ISOMORPHISMS FOR SPACES OF PREDICTABLE PROCESSES AND AN EXTENSION OF THE ITÔ INTEGRAL

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Abstract. Our goal of this note is to give an easy proof that spaces of predictable processes with values in a Banach space are isomorphic to spaces of progressive resp. adapted, measurable processes. This provides a straightforward extension of the Itô integral in infinite dimensions. We also outline an application to stochastic partial differential equations.

1. Introduction

The Itô integral for predictable processes is well-established in the literature, see, e.g., [11, 20]. For non-predictable integrands, there is also an integration theory if the driving process is a continuous semimartingale, see, e.g., [12, 21, 9], and some references, such as [23, 13, 22], also consider the situation where the driving noise has jumps.

Another approach to stochastic integration has been presented in [7, 8]. Here the connection to the usual Itô integral was established in [24], see also Appendix B in [5].

The goal of the note is to give an easy proof that spaces of predictable processes with values in a Banach space are isomorphic to spaces of progressive resp. adapted, measurable processes, which provides a straightforward extension of the Itô integral for Banach space valued processes. We also compute the inverse of the embedding operator of these spaces in particular situations.

The remainder of this text is organized as follows: In Section 2 we prove the announced result and compute the inverse of the embedding operator. In Section 3 we consider the Itô integral in various situations and sketch an application to stochastic partial differential equations.

2. Isomorphisms for spaces of predictable processes

Let $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}})$ be a measure space. In view of our applications in Section 3.3, we do not demand that $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}})$ is a probability space. Moreover, let $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ be a filtration satisfying the usual conditions.

Fix $T > 0$ and let μ be a measure on $(\tilde{\Omega} \times [0,T], \tilde{\mathcal{F}}_T \otimes \mathcal{B}([0,T]))$ with marginales

$$\mu(A \times [0,T]) = \tilde{\mathbb{P}}(A), \quad A \in \tilde{\mathcal{F}}_T.$$

We assume that there exists a sequence $(A_n)_{n \in \mathbb{N}} \subset \tilde{\mathcal{F}}_0$ such that $A_n \uparrow \tilde{\Omega}$ and $\tilde{\mathbb{P}}(A_n) < \infty$ for all $n \in \mathbb{N}$. In particular, the measures $\tilde{\mathbb{P}}$ and μ are σ-finite.

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There exists a transition kernel $K : \tilde{\Omega} \times \mathcal{B}([0,T]) \to \mathbb{R}_+$ from $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$ to $([0,T], \mathcal{B}([0,T]))$ such that

$$\mu(B) = \int_{\tilde{\Omega}} \int_0^T \mathbb{1}_B(\tilde{\omega}, t) K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}), \quad B \in \tilde{\mathcal{F}}_T \otimes \mathcal{B}([0,T]),$$

see [11, Sec. II.1a]. We denote by $\tilde{\mathcal{P}}_T$ the predictable σ-algebra on $\tilde{\Omega} \times [0,T]$. Let F be a separable Banach space. Fix an arbitrary $p \geq 1$ and define the spaces

$$\begin{aligned} L^p_{T,\mathrm{pred}}(F) &:= L^p(\tilde{\Omega} \times [0,T], \tilde{\mathcal{P}}_T, \mu; F), \\ L^p_{T,\mathrm{prog}}(F) &:= L^p(\tilde{\Omega} \times [0,T], \tilde{\mathcal{F}}_T \otimes \mathcal{B}([0,T]), \mu; F) \cap \mathrm{Prog}_T(F), \\ L^p_{T,\mathrm{ad}}(F) &:= L^p(\tilde{\Omega} \times [0,T], \tilde{\mathcal{F}}_T \otimes \mathcal{B}([0,T]), \mu; F) \cap \mathrm{Ad}_T(F), \end{aligned}$$

where $\mathrm{Prog}_T(F)$ denotes the linear space of all F-valued progressively measurable processes $(\Phi_t)_{t\in[0,T]}$ and $\mathrm{Ad}_T(F)$ denotes the linear space of all F-valued adapted processes $(\Phi_t)_{t\in[0,T]}$. Then we have the inclusions

$$L^p_{T,\mathrm{pred}}(F) \subset L^p_{T,\mathrm{prog}}(F) \subset L^p_{T,\mathrm{ad}}(F).$$

In the upcoming theorem, we will show that these three spaces actually are isometrically isomorphic, provided the measures $A \mapsto K(\omega, A)$ are absolutely continuous. In particular, the latter two spaces are Banach spaces, too.

2.1. **Theorem.** *Suppose there is a nonnegative, measurable function $f : \tilde{\Omega} \times [0,T] \to \mathbb{R}$ such that for each $\tilde{\omega} \in \tilde{\Omega}$ we have $K(\tilde{\omega}, dt) = f(\tilde{\omega}, t)dt$. Then we have*

$$L^p_{T,\mathrm{pred}}(F) \cong L^p_{T,\mathrm{prog}}(F) \cong L^p_{T,\mathrm{ad}}(F).$$

Proof. It suffices to prove that for each $\Phi \in L^p_{T,\mathrm{ad}}(F)$ there exists a process $\pi(\Phi) \in L^p_{T,\mathrm{pred}}(F)$ such that $\Phi = \pi(\Phi)$ almost everywhere with respect to μ.

Let $\Phi \in L^p_{T,\mathrm{ad}}(F)$ be arbitrary. We will show that there is a sequence $(\Phi^n)_{n\in\mathbb{N}} \subset L^p_{T,\mathrm{pred}}(F)$ such that $\Phi^n \to \Phi$ in $L^p_{T,\mathrm{ad}}(F)$. Then, $(\Phi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p_{T,\mathrm{pred}}(F)$ and thus has a limit $\pi(\Phi) \in L^p_{T,\mathrm{pred}}(F)$. But this limit has the property $\Phi = \pi(\Phi)$ almost everywhere with respect to μ, which will finish the proof.

The proof of the existence of a sequence $(\Phi^n)_{n\in\mathbb{N}} \subset L^p_{T,\mathrm{pred}}(F)$ satisfying $\Phi^n \to \Phi$ in $L^p_{T,\mathrm{ad}}(F)$ is divided into two steps:

(1) First of all, we may assume that

$$\tilde{\mathbb{P}}(\tilde{\Omega}) = \mu(\tilde{\Omega} \times [0,T]) < \infty \tag{2.1}$$

and that there is a constant $M > 0$ such that

$$\|\Phi\| \leq M \quad \text{everywhere.} \tag{2.2}$$

Indeed, by assumption, there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{F}}_0$ with $A_n \uparrow \tilde{\Omega}$ and $\tilde{\mathbb{P}}(A_n) < \infty$ for all $n \in \mathbb{N}$. Defining the sequence $(\Phi^n)_{n\in\mathbb{N}} \subset L^p_{T,\mathrm{ad}}(F)$ by $\Phi^n := (\Phi \wedge n)\mathbb{1}_{A_n}$, Lebesgue's dominated convergence theorem yields that $\Phi^n \to \Phi$ in $L^p_{T,\mathrm{ad}}(F)$.

(2) Now, we proceed with a similarly technique as in [13, pp. 97–99]. We extend Φ to a process $(\Phi_t)_{t\in\mathbb{R}}$ by setting

$$\Phi_t(\tilde{\omega}) := 0 \quad \text{for } (\tilde{\omega}, t) \in \tilde{\Omega} \times \mathbb{R} \setminus [0,T].$$

Defining for $n \in \mathbb{N}$ the function $\theta_n : \mathbb{R} \to \mathbb{R}$ by

$$\theta_n(t) := \sum_{j\in\mathbb{Z}} \frac{j-1}{2^n} \mathbb{1}_{(\frac{j-1}{2^n}, \frac{j}{2^n}]}(t),$$

we have $\theta_n(t) \uparrow t$ for all $t \in \mathbb{R}$. The shift semigroup $(S_t)_{t\geq 0}$, $S_t f = f(t + \cdot)$ is strongly continuous on $L^p(\mathbb{R}; F)$. Thus, performing integration by the substitution $t \rightsquigarrow t+s$, using Fubini's theorem, Lebesgue's dominated convergence theorem and noting (2.1), (2.2) we obtain

$$\begin{aligned}
&\int_{\tilde{\Omega}} \int_0^T \int_0^T \|\Phi_{s+\theta_n(t-s)}(\tilde{\omega}) - \Phi_t(\tilde{\omega})\|^p dt ds \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \int_0^T \int_{-s}^{T-s} \|\Phi_{s+\theta_n(t)}(\tilde{\omega}) - \Phi_{s+t}(\tilde{\omega})\|^p dt ds \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \int_0^T \int_0^{T-t} \|\Phi_{s+\theta_n(t)}(\tilde{\omega}) - \Phi_{s+t}(\tilde{\omega})\|^p ds dt \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&\quad + \int_{\tilde{\Omega}} \int_{-T}^0 \int_{-t}^{T} \|\Phi_{s+\theta_n(t)}(\tilde{\omega}) - \Phi_{s+t}(\tilde{\omega})\|^p ds dt \tilde{\mathbb{P}}(d\tilde{\omega}) \to 0.
\end{aligned}$$

After passing to a subsequence, if necessary, for $\tilde{\mathbb{P}} \otimes \lambda \otimes \lambda$–almost all $(\tilde{\omega}, s, t) \in \tilde{\Omega} \times [0,T] \times [0,T]$ we have

$$\|\Phi_{s+\theta_n(t-s)}(\tilde{\omega}) - \Phi_t(\tilde{\omega})\|^p \to 0,$$

where λ denotes the Lebesgue measure. Thus, there exists $s \in [0,T]$ such that

$$\|\Phi_{s+\theta_n(t-s)}(\tilde{\omega}) - \Phi_t(\tilde{\omega})\|^p \to 0 \quad \text{for } \tilde{\mathbb{P}} \otimes \lambda\text{–almost all } (\tilde{\omega}, t) \in \tilde{\Omega} \times [0,T]. \tag{2.3}$$

For $n \in \mathbb{N}$ we define the process $\Phi^n = (\Phi_t^n)_{t \in [0,T]}$ by

$$\Phi_t^n := \Phi_{s+\theta_n(t-s)} = \sum_{j \in \mathbb{Z}} \Phi_{s+\frac{j-1}{2^n}} \mathbb{1}_{(s+\frac{j-1}{2^n}, s+\frac{j}{2^n}]}(t), \quad t \in [0,T].$$

Note that Φ^n is predictable, because Φ is adapted. Hence, we have $(\Phi^n)_{n \in \mathbb{N}} \subset L^p_{T,\mathrm{pred}}(F)$. By assumption, there is a nonnegative, measurable function $f : \tilde{\Omega} \times [0,T] \to \mathbb{R}$ such that for each $\tilde{\omega} \in \tilde{\Omega}$ we have $K(\tilde{\omega}, dt) = f(\tilde{\omega}, t)dt$. Using (2.1) we have

$$\int_{\tilde{\Omega}} \int_0^T f(\tilde{\omega}, t) dt \tilde{\mathbb{P}}(d\tilde{\omega}) = \int_{\tilde{\Omega}} \int_0^T K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) = \mu(\tilde{\Omega} \times [0,T]) < \infty.$$

Noting (2.1), (2.2), we obtain by (2.3) and Lebesgue's dominated convergence theorem

$$\begin{aligned}
&\iint_{\tilde{\Omega} \times [0,T]} \|\Phi^n - \Phi\|^p d\mu = \int_{\tilde{\Omega}} \int_0^T \|\Phi_{s+\theta_n(t-s)}(\tilde{\omega}) - \Phi_t(\tilde{\omega})\|^p K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \int_0^T \|\Phi_{s+\theta_n(t-s)}(\tilde{\omega}) - \Phi_t(\tilde{\omega})\|^p f(\tilde{\omega}, t) dt \tilde{\mathbb{P}}(d\tilde{\omega}) \to 0,
\end{aligned}$$

showing that $\Phi^n \to \Phi$ in $L^p_{T,\mathrm{ad}}(F)$.

□

2.2. **Remark.** *Let $\pi : L^p_{T,\mathrm{ad}}(F) \to L^p_{T,\mathrm{pred}}(F)$ be the inverse of the embedding operator* $\mathrm{Id} : L^p_{T,\mathrm{pred}}(F) \hookrightarrow L^p_{T,\mathrm{ad}}(F)$. *Then, for $\Phi \in L^p_{T,\mathrm{ad}}(F)$ the process $\pi(\Phi)$ coincides with the conditional expectation of Φ given the predictable σ-algebra $\tilde{\mathcal{P}}_T$, that is*

$$\pi(\Phi) = \tilde{\mathbb{E}}[\Phi \,|\, \tilde{\mathcal{P}}_T],$$

because $\pi(\Phi) = \Phi$ almost everywhere with respect to μ.

If the process Φ is càdlàg, then $\pi(\Phi)$ is easy to determine:

2.3. **Proposition.** *Suppose there is a nonnegative, measurable function $f : \tilde{\Omega} \times [0,T] \to \mathbb{R}$ such that for each $\tilde{\omega} \in \tilde{\Omega}$ we have $K(\tilde{\omega}, dt) = f(\tilde{\omega}, t)dt$, and suppose that $\Phi \in \mathcal{L}^p_{T,\mathrm{ad}}(F)$ is càdlàg. Then we have $\pi(\Phi) = \Phi_-$.*

Proof. First, we note that the process Φ_- is predictable. Let $\tilde{\omega} \in \tilde{\Omega}$ be arbitrary. The set $\mathcal{N}_{\tilde{\omega}} = \{t \in [0,T] : \Delta\Phi_t(\tilde{\omega}) \neq 0\}$ is countable. Hence, by the continuity of the measure $A \mapsto K(\tilde{\omega}, A)$ we have

$$K(\tilde{\omega}, \mathcal{N}_{\tilde{\omega}}) = 0 \quad \text{for all } \tilde{\omega} \in \tilde{\Omega}.$$

Therefore, we obtain

$$\begin{aligned}
&\iint_{\tilde{\Omega}\times[0,T]} \|\Phi_-\|^p d\mu = \int_{\tilde{\Omega}} \int_0^T \|\Phi_{t-}(\tilde{\omega})\|^p K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \int_0^T \|\Phi_t(\tilde{\omega})\|^p K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) = \iint_{\tilde{\Omega}\times[0,T]} \|\Phi\|^p d\mu < \infty,
\end{aligned}$$

because $\Phi \in \mathcal{L}^p_{T,\mathrm{ad}}(F)$, showing that $\Phi_- \in \mathcal{L}^p_{T,\mathrm{pred}}(F)$. Moreover, we get

$$\begin{aligned}
&\iint_{\tilde{\Omega}\times[0,T]} \|\Phi - \Phi_-\|^p d\mu = \int_{\tilde{\Omega}} \int_0^T \|\Phi_t(\tilde{\omega}) - \Phi_{t-}(\tilde{\omega})\|^p K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \int_0^T \|\Delta\Phi_t(\tilde{\omega})\|^p K(\tilde{\omega}, dt) \tilde{\mathbb{P}}(d\tilde{\omega}) = 0,
\end{aligned}$$

proving that $\Phi = \Phi_-$ almost everywhere with respect to μ. □

2.4. **Remark.** *Let $F := \mathbb{R}$, let*

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}) := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$$

be a filtered probability space and let μ be the product measure $\mu = \mathbb{P} \otimes K$, where K is an absolutely continuous, finite measure on $([0,T], \mathcal{B}[0,T])$. Suppose that $\Phi \in \mathcal{L}^p_{T,\mathrm{ad}}(\mathbb{R})$ is a bounded process. According to [4, Thm. VI.43]*, there exist (up to an evanescent set) unique processes ${}^o\Phi$ and ${}^p\Phi$ such that ${}^o\Phi$ is optional (and hence progressively measurable), ${}^p\Phi$ is predictable and we have*

$$\begin{aligned}
\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathcal{F}_\tau] &= {}^o\Phi_\tau \mathbb{1}_{\{\tau<\infty\}} \quad \textit{almost surely for every stopping time } \tau, \\
\mathbb{E}[X_\tau \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathcal{F}_{\tau-}] &= {}^p\Phi_\tau \mathbb{1}_{\{\tau<\infty\}} \quad \textit{almost surely for every predictable time } \tau.
\end{aligned}$$

They are called the optional and the predictable projection of Φ. By [4, Remark VI.44.g] *the optional projection ${}^o\Phi$ is a modification of Φ, and hence*

$$\mu(\Phi \neq {}^o\Phi) = (\mathbb{P} \otimes K)(\Phi \neq {}^o\Phi) = \int_0^T \mathbb{P}(\Phi_t \neq {}^o\Phi_t) K(dt) = 0.$$

Moreover, by [4, Thm. VI.46] *we have*

$$\{{}^o\Phi \neq {}^p\Phi\} = \bigcup_{n\in\mathbb{N}} [\![\tau_n]\!],$$

where $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times and $[\![\tau_n]\!]$ denotes the graph

$$[\![\tau_n]\!] = \{(\omega, \tau_n(\omega)) : \omega \in \Omega\}.$$

Consequently, by the continuity of the measure K we obtain

$$\mu({}^o\Phi \neq {}^p\Phi) = (\mathbb{P} \otimes K)({}^o\Phi \neq {}^p\Phi) \leq \sum_{n\in\mathbb{N}} \mathbb{E}[K(\{\tau_n\})] = 0,$$

showing that the inverse of the embedding operator is given by $\pi(\Phi) = {}^p\Phi$.

Theorem 2.1 provides a straightforward extension of the Itô integral. Usually, one defines the Itô integral as a continuous linear operator

$$\mathcal{I} : L^2_{T,\mathrm{pred}}(F) \to M^2_T(G), \tag{2.4}$$

where G is another separable Banach space and $M^2_T(G)$ denotes the Banach space of all G-valued square-integrable martingales $M = (M_t)_{t\in[0,T]}$. In fact, if F and G

are Hilbert spaces, then the integral operator (2.4) is even an isometry. Using that $L^2_{T,\mathrm{ad}}(F) \cong L^2_{T,\mathrm{pred}}(F)$ according to Theorem 2.1, we can define the Itô integral as continuous linear operator

$$\mathcal{I} : L^2_{T,\mathrm{ad}}(F) \to M^2_T(G). \tag{2.5}$$

By localization, we can further extend the Itô integral to all F-valued progressively measurable processes $\Phi = (\Phi_t)_{t\geq 0}$ such that $\tilde{\mathbb{P}}$–almost surely

$$\int_0^T \|\Phi_s\|^2 K(ds) < \infty \quad \text{for all } T > 0,$$

and then the integral process $\mathcal{I}(\Phi)$ is a local martingale. We shall outline some concrete situations in the upcoming section.

3. The Itô integral for adapted, measurable processes

We shall now outline the extension of the Itô integral in various situations. In what follows, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denotes a filtered probability space satisfying the usual conditions.

3.1. **The Itô integral with respect to martingales.** Let M be a real-valued, square-integrable martingale. Recall that the predictable quadratic variation $\langle M, M \rangle$ is the unique adapted, non-decreasing process such that $M^2 - \langle M, M \rangle$ is a martingale, see [11, Thm. I.4.2]. We assume that the quadratic variation $\langle M, M \rangle$ is absolutely continuous, which is in particular the case for Lévy processes. We set

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}) := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \quad \text{and} \quad \mu := \mathbb{P} \otimes \langle M, M \rangle$$

and let $F = G$ be a separable Hilbert space. Proceeding as in [11, Sec. I.4d], we define the Itô integral $\Phi \cdot M$ as the isometry (2.4). Using Theorem 2.1, we extend it to the isometry (2.5). The resulting Itô integral coincides with the stochastic integral constructed in [13].

We remark that this construction is still possible if F, G are separable Banach spaces of M-type 2 (see, e.g., [18, Chap. 6]). Moreover, the martingale M may even be infinite dimensional. Then, Φ has values in the space $L(F, G)$ of bounded linear operators from F to G, see [19].

As pointed out in [14, Sec. 18.4.1], for non-predictable integrands the just defined integral may not coincide with the pathwise Lebesgue-Stieltjes integral, provided the latter exists. For example, let N be a standard Poisson process and let M be the martingale $M_t = N_t - t$. By Proposition 2.3, the inverse of N under the embedding operator is given by $\pi(N) = N_-$, and therefore we obtain the Itô integral

$$(N \cdot M)_t = (N_- \cdot M)_t = \frac{1}{2}(N_t^2 - N_t) - \int_0^t N_{s-} ds,$$

because the Itô integral of the predictable process N_- coincides with the pathwise Lebesgue-Stieltjes integral. On the other hand, we obtain the pathwise Lebesgue-Stieltjes integral

$$\int_0^t N_s dM_s = \frac{1}{2}(N_t^2 + N_t) - \int_0^t N_s ds = N_t + (N \cdot M)_t,$$

which cannot be a martingale, because $\mathbb{E}[N_t] = t$.

Thus, our extension of the Itô integral may not coincide with the pathwise Lebesgue-Stieltjes integral, but it preserves the martingale property of the integral process, which makes it interesting for applications.

3.2. **The Itô integral with respect to infinite dimensional Wiener processes.** Let H, U be separable Hilbert spaces and let $Q \in L(U)$ be a compact, self-adjoint, strictly positive linear operator. Then, there exist an orthonormal basis $\{e_j\}$ of U and a bounded sequence (λ_j) of strictly positive real numbers such that

$$Qu = \sum_j \lambda_j \langle u, e_j \rangle e_j, \quad u \in U$$

namely, the λ_j are the eigenvalues of Q, and each e_j is an eigenvector corresponding to λ_j. The space $U_0 := Q^{1/2}(U)$, equipped with inner product $\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U$, is another separable Hilbert space and $\{\sqrt{\lambda_j} e_j\}$ is an orthonormal basis.

Let W be a Q-Wiener process [3, p. 86,87] such that Q is a trace class operator, that is $\operatorname{tr}(Q) = \sum_j \lambda_j < \infty$. We denote by $L_2^0 := L_2(U_0, H)$ the space of Hilbert-Schmidt operators from U_0 into H, which, endowed with the Hilbert-Schmidt norm

$$\|\Phi\|_{L_2^0} := \sqrt{\sum_j \lambda_j \|\Phi e_j\|^2}, \quad \Phi \in L_2^0$$

itself is a separable Hilbert space. We set

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}) := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), \quad \mu := \mathbb{P} \otimes \lambda, \quad F := L_2^0, \quad G := H,$$

where λ denotes the Lebesgue measure. Following [3, Chap. 4.2], we define the Itô integral $\Phi \cdot W$ as the isometry (2.4). Using Theorem 2.1, we extend it to the isometry (2.5). The resulting Itô integral coincides with the stochastic integral constructed in [9].

Analogously, we define the stochastic integral with respect to an infinite dimensional Lévy process (see [17]) for adapted, measurable integrands.

3.3. **The Itô integral with respect to compensated Poisson random measures.** Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see, e.g., [10]). We remark that every Polish space with its Borel σ-field is a Blackwell space. Let N be a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [11, Def. II.1.20]. Then its compensator is of the form $dt \otimes \beta(dx)$, where β is a σ-finite measure on $(E, \mathcal{E})$. We define the compensated Poisson random measure $q(dt, dx) := N(dt, dx) - \beta(dx)dt$ and set

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}) := (\Omega \times E, \mathcal{F} \times \mathcal{E}, (\mathcal{F}_t \times \mathcal{E})_{t \geq 0}, \mathbb{P} \otimes \beta), \quad \mu := \tilde{\mathbb{P}} \otimes \lambda,$$

where λ denotes the Lebesgue measure, and let $F = G$ be separable Hilbert spaces. Proceeding as in [2, Sec. 4], we define the Itô integral $\Phi \cdot q$ as the isometry (2.4). Using Theorem 2.1, we extend it to the isometry (2.5).

We remark that this construction is still possible if F is a separable Banach space of M-type 2 (see, e.g., [18, Chap. 6]). The resulting Itô integral coincides with the stochastic integral constructed in [22].

Moreover, such a construction is also possible on general separable Banach spaces, provided that the inequality

$$\mathbb{E}\left[\left\| \int_0^T \int_E \Phi(s,x) q(ds,dx) \right\|^2\right] \leq K_\beta \mathbb{E}\left[\int_0^T \int_E \|\Phi(s,x)\|^2 \beta(dx) ds\right]$$

holds for all simple processes Φ, with a constant $K_\beta > 0$ only depending on β, see [15].

3.4. **Stochastic partial differential equations.** Finally, we mention that we can use the extension of the stochastic integral provided in this paper in order to solve stochastic differential equations or even stochastic partial differential equations

$$\left\{ \begin{array}{rcl} dr_t & = & (Ar_t + \alpha(t, r_t))dt + \sigma(t, r_t)dW_t + \int_E \gamma(t, x, r_t)q(dt, dx) \\ r_0 & = & h_0 \end{array} \right. \tag{3.1}$$

on Hilbert spaces driven by an infinite dimensional Wiener process and a compensated Poisson random measure. Such equations have been studied, e.g., in [1, 6, 16]. In equation (3.1), the operator $A : \mathcal{D}(A) \subset H \to H$ denotes the generator of a strongly continuous semigroup.

Using our extension of the Itô integral, under appropriate Lipschitz conditions on the vector fields we can prove the existence if a unique mild solution for (3.1) by performing a fixed point argument on appropriate spaces of progressively measurable processes.

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