

A NOTE ON STOCHASTIC INTEGRALS AS L^2 -CURVES

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ABSTRACT. In a work of van Gaans (2005a) stochastic integrals are regarded as L^2 -curves. In Filipović and Tappe (2008) we have shown the connection to the usual Itô-integral for càdlàg-integrands. The goal of this note is to complete this result and to provide the full connection to the Itô-integral. We also sketch an application to stochastic partial differential equations.

Key Words: Stochastic integrals, L^2 -curves, connection to the Itô-integral, stochastic partial differential equations.

60H05, 60H15

1. INTRODUCTION

In the paper Filipović and Tappe (2008) we have established an existence and uniqueness result for stochastic partial differential equations, driven by Lévy processes, by applying a result from (van Gaans 2005a, Thm. 4.1). In van Gaans (2005a) stochastic integrals are regarded as L^2 -curves. It was therefore necessary to establish the connection to the usual Itô-integral (developed, e.g., in Jacod and Shiryaev (2003) or Protter (2005)) for càdlàg-integrands, which we have provided in (Filipović and Tappe 2008, Appendix B).

The goal of the present note is to complete this result and to provide the full connection to the Itô-integral. More precisely, we will show that the space of adapted L^2 -curves is embedded into the space of Itô-integrable processes (see Proposition 2.5 below), and that the corresponding Itô-integral is a càdlàg-version of the stochastic integral in the sense of van Gaans (2005a) (see Proposition 2.9 below).

This is the content of Section 2. Afterwards, we outline an application to stochastic partial differential equations in Section 3.

2. STOCHASTIC INTEGRALS AS L^2 -CURVES

Throughout this text, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Furthermore, let $(H, \|\cdot\|)$ denote a separable Hilbert space.

For any $T \in \mathbb{R}_+$ the space $C[0, T] := C([0, T]; L^2(\Omega; H))$ of all continuous curves from $[0, T]$ into $L^2(\Omega; H)$ is a Banach space with respect to the norm

$$\|r\|_T := \sup_{t \in [0, T]} \|r_t\|_{L^2(\Omega; H)} = \sqrt{\sup_{t \in [0, T]} \mathbb{E}[\|r_t\|^2]}.$$

The subspace $C_{\text{ad}}[0, T]$ consisting of all adapted processes from $C[0, T]$ is closed with respect to this norm. Note that, by the completeness of the filtration $(\mathcal{F}_t)_{t \geq 0}$, adaptedness is independent of the choice of the representative.

2.1. Stochastic integral with respect to a Lévy martingale. Let M be a real-valued, square-integrable Lévy martingale. We recall how in this case the stochastic integral $(G\text{-})(\Phi \cdot M)$ in the sense of (van Gaans 2005a, Sec. 3) is defined for $\Phi \in C_{\text{ad}}[0, T]$.

2.1. Lemma. (van Gaans 2005a, Prop. 3.2.1) *Let $\Phi \in C_{\text{ad}}[0, T]$ be arbitrary. For each $t \in [0, T]$ there exists a unique random variable $Y_t \in L^2(\Omega)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(2.1) \quad \mathbb{E} \left[\left\| Y_t - \sum_{i=0}^{n-1} \Phi_{t_i} (M_{t_{i+1}} - M_{t_i}) \right\|^2 \right] < \varepsilon$$

for every partition $0 = t_0 < t_1 < \dots < t_n = t$ with $\sup_{i=0, \dots, n-1} |t_{i+1} - t_i| < \delta$.

2.2. Definition. (van Gaans (2005a)) *Let $\Phi \in C_{\text{ad}}[0, T]$ be arbitrary. Then the stochastic integral $Y = (G\text{-})(\Phi \cdot M)$ is the stochastic process $Y = (Y_t)_{t \in [0, T]}$ where every Y_t is the unique element from $L^2(\Omega)$ such that (2.1) is valid.*

We observe that the integrand Φ as well as the stochastic integral $(G\text{-})(\Phi \cdot M)$ are only determined up to a version. In particular, it is not clear if the integral process has a càdlàg-version.

2.3. Lemma. (van Gaans 2005a, Thm. 3.3.2) *For each $\Phi \in C_{\text{ad}}[0, T]$ we have*

$$(G\text{-})(\Phi \cdot M) \in C_{\text{ad}}[0, T].$$

We are now interested in finding the connection between the stochastic integral $(G\text{-})(\Phi \cdot M)$ and the usual Itô-integral (developed, e.g., in Jacod and Shiryaev (2003), Protter (2005), Applebaum (2005) for the finite dimensional case and in Da Prato and Zabczyk (1992), Peszat and Zabczyk (2007) for the infinite dimensional case). We use the abbreviation

$$L^2(\mathcal{P}_T) := L^2(\Omega \times [0, T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; H),$$

where \mathcal{P}_T denotes the predictable σ -algebra on $\Omega \times [0, T]$ and λ the Lebesgue measure. Since for any square-integrable Lévy martingale M the predictable quadratic covariation $\langle M, M \rangle$ is linear, $L^2(\mathcal{P}_T)$ is the space of all L^2 -processes Φ , for which the Itô-integral $\Phi \cdot M$ exists, independent of the choice of M .

2.4. Lemma. *For each $\Phi \in C_{\text{ad}}[0, T]$ there exists a predictable version ${}^p\Phi \in L^2(\mathcal{P}_T)$ of Φ .*

Proof. By (Da Prato and Zabczyk 1992, Prop. 3.6.ii) there exists a predictable version ${}^p\Phi$ of Φ . Since $\Phi \in C_{\text{ad}}[0, T]$, we also have

$$\int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2] dt = \int_0^T \mathbb{E}[\|\Phi_t\|^2] dt \leq T \sup_{t \in [0, T]} \mathbb{E}[\|\Phi_t\|^2] < \infty,$$

that is ${}^p\Phi \in L^2(\mathcal{P}_T)$. □

2.5. Proposition. *The map $\Phi \mapsto {}^p\Phi$ defines an embedding from $C_{\text{ad}}[0, T]$ into $L^2(\mathcal{P}_T)$.*

Proof. For two predictable versions $\Phi^1, \Phi^2 \in L^2(\mathcal{P}_T)$ of Φ we have

$$\int_0^T \mathbb{E}[\|\Phi_t^1 - \Phi_t^2\|^2] dt = 0,$$

whence $\Phi^1 = \Phi^2$ in $L^2(\mathcal{P}_T)$. Therefore, the map $\Phi \mapsto {}^p\Phi$ is well-defined. The linearity of $\Phi \mapsto {}^p\Phi$ is immediately checked, and the estimate

$$\int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2] dt = \int_0^T \mathbb{E}[\|\Phi_t\|^2] dt \leq T \sup_{t \in [0, T]} \mathbb{E}[\|\Phi_t\|^2], \quad \Phi \in C_{\text{ad}}[0, T]$$

proves the continuity of $\Phi \mapsto {}^p\Phi$. For $\Phi \in C_{\text{ad}}[0, T]$ with ${}^p\Phi = 0$ in $L^2(\mathcal{P}_T)$ we have

$$\int_0^T \mathbb{E}[\|\Phi_t\|^2] dt = \int_0^T \mathbb{E}[\|{}^p\Phi_t\|^2] dt = 0.$$

Since $\Phi \in C_{\text{ad}}[0, T]$, the map $t \mapsto \mathbb{E}[\|\Phi_t\|^2]$ is continuous, which implies $\Phi = 0$ in $C_{\text{ad}}[0, T]$, showing that $\Phi \mapsto {}^p\Phi$ is injective. \square

The notation ${}^p\Phi$ reminds of the *predictable projection* of a process Φ , which we shall briefly recall. In the real-valued case one defines, for every \mathbb{R} -valued and $\mathcal{F}_T \otimes \mathcal{B}[0, T]$ -measurable process Φ the predictable projection ${}^\pi\Phi$ of Φ , according to (Jacod and Shiryaev 2003, Thm. I.2.28), as the (up to an evanescent set) unique $(-\infty, \infty)$ -valued process satisfying the following two conditions:

- (1) It is predictable;
- (2) $({}^\pi\Phi)_\tau = \mathbb{E}[\Phi_\tau | \mathcal{F}_{\tau-}]$ on $\{\tau \leq T\}$ for all predictable times τ .

Note that for every predictable process Φ we have ${}^\pi\Phi = \Phi$.

We transfer this definition to any H -valued process $\tilde{\Phi}$, which is an $\mathcal{F}_T \otimes \mathcal{B}[0, T]$ -measurable version of a process $\Phi \in C_{\text{ad}}[0, T]$ by using the notion of conditional expectation from (Da Prato and Zabczyk 1992, Sec. 1.3). Then, the second property of the predictable projection ensures that ${}^\pi\tilde{\Phi}$ is finite, i.e. H -valued.

We obtain the following relation between the embedding ${}^p\Phi$ and the predictable projection ${}^\pi\tilde{\Phi}$:

2.6. Lemma. *For each $\Phi \in C_{\text{ad}}[0, T]$ and every $\mathcal{F}_T \otimes \mathcal{B}[0, T]$ -measurable version $\tilde{\Phi}$ we have*

$${}^\pi\tilde{\Phi} = {}^p\Phi \quad \text{in } L^2(\mathcal{P}_T).$$

Proof. For each $t \in [0, T]$ the identities

$${}^\pi\tilde{\Phi}_t = \mathbb{E}[\tilde{\Phi}_t | \mathcal{F}_{t-}] = \mathbb{E}[\Phi_t | \mathcal{F}_{t-}] = \mathbb{E}[{}^p\Phi_t | \mathcal{F}_{t-}] = {}^p\Phi_t \quad \mathbb{P}\text{-a.s.}$$

are valid, which gives us

$$\int_0^T \mathbb{E}[\|{}^\pi\tilde{\Phi}_t - {}^p\Phi_t\|^2] dt = 0,$$

proving the claimed result. \square

2.7. Lemma. *If $\Phi \in C_{\text{ad}}[0, T]$ has a càdlàg-version, then we have*

$${}^p\Phi = \Phi_- \quad \text{in } L^2(\mathcal{P}_T).$$

Proof. The process Φ_- is predictable and we have

$$\mathbb{E} \left[\int_0^T \|{}^p\Phi_t - \Phi_{t-}\|^2 dt \right] = \mathbb{E} \left[\int_0^T \|\Phi_t - \Phi_{t-}\|^2 dt \right] = \mathbb{E} \left[\int_0^T \|\Delta\Phi_t\|^2 dt \right] = 0,$$

because $\mathcal{N}_\omega = \{t \in [0, T] : \Delta\Phi_t(\omega) \neq 0\}$ is countable for all $\omega \in \Omega$. \square

2.8. Proposition. *For each $\Phi \in C_{\text{ad}}[0, T]$ we have*

$$(G-)(\Phi \cdot M) = {}^p\Phi \cdot M \quad \text{in } C_{\text{ad}}[0, T].$$

In particular, $(G-)(\Phi \cdot M)$ has a càdlàg-version.

Proof. Let $t \in [0, T]$ and $\epsilon > 0$ be arbitrary. Since $\Phi \in C_{\text{ad}}[0, T]$, it is uniformly continuous on the compact interval $[0, t]$, and thus there exists $\delta > 0$ such that

$$(2.2) \quad \mathbb{E}[\|\Phi_u - \Phi_v\|^2] < \frac{\epsilon}{\langle M, M \rangle_t}$$

for all $u, v \in [0, t]$ with $|u - v| < \delta$. Let $\mathcal{Z} = \{0 = t_0 < t_1 < \dots < t_n = t\}$ be an arbitrary decomposition with $\sup_{i=0, \dots, n-1} |t_{i+1} - t_i| < \delta$. Defining

$$\Phi^{\mathcal{Z}} := \Phi_0 \mathbb{1}_{[0]} + \sum_{i=0}^{n-1} \Phi_{t_i} \mathbb{1}_{(t_i, t_{i+1}]},$$

we obtain, by using the Itô-isometry and (2.2),

$$\begin{aligned} \mathbb{E} \left[\left\| ({}^p\Phi \cdot M)_t - \sum_{i=0}^{n-1} \Phi_{t_i} (M_{t_{i+1}} - M_{t_i}) \right\|^2 \right] &= \mathbb{E} \left[\left\| \int_0^t ({}^p\Phi_s - \Phi_s^{\mathcal{Z}}) dM_s \right\|^2 \right] \\ &= \mathbb{E} \left[\int_0^t \|{}^p\Phi_s - \Phi_s^{\mathcal{Z}}\|^2 d\langle M, M \rangle_s \right] = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[\|\Phi_s - \Phi_{t_i}\|^2] d\langle M, M \rangle_s < \epsilon, \end{aligned}$$

establishing that ${}^p\Phi \cdot M$ is a version of $(G-)(\Phi \cdot M)$. \square

2.2. Stochastic integral with respect to Lebesgue measure. In an analogous fashion, we introduce the stochastic integral $(G-)(\Phi \cdot \lambda)$ with respect to the Lebesgue measure λ , cf. (van Gaans 2005a, Lemma 3.6). By similar arguments as in the previous subsection, we obtain the same relation between this stochastic integral $(G-)(\Phi \cdot \lambda)$ and the usual Bochner integral $\Phi \cdot \lambda$.

2.3. Stochastic integral with respect to a Lévy process. Now let X be a square-integrable Lévy process with semimartingale decomposition $X_t = M_t + bt$, where M is a square-integrable Lévy martingale and $b \in \mathbb{R}$. According to (van Gaans 2005a, Def. 3.7) we set

$$(G-)(\Phi \cdot X) := (G-)(\Phi \cdot M) + b(G-)(\Phi \cdot \lambda).$$

As a direct consequence of our previous results, we obtain:

2.9. Proposition. *For each $\Phi \in C_{\text{ad}}[0, T]$ we have*

$$(G-)(\Phi \cdot X) = {}^p\Phi \cdot X \quad \text{in } C_{\text{ad}}[0, T].$$

In particular, $(G-)(\Phi \cdot X)$ has a càdlàg-version.

Summing up, we have seen that the space $C_{\text{ad}}[0, T]$ of all adapted continuous curves from $[0, T]$ into $L^2(\Omega; H)$ is embedded into $L^2(\mathcal{P}_T)$ via $\Phi \mapsto {}^p\Phi$, see Proposition 2.5, and that the Itô-integral ${}^p\Phi \cdot X$ is a càdlàg-version of $(G-)(\Phi \cdot X)$, see Proposition 2.9. Moreover, we have seen the relation to the predictable projection in Lemma 2.6.

We close this section with an example, which seems surprising at a first view. Let X be a standard Poisson process with values in \mathbb{R} . In Ex. 3.9 in van Gaans (2005a) it is derived that

$$(G-) \int_0^t X_s dX_s = \frac{1}{2}(X_t^2 - X_t).$$

Apparently, this does not coincide with the pathwise Lebesgue-Stieltjes integral

$$\int_0^t X_s dX_s = \frac{1}{2}(X_t^2 + X_t).$$

The explanation for this seemingly inconsistency is easily provided. The process X is not predictable, whence it is not Itô-integrable, and a straightforward calculation shows that

$$(G-) \int_0^t X_s dX_s = \int_0^t X_{s-} dX_s.$$

This, however, is exactly what an application of Proposition 2.9 and Lemma 2.7 yields.

3. SOLUTIONS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AS L^2 -CURVES

Regarding stochastic integrals as L^2 -curves provides an existence and uniqueness proof for stochastic partial differential equations. Of course, this result is well-known in the literature (see, e.g., Albeverio et al. (2008), Da Prato and Zabczyk (1992), Filipović et al. (2010), Marinelli et al. (2010), Peszat and Zabczyk (2007)), whence we only give an outline.

Consider the stochastic partial differential equation

$$(3.1) \quad \begin{cases} dr_t &= (Ar_t + \alpha(t, r_t))dt + \sum_{i=1}^n \sigma_i(t, r_{t-})dX_t^i \\ r_0 &= h_0, \end{cases}$$

where $A : \mathcal{D}(A) \subset H \rightarrow H$ denotes the infinitesimal generator of a C_0 -semigroup $(S_t)_{t \geq 0}$ on H , and where X^1, \dots, X^n are real-valued, square-integrable Lévy processes as in Section 2.3. We assume that the standard Lipschitz conditions are satisfied.

Then, there exists a unique solution $r \in C_{\text{ad}}[0, T]$ of the equation

$$r_t := S_t h_0 + (\text{G-}) \int_0^t S_{t-s} \alpha(s, r_s) ds + \sum_{i=1}^n (\text{G-}) \int_0^t S_{t-s} \sigma_i(s, r_s) dX_s^i,$$

see van Gaans (2005b) for the Wiener case and van Gaans (2005a) for the Lévy case. It is remarkable that the proof is established by means of precisely the same arguments as in the classical Picard-Lindelöf iteration scheme for ordinary differential equations, where one works on the Banach space $C([0, T]; H)$ instead of $C_{\text{ad}}[0, T]$.

Applying Proposition 2.9 for any fixed $t \in [0, T]$, we obtain the existence of a (up to a version) unique, predictable mild solution for the SPDE

$$\begin{cases} dr_t &= (Ar_t + \alpha(t, r_t))dt + \sum_{i=1}^n \sigma_i(t, r_t)dX_t^i \\ r_0 &= h_0, \end{cases}$$

which, in addition, is mean-square continuous.

Observe that we have no statement on path properties of the solution. If, however, the semigroup is pseudo-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0$$

then the stochastic convolution (Itô-)integrals have a càdlàg-version. This can be shown by using the Kotelenez inequality (see Kotelenez (1982)) or by using the Szökefalvi-Nagy theorem on unitary dilations (see, e.g., (Sz.-Nagy and Foias 1970, Thm. I.8.1), or (Davies 1976, Sec. 7.2)). We refer to (Peszat and Zabczyk 2007, Sec. 9.4) for an overview. In this case, we conclude that there even exists a (up to indistinguishability) unique càdlàg, adapted mild solution $(r_t)_{t \geq 0}$ for (3.1), which, in addition, is mean-square continuous.

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REFERENCES

- Albeverio, S., Mandrekar, V., and Rüdiger, B. (2008). Existence of mild solutions for stochastic differential equations and semilinear equations with non Gaussian Lévy noise. *Stochastic Processes and Their Applications*, 119:835–863.
- Applebaum, D. (2005). *Lévy processes and stochastic calculus*. Cambridge University Press, Cambridge.

- Da Prato, G. and Zabczyk, J. (1992). *Stochastic equations in infinite dimensions*. Cambridge University Press, New York.
- Davies, E. B. (1976). *Quantum theory of open systems*. Academic Press, London.
- Filipović, D. and Tappe, S. (2008). Existence of Lévy term structure models. *Finance and Stochastics*, 12:83–115.
- Filipović, D., Tappe, S., and Teichmann, J. (2010). Jump-diffusions in Hilbert spaces: Existence, stability and numerics. *Stochastics*. to appear.
- Jacod, J. and Shiryaev, A. N. (2003). *Limit theorems for stochastic processes*. Springer, Berlin.
- Kotelenez, P. (1982). A submartingale type inequality with applications to stochastic evolution equations. *Stochastics*, 8:139–151.
- Marinelli, C., Prévôt, C., and Röckner, M. (2010). Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. *Journal of Functional Analysis*, 258:616–649.
- Peszat, S. and Zabczyk, J. (2007). *Stochastic partial differential equations with Lévy noise*. Cambridge University Press, Cambridge.
- Protter, P. (2005). *Stochastic integration and differential equations*. Springer, Berlin, second edition. Version 2.1.
- Sz.-Nagy, B. and Foiaş, C. (1970). *Harmonic analysis of operators on Hilbert space*. North-Holland, Amsterdam.
- van Gaans, O. (2005a). Invariant measures for stochastic evolution equations with Hilbert space valued Lévy noise. Technical report, Leiden University. (www.math.leidenuniv.nl/~vangaans/gaansrep1.pdf).
- van Gaans, O. (2005b). A series approach to stochastic differential equations with infinite dimensional noise. *Integral Equations and Operator Theory*, 51:435–458.

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