FLATNESS OF INVARIANT MANIFOLDS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES

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Abstract. The purpose of this note is to prove that the flatness of an invariant manifold for a semilinear stochastic partial differential equation driven by Lévy processes is at least equal to the number of driving sources with small jumps. We illustrate our findings by means of an example.

1. Introduction

The purpose of this note is to show that an invariant manifold for a semilinear stochastic partial differential equation (SPDE)

$$\left\{ \begin{array}{rcl} dr_t & = & (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \gamma(r_{t-})dX_t \\ r_0 & = & h_0 \end{array} \right. \tag{1.1}$$

in the spirit of [14] driven by Lévy processes with small jumps necessarily has a certain amount of flatness, that is, of linear structure.

A result which is related to the findings of our paper has been provided in [9] for the particular case of Wiener process driven Heath-Jarrow-Morton (HJM, see [10]) interest rate term structure models, namely that under suitable conditions an invariant manifold for the HJM equation necessarily is a foliation, that is, a collection of affine spaces.

In this paper, we deal with general SPDEs of the type (1.1) driven by Lévy processes, and the intuitive statement of our main results (Theorems 2.6 and 2.7) is that the flatness of an invariant manifold is at least equal to the number of driving sources with small jumps.

In order to acquaint the reader with the ideas behind these results, let us present the key concepts and ideas of the proof in an informal way. Denoting by H the state space of the SPDE (1.1), which we assume to be a separable Hilbert space, and by $\mathcal{M}$ be a finite dimensional submanifold of H, we have the following concepts, which are explained in more detail in Section 2 and Appendix A:

- We call $\mathcal{M}$ invariant for the SPDE (1.1) if for each starting point $h_0 \in \mathcal{M}$ the mild solution to (1.1) with $r_0 = h_0$ stays on the manifold.
- For a point $h_0 \in \mathcal{M}$ the flatness of $\mathcal{M}$ at h_0 is the largest integer d such that some d-dimensional subspace $\mathcal{L} \subset H$ is contained simultaneously in all tangent spaces of the manifold $\mathcal{M}$ locally around h_0.
- Then, the flatness of $\mathcal{M}$ is defined as the minimum over all these local quantities.

As already indicated, throughout this paper we will assume that $\mathcal{M}$ is an invariant manifold. The volatility $\gamma = (\gamma^k)_{k \in K}$, where $K = \{1, \ldots, q\}$ with q denoting the

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dimension of the Lévy process X, consists of mappings $\gamma^k : H \to H$, $k \in K$. In order to exemplify the ideas behind our result, we assume (for the sake of simplicity) that for each $k \in K$ the Lévy process X^k makes arbitrary small positive jumps. Then, for each $h \in \mathcal{M}$ the flatness of $\mathcal{M}$ at h is of the stated size, and the proof is divided into two steps:

- For an arbitrary $k \in K$ the volatility $\gamma^k(h)$ belongs to the tangent space to $\mathcal{M}$ at h. Indeed, since the manifold $\mathcal{M}$ is invariant, it captures every possible jump of X^k. Since, in addition, the Lévy process X^k makes arbitrary small positive jumps, this means that for some $\epsilon > 0$ we have
$$h + x_k \gamma^k(h) \in \mathcal{M} \quad \text{for all } x_k \in [0, \epsilon].$$
In other words, the line segment $\{h + x_k \gamma^k(h) : x_k \in [0, \epsilon]\}$ is contained in the manifold $\mathcal{M}$. From an intuitive point of view, it is clear that this implies that $\gamma^k(h)$ belongs to the tangent space to $\mathcal{M}$ at h. We refer to Proposition 2.5 for the precise formulation of this statement and its proof.
- Due to the previous step, the linear space $\mathcal{L}$ generated by all $\gamma^k(h)$, $k \in K$ is contained in the tangent space to $\mathcal{M}$ at h, which provides the desired result concerning the flatness of the manifold.

The remainder of this note is organized as follows. In Section 2 we provide the general framework and present our main results. In Section 3 we illustrate our main results by means of an example; namely we apply our results to the Hull-White extension of the Vasiček model from interest rate theory. For convenience of the reader, in Appendix A we provide the crucial definitions and results regarding submanifolds in Hilbert spaces.

2. Flatness of invariant manifolds

In this section, we present our main results concerning the flatness of invariant manifolds for SPDEs driven by Lévy processes.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a filtered probability space with right-continuous filtration. Let W be a p-dimensional Wiener standard process for some $p \in \mathbb{N}_0$, and let X be an q-dimensional Lévy process for some $q \in \mathbb{N}$, which we assume to be a purely discontinuous martingale with canonical representation $X = x * (\mu^X - \nu)$ in the sense of [11, Cor. II.2.38]. Here μ^X denotes the random measure associated to the jumps of X, which is a homogeneous Poisson random measure, and ν denotes its compensator, which is given by $\nu(dt, dx) = dt \otimes F(dx)$ with F denoting the Lévy measure of X. We assume that $X^1, \ldots, X^q$ are independent, which implies that the Lévy measure F is given by

$$F(B) = \sum_{k=1}^{q} \int_{\mathbb{R}} \mathbb{1}_B(x e_k) F^k(dx), \quad B \in \mathcal{B}(\mathbb{R}^q) \tag{2.1}$$

with $e_1, \ldots, e_q$ denoting the unit vectors in $\mathbb{R}^q$, and with F^k denoting the Lévy measure of X^k for $k = 1, \ldots, q$. We assume that

$$\int_{\mathbb{R}} \big(|x|^2 \vee |x|^4 \big) F^k(dx) < \infty \quad \text{for all } k = 1, \ldots, q. \tag{2.2}$$

The following definition identifies the set of all indices such that the corresponding Lévy process makes "small jumps".

2.1. **Definition.** *We denote by K be the set of all indices $k \in \{1, \ldots, q\}$ such that for some $\epsilon > 0$ we have $[0, \epsilon] \subset \operatorname{supp}(F^k)$ or $[-\epsilon, 0] \subset \operatorname{supp}(F^k)$.*

Let H be a separable Hilbert space and let $A : \mathcal{D}(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup $(S_t)_{t \geq 0}$ on H. Furthermore, let $\alpha : H \to H$,

$\sigma : H \to H^p$ and $\gamma : H \to H^q$ be Lipschitz continuous mappings such that $\sigma^j \in C^1(H)$ for all $j = 1, \dots, p$. We suppose that the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive, that is

$$\|S_t\| \leq e^{\beta t}, \quad t \geq 0$$

for some constant $\beta \in \mathbb{R}$. Then, for each $h_0 \in H$ there exists a unique mild solution to the SPDE (1.1), that is, an adapted càdlàg process $r = r^{(h_0)}$ such that

$$\begin{aligned} r_t = S_t h_0 + \int_0^t S_{t-s}\alpha(r_s)ds + \sum_{j=1}^p \int_0^t S_{t-s}\sigma^j(r_s)dW_s^j \\ + \sum_{k=1}^q \int_0^t S_{t-s}\gamma^k(r_{s-})dX_s^k, \quad t \in \mathbb{R}_+, \end{aligned}$$

see, for example, [14], [13] or [7]. For what follows, let $\mathcal{M}$ be a finite dimensional C^3-submanifold of H, which we assume to be closed as a subset of H. We refer to Appendix A for details about submanifolds in Hilbert spaces.

2.2. **Definition.** *The submanifold $\mathcal{M}$ is called* invariant *for (1.1) if for all $h_0 \in \mathcal{M}$ we have $r \in \mathcal{M}$ up to an evanescent set*[1]*, where $r = r^{(h_0)}$ denotes the mild solution to (1.1) with $r_0 = h_0$.*

2.3. **Remark.** *As our first step in order to analyze the flatness of invariant manifolds, we will write the SPDE (1.1) as the SPDE (2.4) below, and apply* [8, Thm. 2.8]*. Let us emphasize those of our previous assumptions, which we have exclusively made for an application of this result:*

- *We assume the integrability condition (2.2), which ensures that condition (2.5) from* [8] *holds true.*
- *We assume that $\mathcal{M}$ is a C^3-submanifold of H, and that it is closed as a subset of H. This assumption is also required for the mentioned result from* [8].

In the sequel, we also assume that the index set K, which identifies all Lévy processes with "small jumps", is nonempty. Otherwise, no statement concerning the flatness of $\mathcal{M}$ is possible, as the following counterexample shows:

2.4. **Example.** *We consider the SPDE*

$$\begin{cases} dr_t &= \gamma(r_{t-})dN_t \\ r_0 &= h_0 \end{cases} \tag{2.3}$$

on the state space $H = \mathbb{R}^2$, which – after rewriting – is of the form (1.1). Here N is a Poisson process, and the volatility $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\gamma(h) = (1, 0)$ for all $h \in \mathbb{R}^2$. Then the one-dimensional submanifold

$$\mathcal{M} = \{(\xi, \sin(2\pi\xi)) : \xi \in \mathbb{R}\}$$

is invariant for (2.3), which follows from [8, Thm. 2.11]*, but we have* $\operatorname{fl}\mathcal{M}(h_0) = 0$ *for all $h_0 \in \mathcal{M}$, showing that the flatness of $\mathcal{M}$ is zero.*

The following result shows that in case of invariance all volatilities associated to Lévy processes with "small jumps" are tangential to the submanifold.

2.5. **Proposition.** *Suppose that the submanifold $\mathcal{M}$ is invariant for (1.1). Then we have*

$$\gamma^k(h) \in T_h\mathcal{M} \quad \textit{for all } k \in K \textit{ and all } h \in \mathcal{M}.$$

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[1] A random set $A \subset \Omega \times \mathbb{R}_+$ is called *evanescent* if the set $\{\omega \in \Omega : (\omega, t) \in A \text{ for some } t \in \mathbb{R}_+\}$ is a $\mathbb{P}$-nullset, cf. [11, 1.1.10].

Proof. We can write the SPDE (1.1) as

(2.4)
$$\begin{cases} dr_t & = & (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_{\mathbb{R}^q} \delta(r_{t-}, x)(\mu^X(dt,dx) - F(dx)dt) \\ r_0 & = & h_0, \end{cases}$$

where $\delta : H \times \mathbb{R}^q \to H$ is given by

$$\delta(h,x) = \sum_{k=1}^q x_k \gamma^k(h), \quad (h,x) \in H \times \mathbb{R}^q.$$

In view of (2.2), all assumptions of [8, Thm. 2.8] are fulfilled, and together with (2.1), for each $k = 1, \ldots, q$ we obtain

$$h + x_k \gamma^k(h) \in \mathcal{M} \quad \text{for all } h \in \mathcal{M} \text{ and all } x_k \in \operatorname{supp}(F^k). \tag{2.5}$$

Now, let $k \in K$ and $h_0 \in \mathcal{M}$ be arbitrary, and let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $T_{h_0}\mathcal{M}$. According to [5, Lemma 6.1.2] there exists a parametrization $\phi : V \subset \mathbb{R}^m \to U \cap \mathcal{M}$ around h_0 such that

$$\phi(\langle e, h \rangle) = h \quad \text{for all } h \in U \cap \mathcal{M}, \tag{2.6}$$

where we use the notation $\langle e, h \rangle := (\langle e_1, h \rangle, \ldots, \langle e_m, h \rangle)$. In view of Definition 2.1 we may assume, without loss of generality, that $[0, \epsilon] \subset \operatorname{supp}(F^k)$ for some $\epsilon > 0$. By (2.5), and since U is an open neighborhood of h_0, we obtain, after reducing $\epsilon > 0$ if necessary, that

$$h_0 + t\gamma^k(h_0) \in U \cap \mathcal{M} \quad \text{for all } t \in [0, \epsilon]. \tag{2.7}$$

Setting $y_0 := \langle e, h_0 \rangle$, by taking into account (2.7) and (2.6) we get

$$\begin{aligned} \gamma^k(h_0) &= \frac{\partial}{\partial x_k}(h_0 + x_k \gamma^k(h_0))|_{x=0} = \lim_{t \to 0} \frac{h_0 + t\gamma^k(h_0) - h_0}{t} \\ &= \lim_{t \to 0} \frac{\phi(y_0 + t\langle e, \gamma^k(h_0) \rangle) - \phi(y_0)}{t} = D\phi(y_0)\langle e, \gamma^k(h_0) \rangle \in T_{h_0}\mathcal{M}, \end{aligned}$$

finishing the proof. □

Now, we are ready to present our main results concerning the flatness of invariant manifolds.

2.6. **Theorem.** *Suppose that the submanifold $\mathcal{M}$ is invariant for (1.1). Suppose there exists $d \in \mathbb{N}_0$ such that for each $h_0 \in \mathcal{M}$ we have*

$$d \leq \dim \bigcap_{h \in U \cap \mathcal{M}} \langle \gamma^k(h) : k \in K \rangle \tag{2.8}$$

for some open neighborhood $U \subset H$ of h_0.

(1) *Then, for each $h_0 \in \mathcal{M}$ the following statements are true:*
   (a) *We have $\operatorname{fl} \mathcal{M}(h_0) \geq d$.*
   (b) *There exist an open neighborhood $U_0 \subset H$ of h_0, a d-dimensional subspace $\mathcal{L} \subset H$ and a finite dimensional C^3-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = \dim \mathcal{M} - d$ such that $U_0 \cap \mathcal{M} = U_0 \cap (\mathcal{N} \oplus \mathcal{L})$.*
   (c) *If $d = \dim \mathcal{M}$, then $\mathcal{M}$ is a local affine space generated by $\mathcal{L}$ around h_0.*
   (d) *If $d = \dim \mathcal{M} - 1$, then $\mathcal{M}$ is a local foliation generated by $\mathcal{L}$ around h_0.*

(2) *If, furthermore, the submanifold $\mathcal{M}$ is connected as a topological subspace of H, and we have $\operatorname{fl} \mathcal{M}(h_0) = d$ for each $h_0 \in \mathcal{M}$, then the following statements are true:*
   (a) *We have $\operatorname{fl} \mathcal{M} = d$.*

(b) *There exist a d-dimensional subspace $\mathcal{L} \subset H$ and a finite dimensional C^3-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = \dim \mathcal{M} - d$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$.*
(c) *If $d = \dim \mathcal{M}$, then $\mathcal{M}$ is an affine space generated by $\mathcal{L}$.*
(d) *If $d = \dim \mathcal{M} - 1$, then $\mathcal{M}$ is a foliation generated by $\mathcal{L}$.*

Proof. Let $h_0 \in \mathcal{M}$ be arbitrary. By assumption, there exists a d-dimensional subspace $\mathcal{L}_{h_0}$ such that

$$\mathcal{L}_{h_0} \subset \bigcap_{h \in U \cap \mathcal{M}} \langle \gamma^k(h) : k \in K \rangle,$$

and hence, by Proposition 2.5 we obtain

$$\mathcal{L}_{h_0} \subset T_h \mathcal{M} \quad \text{for all } h \in U \cap \mathcal{M}.$$

Therefore, Proposition A.7 proves the first statement, and the second statement follows from Proposition A.9. □

Theorem 2.6 shows that under condition (2.8) on the volatilities $(\gamma^k)_{k \in K}$ invariance of the submanifold implies the inequality $\operatorname{fl} \mathcal{M}(h_0) \geq d$ concerning its flatness. Roughly speaking, this means that the flatness of the submanifold is at least equal to the number of driving sources with small jumps. Furthermore, the submanifold admits locally a direct sum decomposition into another manifold and a d-dimensional linear space. If the submanifold $\mathcal{M}$ is connected and we even have equality in $\operatorname{fl} \mathcal{M}(h_0) \geq d$, then the direct sum decomposition holds globally. The following Theorem 2.7 presents another condition, namely (2.9), on the volatilities $(\gamma^k)_{k \in K}$ under which such a global direct sum decomposition of the manifold holds true.

2.7. **Theorem.** *Suppose that the submanifold $\mathcal{M}$ is invariant for (1.1), and let $d \in \mathbb{N}_0$ be such that*

$$d \leq \dim \bigcap_{h \in \mathcal{M}} \langle \gamma^k(h) : k \in K \rangle. \tag{2.9}$$

Then the following statements are true:

(1) *We have $\operatorname{fl} \mathcal{M} \geq d$.*
(2) *There exist a d-dimensional subspace $\mathcal{L} \subset H$ and a finite dimensional C^3-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = \dim \mathcal{M} - d$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$.*
(3) *If $d = \dim \mathcal{M}$, then $\mathcal{M}$ is an affine space generated by $\mathcal{L}$.*
(4) *If $d = \dim \mathcal{M} - 1$, then $\mathcal{M}$ is a foliation generated by $\mathcal{L}$.*

Proof. By assumption, there exists a d-dimensional subspace $\mathcal{L}$ such that

$$\mathcal{L} \subset \bigcap_{h \in \mathcal{M}} \langle \gamma^k(h) : k \in K \rangle,$$

and hence, by Proposition 2.5 we obtain

$$\mathcal{L} \subset T_h \mathcal{M} \quad \text{for all } h \in \mathcal{M}.$$

Therefore, Proposition A.8 concludes the proof. □

3. An example: The Lévy driven Hull-White extension of the Vasiček model

For the sake of illustration of our previous results, we present an example from mathematical finance, which concerns the modeling of interest rate curves, namely the Lévy driven Hull-White extension of the Vasiček model, which is an example of the so-called HJMM (Heath-Jarrow-Morton-Musiela) equation

$$\begin{cases} dr_t &= (\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t))dt + \gamma(r_{t-})dX_t \\ r_0 &= h_0. \end{cases} \tag{3.1}$$

Here the state space is a suitable Hilbert space H consisting of functions $h : \mathbb{R}_+ \to \mathbb{R}$ (see, for example, [5, Sec. 5]), and $\frac{d}{d\xi}$ is the differential operator, which is generated by the translation semigroup on H. We refer, e.g., to [6, 4, 15, 12] for the Lévy driven HJMM equation. In this section, we assume that the Lévy process is one-dimensional and has the canonical representation $X = W + x * (\mu^X - \nu)$ with a standard Wiener process W such that for some $\epsilon > 0$ we have $[0, \epsilon] \subset \mathrm{supp}(F)$ or $[-\epsilon, 0] \subset \mathrm{supp}(F)$, where F denotes the Lévy measure of X. For the Hull-White extension of the Vasiček model the volatility $\gamma : H \to H$ is constant, that is $\gamma(h_1) = \gamma(h_2)$ for all $h_1, h_2 \in H$. Therefore, and since H consists of functions mapping $\mathbb{R}_+$ to $\mathbb{R}$, we agree to write $\gamma(\xi)$ instead of $(\gamma(h))(\xi)$ for $\xi \in \mathbb{R}_+$. With this convention, the volatility $\gamma \in H$ is given by

$$\gamma(\xi) = \rho \cdot \exp(-c\xi), \quad \xi \in \mathbb{R}_+$$

with constants $\rho \neq 0$ and $c \in \mathbb{R}$. The drift $\alpha_{\mathrm{HJM}} \in H$ is constant as well, and it is given by the HJM drift condition

$$\alpha_{\mathrm{HJM}} = -\gamma \cdot \Psi'\bigg(- \int_0^{\bullet} \gamma(\xi) d\xi \bigg),$$

where Ψ denotes the cumulant generating function of the Lévy process X. Now, let $\mathcal{M}$ be a two-dimensional submanifold, which is invariant for (3.1). Then, according to Theorem 2.7 the submanifold $\mathcal{M}$ is a foliation generated by $\mathcal{L} = \langle \xi \mapsto \exp(-c\xi) \rangle$. Consequently, for the Lévy driven Hull-White extension of the Vasiček model with small jumps, every invariant manifold must necessarily be a foliation. It is well-known that, conversely, the Hull-White extension of the Vasiček model admits a two-dimensional realization, that is, for every $h_0 \in \mathcal{D}(d/d\xi)$ there exists a two-dimensional invariant manifold with $h_0 \in \mathcal{M}$, where the invariant manifolds are foliations generated by $\mathcal{L}$. For the Lévy driven case, we refer, for example, to [17].

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Appendix A. Finite dimensional submanifolds in Hilbert spaces

In this appendix, we provide the required results about finite dimensional submanifolds in Hilbert spaces. Let H be a Hilbert space and let $k, m \in \mathbb{N}$ be positive integers.

A.1. **Definition.** *A nonempty subset $\mathcal{M} \subset H$ is a* m-dimensional C^k-submanifold *of H, if for all $h_0 \in \mathcal{M}$ there exist an open neighborhood $U \subset H$ of h_0, an open subset $V \subset \mathbb{R}^m$ and a map $\phi \in C^k(V; H)$ such that*

1. *$\phi : V \to U \cap \mathcal{M}$ is a homeomorphism;*
2. *$D\phi(y)$ is one to one for all $y \in V$.*

The map ϕ is called a parametrization *of $\mathcal{M}$ around h_0.*

For what follows, let $\mathcal{M}$ be a m-dimensional C^k-submanifold of H. For the purpose of this paper, we require the notion of the flatness of $\mathcal{M}$, which is defined as follows.

A.2. **Definition.** *For $h_0 \in \mathcal{M}$ we define the* flatness *of $\mathcal{M}$ at h_0, denoted by $\operatorname{fl} \mathcal{M}(h_0)$, as the largest integer $d \in \{0, \dots, m\}$ such that for some d-dimensional subspace $\mathcal{L} \subset H$ and some open neighborhood U of h_0 we have*

$$\mathcal{L} \subset T_h \mathcal{M} \quad \text{for all } h \in U \cap \mathcal{M}.$$

A.3. **Definition.** *We call $\operatorname{fl} \mathcal{M} := \min_{h \in \mathcal{M}} \operatorname{fl} \mathcal{M}(h)$ the* flatness *of $\mathcal{M}$.*

A.4. **Remark.** *A similar notion, which also measures the amount of flatness of a manifold, is the rank, which is defined for complete Riemannian manifolds. We refer, for example, to* [3], [2] *or* [16] *for the precise definition.*

A.5. **Definition.** *Let $\mathcal{L} \subset H$ be a finite dimensional subspace.*

(1) *$\mathcal{M}$ is an* affine space *generated by $\mathcal{L}$ if there exists an element $g_0 \in \mathcal{L}^\perp$ such that $\mathcal{M} = g_0 \oplus \mathcal{L}$.*
(2) *$\mathcal{M}$ is a* foliation *generated by $\mathcal{L}$ if there exists a one-dimensional C^k-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$.*

A.6. **Definition.** *Let $\mathcal{L} \subset H$ be a finite dimensional subspace, and let $h_0 \in \mathcal{M}$ be arbitrary.*

(1) *$\mathcal{M}$ is a* local affine space *generated by $\mathcal{L}$ around h_0 if there exist an open neighborhood U of h_0 and an element $g_0 \in \mathcal{L}^\perp$ such that $U \cap \mathcal{M} = U \cap (g_0 \oplus \mathcal{L})$.*
(2) *$\mathcal{M}$ is a* local foliation *generated by $\mathcal{L}$ around h_0 if there exist an open neighborhood U of h_0 and a one-dimensional C^k-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ such that $U \cap \mathcal{M} = U \cap (\mathcal{N} \oplus \mathcal{L})$.*

A.7. **Proposition.** *Let $h_0 \in \mathcal{M}$ be arbitrary, let $\mathcal{L} \subset H$ be a subspace and let $U \subset H$ be an open neighborhood of h_0 such that*

$$\mathcal{L} \subset T_h \mathcal{M} \quad \text{for all } h \in U \cap \mathcal{M}. \tag{A.1}$$

Then, denoting by $h_0 = h_1 + h_2$ the direct sum decomposition of h_0 according to $H = \mathcal{L}^\perp \oplus \mathcal{L}$, there exist open neighborhoods $U_1 \subset \mathcal{L}^\perp$ of h_1 and $U_2 \subset \mathcal{L}$ of h_2 such that $U_0 := U_1 \oplus U_2$ is an open neighborhood of h_0 satisfying the following conditions:

(1) *We have $U_0 \cap \mathcal{M} = U_0 \cap ((U_0 \cap \mathcal{M}) + \mathcal{L})$.*
(2) *The subset $\mathcal{N} := U_1 \cap \Pi_{\mathcal{L}^\perp} \mathcal{M}$ is a C^k-submanifold of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = \dim \mathcal{M} - \dim \mathcal{L}$, and we have $U_0 \cap \mathcal{M} = U_0 \cap (\mathcal{N} \oplus \mathcal{L})$.*

Proof. Setting $p := \dim \mathcal{L}$, there exists an orthonormal basis $\{e_1, \dots, e_m\}$ of $T_{h_0}\mathcal{M}$ such that $\{e_1, \dots, e_p\}$ is an orthonormal basis of $\mathcal{L}$. According to [5, Lemma 6.1.2] there exists a parametrization $\phi : V' \subset \mathbb{R}^m \to U' \cap \mathcal{M}$ around h_0 with $U' \subset U$ such that

$$\phi(\langle e, h \rangle) = h \quad \text{for all } h \in U' \cap \mathcal{M}, \tag{A.2}$$

where we use notation $\langle e, h \rangle := (\langle e_1, h \rangle, \dots, \langle e_m, h \rangle) \in \mathbb{R}^m$. Since $U' \subset H$ is an open neighborhood of h_0, there exist open neighborhoods $U_1' \subset \mathcal{L}^\perp$ of h_1 and $U_2' \subset \mathcal{L}$ of h_2 such that $U_1' \oplus U_2' \subset U'$. By (A.2) we have

$$\phi^{-1}(U_1' \cap \mathcal{M}) \subset \mathbb{R}^{m-p} \quad \text{and} \quad \phi^{-1}(U_2' \cap \mathcal{M}) \subset \mathbb{R}^p \tag{A.3}$$

with respect to the direct sum decomposition $\mathbb{R}^m = \mathbb{R}^{m-p} \oplus \mathbb{R}^p$. Since V' is open in $\mathbb{R}^m$, there are open subsets $V_1 \subset \mathbb{R}^{m-p}$ and $V_2 \subset \mathbb{R}^p$ such that $V_0 \subset V'$, where $V_0 := V_1 \oplus V_2$. Since ϕ is a homeomorphism, there exists an open neighborhood U_0' of h_0 such that $\phi(V_0) = U_0' \cap \mathcal{M}$. By (A.3) there exist open neighborhoods $U_1 \subset \mathcal{L}^\perp$ of h_1 and $\widetilde{U}_2 \subset \mathcal{L}$ of h_2 such that $(U_1 \oplus \widetilde{U}_2) \cap \mathcal{M} = U_0' \cap \mathcal{M}$. Setting $\mathcal{N} := U_1 \cap \Pi_{\mathcal{L}^\perp} \mathcal{M}$, $U_2 := \widetilde{U}_2 \cap \Pi_{\mathcal{L}} \mathcal{M}$ and $U_0 := U_1 \oplus U_2$, we have $\Pi_{\mathcal{L}} U_0 = U_2$ and

$$\phi(V_0) = U_0 \cap \mathcal{M} = \mathcal{N} \oplus U_2,$$

and it follows that

$$U_0 \cap \mathcal{M} = U_0 \cap (\mathcal{N} \oplus U_2) \subset U_0 \cap (\mathcal{N} \oplus \mathcal{L}). \tag{A.4}$$

Defining the mappings $\phi_1 := \phi|_{V_1}$ and $\phi_2 := \phi|_{V_2}$, we obtain:

- $\phi_1 \in C^k(V_1; \mathcal{L}^\perp)$ and $\phi_2 \in C^k(V_2; \mathcal{L})$, because $\phi \in C^k(V_0; H)$.
- $\phi_1 : V_1 \to \mathcal{N}$ and $\phi_2 : V_2 \to U_2$ are homeomorphisms, because $\phi : V_0 \to \mathcal{N} \oplus U_2$ is a homeomorphism.
- For all $y_1 \in V_1$ and $y_2 \in V_2$ the mappings $D\phi_1(y_1)$ and $D\phi_2(y_2)$ are one to one, because
$$D\phi(y_1 + y_2) = D\phi_1(y_1) + D\phi_2(y_2)$$
is one to one.

Therefore, $\mathcal{N}$ is a $(m-p)$-dimensional submanifold of $\mathcal{L}^\perp$ with parametrization ϕ_1, and U_2 is a p-dimensional submanifold of $\mathcal{L}$ with parametrization ϕ_2. Furthermore, by (A.2) there is an isomorphism $T : \mathbb{R}^p \to \mathcal{L}$ such that $\phi_2 = T|_{V_2}$, and hence, we have

$$\phi(y_1 + y_2) = \phi_1(y_1) + Ty_2 \quad \text{for all } y_1 \in V_1 \text{ and } y_2 \in V_2.$$

Now, we will show that

$$U_0 \cap ((U_0 \cap \mathcal{M}) + \mathcal{L}) \subset U_0 \cap \mathcal{M}. \tag{A.5}$$

Indeed, let $h \in U_0 \cap \mathcal{M}$ and $g \in \mathcal{L}$ be such that $h + g \in U_0$. Then there exist unique $y_1 \in V_1$, $y_2 \in V_2$ and $z_2 \in \mathbb{R}^p$ such that $h = \phi_1(y_1) + Ty_2$ and $g = Tz_2$, and we obtain

$$h + g = \phi_1(y_1) + T(y_2 + z_2).$$

Since $h + g \in U_0$ and $\Pi_{\mathcal{L}} U_0 = U_2$, we have $T(y_2 + z_2) \in U_2$. Therefore, and since $T : \mathbb{R}^p \to \mathcal{L}$ is an isomorphism, we obtain $y_2 + z_2 \in V_2$, and hence

$$h + g = \phi(y_1 + (y_2 + z_2)) \in U_0 \cap \mathcal{M},$$

proving (A.5). In order to prove the converse inclusion of (A.4), let $h \in \mathcal{N}$ and $g \in \mathcal{L}$ be such that $h + g \in U_0$. There exists $f \in \mathcal{L}$ such that $h + f \in U_0 \cap \mathcal{M}$. Thus, we have $h + g = (h + f) + (g - f) \in U_0 \cap \mathcal{M} + \mathcal{L}$. Since $h + g \in U_0$, by (A.5) we obtain $h + g \in U_0 \cap \mathcal{M}$, completing the proof. □

A.8. **Proposition.** *Suppose that $\mathcal{M}$ is closed as a subset of H, and let $\mathcal{L} \subset H$ be a subspace such that*

$$\mathcal{L} \subset T_h \mathcal{M} \quad \textit{for all } h \in \mathcal{M}. \tag{A.6}$$

Then the following statements are true:

1. *We have $\mathcal{M} = \mathcal{M} + \mathcal{L}$.*
2. *The subset $\mathcal{N} := \Pi_{\mathcal{L}^\perp} \mathcal{M}$ is a C^k-submanifold of $\mathcal{L}^\perp$ with* $\dim \mathcal{N} = \dim \mathcal{M} - \dim \mathcal{L}$, *and we have $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$.*

Proof. In order to prove $\mathcal{M} + \mathcal{L} \subset \mathcal{M}$, let $h \in \mathcal{M}$ and $g \in \mathcal{L}$ be arbitrary, and suppose that $h + g \notin \mathcal{M}$. We define $t \in [0, 1]$ as

$$t := \inf\{s \in [0, 1] : h + sg \notin \mathcal{M}\},$$

and set $h_0 := h + tg$. Since $\mathcal{M}$ is closed as a subset of H, we have $h_0 \in \mathcal{M}$, which implies $t < 1$. Furthermore, there exists a sequence $(s_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $s_n \to 0$ such that $h_0 + s_n g \notin \mathcal{M}$ for all $n \in \mathbb{N}$. By Proposition A.7 there exists an open neighborhood U of h_0 such that

$$U \cap \mathcal{M} = U \cap ((U \cap \mathcal{M}) + \mathcal{L}),$$

which contradicts $h_0 + s_n g \notin \mathcal{M}$ for all $n \in \mathbb{N}$, establishing the first statement.

According to Proposition A.7, the subset $\mathcal{N} := \Pi_{\mathcal{L}^\perp}\mathcal{M}$ is a C^k-submanifold of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = \dim \mathcal{M} - \dim \mathcal{L}$. Furthermore, we have $\mathcal{M} \subset \mathcal{N} \oplus \mathcal{L}$. In order to prove the converse inclusion $\mathcal{N} \oplus \mathcal{L} \subset \mathcal{M}$, let $h \in \mathcal{N}$ and $g \in \mathcal{L}$ be arbitrary. There exists $f \in \mathcal{L}$ such that $h + f \in \mathcal{M}$. Thus, we have $h+g = (h+f)+(g-f) \in \mathcal{M}+\mathcal{L}$, and we obtain $\mathcal{N} \oplus \mathcal{L} \subset \mathcal{M} + \mathcal{L} = \mathcal{M}$, establishing the second statement. □

A.9. **Proposition.** *Suppose that the submanifold $\mathcal{M}$ is connected as a topological subspace of H, and let $d \in \mathbb{N}_0$ be such that $\operatorname{fl}\mathcal{M}(h_0) = d$ for each $h_0 \in \mathcal{M}$. Then there exist a subspace $\mathcal{L} \subset H$ with $\dim \mathcal{L} = d$ and a finite dimensional C^k-submanifold $\mathcal{N}$ of $\mathcal{L}^\perp$ with $\dim \mathcal{N} = m - d$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$.*

Proof. For each $h_0 \in \mathcal{M}$ there exist a d-dimensional subspace $\mathcal{L}_{h_0} \subset H$ and an open neighborhood U_{h_0} of h_0 such that

$$\mathcal{L}_{h_0} \subset T_h\mathcal{M} \quad \text{for all } h \in U_{h_0} \cap \mathcal{M}. \tag{A.7}$$

We will show that

$$\mathcal{L}_{g_0} = \mathcal{L}_{h_0} \quad \text{for all } g_0, h_0 \in \mathcal{M}. \tag{A.8}$$

Let $g_0, h_0 \in \mathcal{M}$ be arbitrary. Since the submanifold $\mathcal{M}$ is locally path-connected and connected, it is even path-connected, see, for example, [1, Prop. 1.6.7]. Thus, there exists a continuous function $f : I \to \mathcal{M}$ with $f(0) = g_0$ and $f(1) = h_0$, where $I = [0,1]$. Since the graph $f(I) \subset \mathcal{M}$ is compact, there exist an integer $n \in \mathbb{N}$ and elements $g_1, \ldots, g_n \in f(I)$ with $g_n = h_0$ such that

$$f(I) = f(I) \cap \left(\bigcup_{k=0}^n U_{g_k} \right).$$

We define an integer $e \in \{1, \ldots, n\}$, elements $0 = t_0 < \ldots < t_e \leq 1$ and pairwise different $\pi(0), \ldots, \pi(e) \in \{0, \ldots, n\}$ with $\pi(0) = 0$, $\pi(e) = n$ and $f(t_k) \in \bigcup_{i=0}^k U_{g_{\pi(i)}}$, $f(t_k) \notin \bigcup_{i=0}^{k-1} U_{g_{\pi(i)}}$ for $k = 0, \ldots, e$ inductively as follows:

- We set $t_0 := 0$ and $\pi(0) := 0$.
- For the induction step $k \to k+1$ let $k \in \{0, \ldots, n-1\}$ be arbitrary.
  - If $f(t_k) \in U_{h_0}$, then we set $e := k$.
  - Otherwise, we define $t_{k+1} \in [t_k, 1]$ as

$$t_{k+1} := \inf \left\{ t \in [t_k, 1] : f(t) \notin \bigcup_{i=0}^k U_{g_{\pi(i)}} \right\}. \tag{A.9}$$

    By the continuity of f we have

$$t_{k+1} > t_k \quad \text{and} \quad f(t_{k+1}) \notin \bigcup_{i=0}^k U_{g_{\pi(i)}}.$$

    Thus, there exists an index $l \in \{1, \ldots, n\}$ with $l \notin \{\pi(1), \ldots, \pi(k)\}$ such that $f(t_{k+1}) \in U_{g_l}$. We set $\pi(k+1) := l$.

Now, by induction we prove that

$$\mathcal{L}_{g_{\pi(0)}} = \mathcal{L}_{g_{\pi(k)}} \quad \text{for all } k = 0, \ldots, e. \tag{A.10}$$

For the induction step $k \to k+1$, by the definition (A.9) of t_{k+1} we have

$$f(s) \in \bigcup_{i=0}^k U_{g_{\pi(i)}} \quad \text{for all } s \in [t_k, t_{k+1}).$$

Moreover, by the continuity of f there exists $\delta > 0$ with $t_k < t_{k+1} - \delta$ such that

$$f(s) \in U_{g_{\pi(k+1)}} \quad \text{for all } s \in (t_{k+1} - \delta, t_{k+1}].$$

Therefore, we obtain

$$f(s) \in \bigcup_{i=0}^{k} \big(U_{g_{\pi(i)}} \cap U_{g_{\pi(k+1)}} \big) \quad \text{for all } s \in (t_{k+1} - \delta, t_{k+1}).$$

Hence, there exist $i \in \{0, \dots, k\}$ and $s \in (t_{k+1} - \delta, t_{k+1})$ such that $U := U_{g_{\pi(i)}} \cap U_{g_{\pi(k+1)}}$ is an open neighborhood of $f(s)$. By (A.7) we obtain

$$\mathcal{L}_{g_{\pi(i)}} + \mathcal{L}_{g_{\pi(k+1)}} \subset T_h \mathcal{M} \quad \text{for all } h \in U \cap \mathcal{M}.$$

Since $\operatorname{fl} \mathcal{M}(f(s)) = d$, we deduce that $\mathcal{L}_{g_{\pi(i)}} = \mathcal{L}_{g_{\pi(k+1)}}$, which completes the induction step, and establishes (A.10), whence we arrive at (A.8). Therefore, and by (A.7) there exists a d-dimensional subspace $\mathcal{L}$ such that (A.6) is fulfilled. Consequently, applying Proposition A.8 finishes the proof. □

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