n\in\mathbb{N}}$ of nonnegative measurable functions, taking only a finite number of values, such that $\Phi_n^+ \uparrow \Phi^+$ and $\Phi_n^- \uparrow \Phi^-$, see, e.g., [4, Satz 11.6].

Moreover, since μ_1 and μ_2 are σ-finite measures, there exist sequences $(C_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and $(D_n)_{n\in\mathbb{N}} \subset \mathcal{F}_2$ such that $\mu_1(C_n) < \infty$, $\mu_2(D_n) < \infty$ for all $n \in \mathbb{N}$ and $C_n \uparrow \Omega_1$, $D_n \uparrow \Omega_2$ as $n \to \infty$. By Lebesgue's dominated convergence theorem we have $(\Phi_n^+ - \Phi_n^-)\mathbb{1}_{C_n \times D_n} \to \Phi$ in $L^2(\Omega, \mathcal{F}, \mu)$.

Therefore, we may, without loss of generality, assume that $\Phi = \sum_{j=1}^m c_j \mathbb{1}_{A_j}$, where $m \in \mathbb{N}$, $c_j \in \mathbb{R} \setminus \{0\}$, $j = 1, \dots, m$ and $A_j \in (\mathcal{F}_1 \otimes \mathcal{F}_2) \cap (C_1 \times C_2)$, $j = 1, \dots, m$, where $C_i \in \mathcal{F}_i$, $i = 1, 2$ and $\mu_i(C_i) < \infty$, $i = 1, 2$.

Note that the trace σ-algebra $(\mathcal{F}_1 \otimes \mathcal{F}_2) \cap (C_1 \times C_2)$ is generated by the algebra

$$\mathcal{A} = \left\{ \biguplus_{k=1}^{p} D_k \times E_k \,|\, p \in \mathbb{N} \text{ and } D_k \in \mathcal{F}_1 \cap C_1,\ E_k \in \mathcal{F}_2 \cap C_2 \text{ for } k = 1, \dots, p \right\}.$$

By [4, Satz 5.7] there exists, for each $j \in \{1, \dots, m\}$ and each $n \in \mathbb{N}$, a set $B_j^n \in \mathcal{A}$ such that $\mu(A_j \Delta B_j^n) < \frac{1}{m^2 n c_j^2}$.

Setting $\Phi_n := \sum_{j=1}^m c_j \mathbb{1}_{B_j^n}$ for $n \in \mathbb{N}$ we have (A.1) and

$$\begin{aligned}
\int_\Omega |\Phi(\omega) - \Phi_n(\omega)|^2 d\mu(\omega) &\leq \int_\Omega \left(\sum_{j=1}^m |c_j| \cdot |\mathbb{1}_{A_j}(\omega) - \mathbb{1}_{B_j^n}(\omega)| \right)^2 d\mu(\omega) \\
&\leq m \sum_{j=1}^m \int_\Omega c_j^2 \mathbb{1}_{A_j \Delta B_j^n}(\omega) d\mu(\omega) = m \sum_{j=1}^m c_j^2 \mu(A_j \Delta B_j^n) < \frac{1}{n}, \quad n \in \mathbb{N}
\end{aligned}$$

showing that $\Phi_n \to \Phi$ in $L^2(\Omega, \mathcal{F}, \mu)$. □

Let $T \in \mathbb{R}_+$ be a finite time horizon. In order to have a more convenient notation in the following stochastic Fubini theorem, we introduce the spaces

$$\begin{aligned}
L_T^2(\mu) &:= L_T^2(\mu; \mathbb{R}), \\
L_T^2(\lambda) &:= L_T^2([0, T], \mathcal{B}[0, T], \lambda), \\
L_T^2(\mathbb{P} \otimes \lambda) &:= L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}[0, T], \mathbb{P} \otimes \lambda),
\end{aligned}$$

where $L_T^2(\mu; H)$ for a separable Hilbert space H was defined in (2.2), and

$$L_T^p(\mu \otimes \lambda) := L^p(\Omega \times [0, T] \times E \times [0, T], \mathcal{P}_T \otimes \mathcal{E} \otimes \mathcal{B}[0, T], \mathbb{P} \otimes \lambda \otimes F \otimes \lambda) \tag{A.2}$$

for all $p \geq 1$.

A.2. **Theorem.** *For each $\Phi \in L_T^2(\mu \otimes \lambda)$ we have*

$$\int_0^T \Phi(\cdot, \cdot, s) ds \in L_T^2(\mu), \tag{A.3}$$

there exists $\phi \in L^2_T(\mathbb{P} \otimes \lambda)$ *such that for* λ*-almost all* $s \in [0,T]$

$$\phi(s) = \int_0^T \int_E \Phi(t,x,s)(\mu(dt,dx) - F(dx)dt) \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \tag{A.4}$$

and we have the identity

$$\int_0^T \phi(s)ds = \int_0^T \int_E \bigg(\int_0^T \Phi(t,x,s)ds \bigg) (\mu(dt,dx) - F(dx)dt) \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}). \tag{A.5}$$

Proof. Let $V \subset L^2_T(\mu \otimes \lambda)$ be the vector space

$$V := \operatorname{span}\{Kf \mid K \in L^2_T(\mu), f \in L^2_T(\lambda)\}.$$

Let $\Phi \in V$ be arbitrary. Then there exist $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$, $K_i \in L^2_T(\mu)$, $f_i \in L^2_T(\lambda)$, $i = 1, \ldots, n$ such that $\Phi = \sum_{i=1}^n c_i K_i f_i$. Moreover we have

$$\begin{aligned} \phi &:= \int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \\ &= \sum_{i=1}^n c_i f_i(\cdot) \int_0^T \int_E K_i(t,x)(\mu(dt,dx) - F(dx)dt) \in L^2_T(\mathbb{P} \otimes \lambda), \\ \int_0^T \Phi(\cdot,\cdot,s)ds &= \sum_{i=1}^n c_i K_i(\cdot,\cdot) \int_0^T f_i(s)ds \in L^2_T(\mu) \end{aligned}$$

and identity (A.5) is valid.

For each $\Phi \in L^2_T(\mu \otimes \lambda) \cap L^1_T(\mu \otimes \lambda)$ we have, according to [18, Prop. II.1.14],

$$\begin{aligned} &\int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \\ &= \sum_{n \in \mathbb{N}} \Phi(\tau_n, \beta_{\tau_n}, \cdot) \mathbb{1}_{\{\tau_n \leq T\}} - \int_0^T \int_E \Phi(t,x,\cdot)F(dx)dt, \end{aligned} \tag{A.6}$$

where $(\tau_n)_{n \in \mathbb{N}}$ is a sequence of stopping times and β denotes an E-valued optional process. By the classical Fubini theorem we deduce that the stochastic integral in (A.6) is $\mathcal{F}_T \otimes \mathcal{B}[0,T]$-measurable. Using the Itô-isometry (2.3) we obtain

$$\begin{aligned} &\int_0^T \mathbb{E} \bigg[\bigg(\int_0^T \int_E \Phi(t,x,s)(\mu(dt,dx) - F(dx)dt) \bigg)^2 \bigg] ds \\ &= \int_0^T \mathbb{E} \bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 F(dx)dt \bigg] ds < \infty, \end{aligned}$$

because $\Phi \in L^2_T(\mu \otimes \lambda)$ by hypothesis, and we conclude

$$\int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \in L^2_T(\mathbb{P} \otimes \lambda), \quad \Phi \in L^2_T(\mu \otimes \lambda) \cap L^1_T(\mu \otimes \lambda). \tag{A.7}$$

Now let $\Phi \in L^2_T(\mu \otimes \lambda)$ be arbitrary. By the classical Fubini theorem the integral appearing in (A.3) is $\mathcal{P}_T \otimes \mathcal{E}$-measurable. Hölder's inequality and the hypothesis $\Phi \in L^2_T(\mu \otimes \lambda)$ yield

$$\begin{aligned} &\mathbb{E} \bigg[\int_0^T \int_E \bigg(\int_0^T \Phi(t,x,s)ds \bigg)^2 F(dx)dt \bigg] \\ &\leq T \mathbb{E} \bigg[\int_0^T \int_E \int_0^T |\Phi(t,x,s)|^2 ds F(dx)dt \bigg] < \infty, \end{aligned}$$

and hence (A.3) is valid.

Since the measure F is σ-finite, there exists a sequence $(B_n)_{n\in\mathbb{N}} \subset E$ with $F(B_n) < \infty$, $n \in \mathbb{N}$ and $B_n \uparrow E$. We define

$$\phi_n := \int_0^T \int_E \Phi(t,x,\cdot)\mathbb{1}_{B_n}(x)(\mu(dt,dx) - F(dx)dt), \quad n \in \mathbb{N}.$$

By (A.7) we have $\phi_n \in L_T^2(\mathbb{P} \otimes \lambda)$ for all $n \in \mathbb{N}$. Now, we shall prove that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_T^2(\mathbb{P} \otimes \lambda)$.

Let $\epsilon > 0$ be arbitrary. By Lebesgue's theorem, there exists an index $n_0 \in \mathbb{N}$ such that

$$\int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg]ds < \epsilon, \quad n \geq n_0$$

For all $m > n \geq n_0$ we obtain by the Itô-isometry (2.3)

$$\begin{aligned}\int_0^T \mathbb{E}[|\phi_n(s) - \phi_m(s)|^2]ds &= \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{B_m\setminus B_n}(x)F(dx)dt\bigg]ds \\ &\leq \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg]ds < \epsilon,\end{aligned}$$

establishing that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_T^2(\mathbb{P} \otimes \lambda)$. Thus, there exists $\phi \in L_T^2(\mathbb{P} \otimes \lambda)$ such that $\phi_n \to \phi$ in $L_T^2(\mathbb{P} \otimes \lambda)$. The relation

$$\int_0^T \mathbb{E}[|\phi_n(s) - \phi(s)|^2]ds \to 0 \quad \text{for } n \to \infty$$

implies that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\mathbb{E}[|\phi_{n_k}(s) - \phi(s)|^2] \to 0 \quad \text{for } \lambda\text{-almost all } s \in [0,T],$$

that is $\phi_{n_k}(s) \to \phi(s)$ in $L_T^2(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$. We define

$$\psi := \int_0^T \int_E \Phi(t,x,\cdot)(\mu(dt,dx) - F(dx)dt).$$

By the classical Fubini theorem we have $\Phi(\cdot,\cdot,s) \in L_T^2(\mu)$ for λ-almost all $s \in [0,T]$. The Itô-isometry (2.3) and Lebesgue's theorem yield

$$\mathbb{E}[|\psi(s) - \phi_n(s)|^2] = \mathbb{E}\bigg[\int_0^T \int_E |\Phi(t,x,s)|^2 \mathbb{1}_{E\setminus B_n}(x)F(dx)dt\bigg] \to 0 \quad \text{for } n \to \infty,$$

implying $\phi_n(s) \to \psi(s)$ in $L_T^2(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$. We infer that $\phi(s) = \psi(s)$ in $L_T^2(\Omega, \mathcal{F}_T, \mathbb{P})$ for λ-almost all $s \in [0,T]$, proving (A.4).

According to Lemma A.1 there exists a sequence $(\Phi_n)_{n\in\mathbb{N}} \subset V$ such that $\Phi_n \to \Phi$ in $L_T^2(\mu \otimes \lambda)$. From the beginning of the proof we know that for each $n \in \mathbb{N}$ we have

$$\int_0^T \Phi_n(\cdot,\cdot,s)ds \in L_T^2(\mu), \qquad \int_0^T \int_E \Phi_n(t,x,\cdot)(\mu(dt,dx) - F(dx)dt) \in L_T^2(\mathbb{P} \otimes \lambda)$$

and the identity

$$\begin{aligned}&\int_0^T \bigg(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt)\bigg)ds \\ &= \int_0^T \int_E \bigg(\int_0^T \Phi_n(t,x,s)ds\bigg)(\mu(dt,dx) - F(dx)dt)\end{aligned} \tag{A.8}$$

in $L^2_T(\Omega, \mathcal{F}_T, \mathbb{P})$. By Hölder's inequality, (A.4), the Itô-isometry (2.3) and the convergence $\Phi_n \to \Phi$ in $L^2_T(\mu \otimes \lambda)$ we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt)\right)ds - \int_0^T \phi(s)ds\right)^2\right] \\
&\leq T \int_0^T \mathbb{E}\left[\left(\int_0^T \int_E \Phi_n(t,x,s)(\mu(dt,dx) - F(dx)dt) - \phi(s)\right)^2\right] ds \qquad \text{(A.9)} \\
&= T \int_0^T \mathbb{E}\bigg[\int_0^T \int_E |\Phi_n(t,x,s) - \Phi(t,x,s)|^2 F(dx)dt\bigg] ds \to 0.
\end{aligned}$$

The Itô-isometry (2.3), Hölder's inequality and the convergence $\Phi_n \to \Phi$ in $L^2_T(\mu \otimes \lambda)$ yield

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^T \int_E \left(\int_0^T \Phi_n(t,x,s)ds\right)(\mu(dt,dx) - F(dx)dt) \\
&\quad - \int_0^T \int_E \left(\int_0^T \Phi(t,x,s)ds\right)(\mu(dt,dx) - F(dx)dt)\bigg)^2\bigg] \qquad \text{(A.10)} \\
&= \mathbb{E}\left[\int_0^T \int_E \left(\int_0^T (\Phi_n(t,x,s) - \Phi(t,x,s))ds\right)^2 F(dx)dt\right] \\
&\leq T\mathbb{E}\bigg[\int_0^T \int_E \int_0^T |\Phi_n(t,x,s) - \Phi(t,x,s)|^2 F(dx)dtds\bigg] \to 0.
\end{aligned}$$

Combining (A.8), (A.9) and (A.10) we arrive at (A.5). □

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