

INFINITE DIMENSIONAL AFFINE PROCESSES

THORSTEN SCHMIDT, STEFAN TAPPE, AND WEIJUN YU

ABSTRACT. The goal of this article is to investigate infinite dimensional affine diffusion processes on the canonical state space. This includes a derivation of the corresponding system of Riccati differential equations and an existence proof for such processes, which has been missing in the literature so far. For the existence proof, we will regard affine processes as solutions to infinite dimensional stochastic differential equations with values in Hilbert spaces. This requires a suitable version of the Yamada-Watanabe theorem, which we will provide in this paper. Several examples of infinite dimensional affine processes accompany our results.

1. INTRODUCTION

Affine processes constitute an important model class due to their analytical tractability; in particular regarding applications in the field of mathematical finance. There is a substantial literature about affine processes in finite dimension. We refer, for example, to [13, 16, 19, 28, 20] for affine processes on the canonical state space, and to [6, 35, 29, 11, 27, 7] for affine processes on more general state spaces. Some recent and related developments are affine processes with stochastic discontinuities (see [30]), affine processes under parameter uncertainty (see [15]) and polynomial processes (see [8, 18, 5, 9]).

Only recently, increasing interest evolved in infinite dimensional affine processes: the theory of probability measure-valued processes has been utilized in [10] for the study of polynomial diffusions. We also mention the works [25], [38] and [24], where some examples, such as infinite dimensional square-root processes and infinite dimensional Heston type processes, are treated within the framework of probability measure-valued stochastic processes. Another recent approach to polynomial processes in infinite dimension is the paper [1], where the notion of a polynomial process – in the sense that polynomials are preserved under conditional expectations – is extended to a Banach space.

The general study in [23] deals with affine processes in infinite dimension on general state spaces; more precisely affine processes are understood as processes with an exponential affine structure of the characteristic exponent, and they are studied on topological vector spaces, which do not need to be separable or metrizable. The work [43], of which the present paper constitutes a further development in certain aspects, studies the special case of affine processes with values in separable Hilbert spaces; with a special focus to applications in finance. Some recent articles deal with particular examples of affine processes with values in

Date: July 24, 2019.

2010 Mathematics Subject Classification. 60J25, 60H10.

Key words and phrases. Infinite dimensional affine process, canonical state space, Riccati equation, stochastic differential equation on a Hilbert space, Yamada-Watanabe theorem, weak solution, retracted subspace with compact embedding, pathwise uniqueness.

We are grateful to Christa Cuchiero, Philipp Harms and Josef Teichmann for valuable discussions. Financial support from the DFG in project number 196379142 and from the Freiburg Institute of Advanced Studies (FRIAS) is gratefully acknowledged.

Hilbert spaces, such as Ornstein-Uhlenbeck processes with stochastic volatility and tensor Heston type processes; see, for example [2] and [3].

The paper [37] is between the finite and the infinite dimensional setting. More precisely, therein it has been investigated when the solutions to a (infinite dimensional) stochastic partial differential equation admit a finite dimensional realization with (finite dimensional) affine state processes.

The goal of the present paper is to explore infinite dimensional affine diffusion processes on the canonical state space. This includes a derivation of the corresponding system of Riccati differential equations and an existence proof for infinite dimensional affine processes, which has been missing in the literature so far. For the existence proof, we regard affine processes as solutions to infinite dimensional stochastic differential equations (SDEs) with values in Hilbert spaces. This requires a suitable version of the Yamada-Watanabe theorem, and – in order to apply the Yamada-Watanabe theorem – sufficient conditions for the existence of weak solutions, and for pathwise uniqueness of solutions. Infinite dimensional versions of the Yamada-Watanabe theorem can be found in [31], [34] and [36]. However, none of these results can directly be applied in our setting, and for this reason we provide a self-contained version in this paper. In order to ensure the existence of weak solutions, we establish a refined version of a result from [22], where the main idea is to consider starting points from an appropriate retracted subspace with compact embedding, and for this reason we need a suitably adjusted version of the Yamada-Watanabe theorem. The pathwise uniqueness follows from a version of the uniqueness result from [42] in infinite dimension.

The remainder of this paper is organized as follows. In Section 2 we introduce affine processes, and derive a general Riccati system for the functions appearing in the characteristic exponent. In Section 3 we provide the existence result for affine processes in the spirit of strong solutions to infinite dimensional SDEs. In Section 4 we present examples, where our existence result applies; this includes infinite dimensional processes of Cox-Ingersoll-Ross type and infinite dimensional processes of Heston type. For convenience of the reader, some auxiliary results are deferred to Appendix A. Moreover, Appendix B contains the required results about SDEs in Hilbert spaces; in particular the adjusted version of the Yamada-Watanabe theorem, and the mentioned results about existence of weak solutions and pathwise uniqueness. Finally, in Appendix C we provide the required results about linear operators in Hilbert spaces.

2. INFINITE DIMENSIONAL AFFINE PROCESSES

Affine models and their applications to dynamic term structure modelling have been intensively studied, mostly focusing on finite-dimensional affine models where the dimension, or the number of factors, is known and fixed. Here, we do not restrict the number of factors to be known or finite but rather study affine processes from an infinite-dimensional perspective. For practical applications, this allows to treat the number of factors as unknown parameter which has to be estimated. For the construction of infinite dimensional affine processes we follow the approaches in [13, 28]. The used techniques for Hilbert-space valued stochastic analysis is taken from [12].

Let $(H, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional and separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. The adjoint of a linear operator $T \in L(H)$ is denoted by T^* . By $\mathcal{B}(H)$ we denote the associated Borel σ -algebra. We fix throughout an orthonormal basis $(e_i)_{i=1}^\infty$ of H .

Affine processes are characterized by the convenient property that their Fourier transforms have exponential affine form. For the study of Fourier transform we introduce the following

complexification of H : set

$$H_{\mathbb{C}} = \{x + iy : x, y \in H\}$$

and equip it with the inner product $\langle x + iy, u + iv \rangle_{H_{\mathbb{C}}} := \langle x, u \rangle + \langle y, v \rangle + i\langle y, u \rangle - i\langle x, v \rangle$. Then $H_{\mathbb{C}}$ is a complex Hilbert space. For $z = x + iy \in H_{\mathbb{C}}$ we call $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ the *real* and *imaginary* part of z . Furthermore, we denote by $\bar{z} := \operatorname{Re}(z) - i\operatorname{Im}(z)$ the complex conjugate of z and the imaginary subspace of H by $iH = \{z \in H_{\mathbb{C}} : \operatorname{Re}(z) = 0\}$. The space of complex numbers with non-positive real part is denoted by $\mathbb{C}_- = \{c \in \mathbb{C} : \operatorname{Re}(c) \leq 0\}$.

2.1. Affine processes. We are interested in *homogeneous* infinite-dimensional continuous affine processes and introduce the following definition. While we do not aim at the greatest level of generality, we use a standard definition of affine processes. For a slightly more general approach (in finite dimensions) see [28]. The time-inhomogeneous case can be treated as in [16] and [30].

Consider a closed subset $\mathcal{X} \subset H$ which will serve as state space of our affine process and assume that the closure of the affine hull of \mathcal{X} is the full space H . Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space on which a family of probability measures $(P_x)_{x \in \mathcal{X}}$ is given. The filtration \mathbb{F} is right-continuous and P_x -complete for all $x \in \mathcal{X}$. Finally, consider a continuous process X with values in \mathcal{X} and denote its transition kernel by

$$p_t(x, A) = P_x(X_t \in A),$$

for $t \geq 0, x \in \mathcal{X}, A \in \mathcal{B}(H)$. We assume that the transition kernel is a Markov transition kernel, i.e. it satisfies the following properties (cf. [14])

- (i) $x \mapsto p_t(x, A)$ is $\mathcal{B}(H)$ -measurable for each $(t, A) \in \mathbb{R}_{\geq 0} \times \mathcal{B}(H)$,
- (ii) $p_0(x, \{x\}) = 1$ for all $x \in \mathcal{X}$,
- (iii) $p_t(x, \mathcal{X}) = 1$ for all $(t, x) \in \mathbb{R}_{\geq 0} \times H$,
- (iv) p satisfies the *Chapman-Kolmogorov equation*, i.e. for each $t, s \geq 0$ and $(x, A) \in H \times \mathcal{B}(H)$, it holds that

$$p_{t+s}(x, A) = \int p_t(y, A) p_s(x, dy). \quad (1)$$

The affine property of the Markov process X is characterized via its Fourier transform. The convex cone where the Fourier transform is defined by

$$\mathcal{U} := \left\{ u \in H_{\mathbb{C}} : \sup_{x \in \mathcal{X}} \operatorname{Re}(\langle u, x \rangle_{H_{\mathbb{C}}}) < \infty \right\}.$$

Then the function $\mathcal{X} \ni x \mapsto e^{\langle u, x \rangle}$ is bounded if and only if $x \in \mathcal{U}$. Moreover, $iH \subset \mathcal{U}$.

For a function $\phi : \mathbb{R}_{\geq 0} \rightarrow H$ the concepts of Fréchet and Gateaux differentiability coincide and we call ϕ *differentiable* with derivative $D_t \phi, t \geq 0$ being a bounded linear operator $D_t \phi : \mathbb{R}_{\geq 0} \rightarrow H$, if for every $t \geq 0$ it holds that

$$\lim_{\varepsilon \rightarrow 0} \frac{\| \phi(t + \varepsilon) - \phi(t) - \varepsilon D_t \phi(t) \|}{|\varepsilon|} = 0.$$

Definition 2.1. An H -valued continuous process X with transition kernel $p_t(x, A)$ is called *affine* with state space \mathcal{X} , if there exist \mathbb{C} and $H_{\mathbb{C}}$ -valued functions ϕ and ψ such that

- (i) $\phi(\cdot, u)$ and $\psi(\cdot, u)$ are differentiable for each $u \in \mathcal{U}$,
- (ii) the derivatives $D_t \phi(t, u)$ and $D_t \psi(t, u)$ are jointly continuous, and
- (iii) the Fourier-transform has exponential affine dependence on the initial value, i.e. for all $t \geq 0, x \in \mathcal{X}$, and $u \in \mathcal{U}$ it holds that

$$\int e^{\langle u, y \rangle_{H_{\mathbb{C}}}} p_t(x, dy) = \exp(\phi(t, u) + \langle \psi(t, u), x \rangle_{H_{\mathbb{C}}}). \quad (2)$$

Uniqueness of ϕ and ψ holds under the normalization $\phi(0, u) = 0$ and $\psi(0, u) = u$. Finite-dimensional affine processes can be viewed as a special case when $H = \mathbb{R}^n$. In this case, Definition 2.1 coincides with the affine class studied in [28].

As a next step we study infinite-dimensional diffusions and classify the affine ones. First, we split the state space in the non-negative part and the unrestricted part. Note that in contrast to the usual procedure in finite dimensions, we gain additional freedom as the basis can be chosen in a suitable way. For any index set $K \subset \mathbb{N}$ we denote the canonical projection to the subspace H_K by $\pi_K : x \mapsto \sum_{k \in K} \langle x, e_k \rangle e_k$ and for $x \in H$ we simply write $x_K = \pi_K x$. Assume that the state space of \mathcal{X} is the direct sum

$$\mathcal{X} = H_I^+ \oplus H_J \quad (3)$$

where $I, J \subset \mathbb{N}$ are two disjoint sets such that $I \cup J = \mathbb{N}$, and $H_I^+ := \{\sum_{i \in I} \langle x, e_i \rangle e_i : x \in H, \langle x, e_k \rangle \geq 0\} \subset H_I$. Then \mathcal{X} is a total set, i.e. the closure of its span is the full space H and for any $x \in \mathcal{X}$ we obtain the unique decomposition $x = x_I + x_J$. Using this structural assumption on the state space \mathcal{X} , the set \mathcal{U} can be determined precisely as follows: for $x \in H$ we write $x \leq 0$ if $\langle x, e_k \rangle \leq 0$ for all $k \in \mathbb{N}$ and similar for $\langle \cdot, \cdot \rangle$ or \geq . It turns out that under (3),

$$\mathcal{U} = \{u \in H_{\mathbb{C}} : \operatorname{Re}(u_I) \leq 0 \text{ and } \operatorname{Re}(u_J) = 0\}. \quad (4)$$

Moreover, the finite-dimensional affine processes studied in [13] can be viewed as special case with $H = \mathbb{R}^n$ and $\mathcal{X} = \mathbb{R}_{\geq 0}^i \oplus \mathbb{R}^j$ and $i + j = n$.

Remark 2.1. Fix $t \geq 0$. If X is affine and the state space satisfies (3), then it follows from Equation (2) that, for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$,

$$\begin{aligned} e^{\operatorname{Re}(\phi(t, u) + \langle \psi(t, u), x \rangle_{H_{\mathbb{C}}})} &= |e^{\phi(t, u) + \langle \psi(t, u), x \rangle_{H_{\mathbb{C}}}}| \\ &\leq \int |e^{\langle u, y \rangle_{H_{\mathbb{C}}}}| p_t(x, dy) \leq \int e^{\operatorname{Re}\langle u, y \rangle_{H_{\mathbb{C}}}} p_t(x, dy) \leq 1, \end{aligned}$$

since $\operatorname{Re}\langle u, y \rangle_{H_{\mathbb{C}}} \leq 0$ for $u \in \mathcal{U}$ and $y \in \mathcal{X}$. Hence, $\operatorname{Re}(\phi(t, u) + \langle \psi(t, u), x \rangle_{H_{\mathbb{C}}}) \leq 0$ for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$ which is equivalent to $(\phi(t, u), \psi(t, u)) \in \mathbb{C}_- \times \mathcal{U}$ for all $u \in \mathcal{U}$.

We are interested in those Markov processes which are strong solutions of stochastic differential equations with respect to an infinite-dimensional Brownian motion. We follow the construction of a stochastic integral laid out in [12]. To this end, denote the trace of a symmetric and non-negative operator Q by $\operatorname{Tr} Q = \sum_{i=1}^{\infty} \langle Q e_i, e_i \rangle$ and call the operator Q trace-class if $\operatorname{Tr} Q < \infty$. Let W be an H -valued \mathbb{F} -Brownian motion with covariance operator Σ_W , i.e. Σ_W is a symmetric and non-negative definite operator Σ_W with $\operatorname{Tr} \Sigma_W < \infty$. Then there exists $\Sigma_W^{1/2}$ such that $\Sigma_W = \Sigma_W^{1/2} (\Sigma_W^{1/2})^*$. Denote $H_0 := \Sigma_W^{1/2} H$ and by $HS(H_0; H)$ the space of all Hilbert-Schmidt operators from H_0 to H , i.e. linear operators Q such that $\sum_{i=1}^{\infty} \langle Q \Sigma_W^{1/2} e_k, \Sigma_W^{1/2} e_k \rangle^2 < \infty$.

We assume, that for each $x_0 \in \mathcal{X}$, $X = X^{x_0}$ is the unique strong solution to the stochastic differential equation

$$\begin{aligned} dX_t &= \mu(X_t) dt + \sigma(X_t) dW_t, \\ X_0 &= x_0 \end{aligned} \quad (5)$$

where $\mu : H \rightarrow H$ and $\sigma : H \rightarrow HS(H_0; H)$ are continuous; compare Theorem 3.1 for precise conditions ensuring the existence of a unique strong solution.

By $S(\cdot) := \sigma(\cdot) \Sigma_W \sigma(\cdot)^*$ we denote the dispersion operator of X , such that $d[X, X]_t = S(X_t) dt$. The next result shows that $S(x)$ is a trace-class operator for each $x \in \mathcal{X}$ and that $x \mapsto \operatorname{Tr} S(x)$ is a real-valued and continuous function.

Lemma 2.1. *For each $x \in \mathcal{X}$ the operator $S(x)$ is non-negative definite and trace-class. Moreover, the mapping $\text{Tr } S(\cdot) : H \rightarrow \mathbb{R}$ is continuous.*

Proof. Note that Σ_W is a symmetric, non-negative definite and trace-class operator. Then, it follows that for $x \in \mathcal{X}$ and $h \in H$

$$hS(x)h^* = (h\sigma(x))\Sigma_W(h\sigma(x))^* \geq 0$$

such that $S(x)$ is also symmetric and non-negative definite. We denote $Q = \Sigma_W^{1/2}$ such that $\Sigma_W = QQ^*$. From the cyclic property of the trace and the Cauchy-Schwartz inequality it follows that

$$\begin{aligned} \text{Tr } S(x) &= \text{Tr}((\sigma(x)Q)(\sigma(x)Q)^*) = \text{Tr}((\sigma(x)Q)^*(\sigma(x)Q)) \\ &\leq \text{Tr}(\sigma(x)\sigma(x)^*) \cdot \text{Tr}(QQ^*) = \text{Tr}(\sigma(x)\sigma(x)^*) \text{Tr } \Sigma_W < \infty, \end{aligned}$$

because for each $x \in H$, $\sigma(x) \in HS(H_0; H)$, and hence $\text{Tr}(\sigma(x)\sigma(x)^*) < \infty$. The continuity from $S(x)$ now follows from the continuity of $\sigma(x)$. \square

Theorem 2.2. *Assume that the process X , given as unique strong solution of (5), is affine. Then for all $x \in \mathcal{X}$ it holds that*

$$\begin{aligned} \mu(x) &= m_0 + Mx \\ S(x) &= n_0 + Nx \end{aligned} \tag{6}$$

with $m_0 \in \mathcal{X}$, $M \in L(H)$, $n_0 \in L(H)$ and $N \in L(H, L(H))$. Denote $n_k = Ne_k$ and $m_k = Me_k$, $k = 1, 2, \dots$. The coefficients n_k are symmetric, non-negative definite and trace-class operators which satisfy $n_j = 0$ for all $j \in J$ and

$$\sum_{i \in I} (\text{Tr } n_i)^2 < \infty. \tag{7}$$

The functions ϕ and $\psi_k(t, u) := \langle \psi(t, u), e_k \rangle_{H_{\mathbb{C}}}$, $k = 1, 2, \dots$ satisfy the general Riccati system

$$\begin{aligned} \partial_t \phi(t, u) &= \langle m_0, \overline{\psi(t, u)} \rangle_{H_{\mathbb{C}}} + \frac{1}{2} \langle n_0 \psi(t, u), \overline{\psi(t, u)} \rangle_{H_{\mathbb{C}}} \\ \phi(0, u) &= 0 \end{aligned} \tag{8}$$

$$\begin{aligned} \partial_t \psi_k(t, u) &= \langle m_k, \overline{\psi(t, u)} \rangle_{H_{\mathbb{C}}} + \frac{1}{2} \langle n_k \psi(t, u), \overline{\psi(t, u)} \rangle_{H_{\mathbb{C}}}, \quad k = 1, 2, \dots \\ \psi(0, u) &= u, \end{aligned} \tag{9}$$

for all $t \geq 0$ and $u \in \mathcal{U}$.

For the proof we need a small result for the interchange of the derivative and the scalar product. It essentially shows that if ψ is differentiable then it is also weakly differentiable.

Lemma 2.3. *Let $\psi : \mathbb{R}_{\geq 0} \rightarrow H$ be a Fréchet differentiable function with derivative $D_t \psi$. Then, for all $h \in H$ it holds that*

$$\partial_t \langle \psi(t), h \rangle = \langle D_t \psi(t), h \rangle. \tag{10}$$

Proof. Fix $h \in H$ and denote $f(t) := \langle \psi(t), h \rangle$. Using the Cauchy-Schwartz inequality we obtain that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{|f(t + \varepsilon) - f(t) - \varepsilon \langle D_t \psi(t), h \rangle|}{|\varepsilon|} &= \lim_{\varepsilon \rightarrow 0} \frac{|\langle \psi(t + \varepsilon) - \psi(t) - \varepsilon D_t \psi(t), h \rangle|}{|\varepsilon|} \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{\| \psi(t + \varepsilon) - \psi(t) - \varepsilon D_t \psi(t) \|}{|\varepsilon|} \| h \| = 0 \end{aligned}$$

and the claim follows. \square

Proof of Theorem 2.2. When X is an affine process, then the processes

$$M_t^u := \exp(\phi(T-t, u) + \langle \psi(T-t, u), X_t \rangle_{H_C}), \quad 0 \leq t \leq T$$

are martingales for all $u \in \mathcal{U}$ since $M_t^u = \mathbb{E}[\exp(\langle u, X_T \rangle_{H_C}) | \mathcal{F}_t] = \mathbb{E}[M_T^u | \mathcal{F}_t]$. Next, we apply the Itô-formula, see Theorem 4.32 in [12], to M_t^u with $f(t, x) = \exp(\phi(T-t, u) + \langle \psi(T-t, u), x \rangle_{H_C})$. Note that, by Lemma 2.1,

$$\partial_t f(t, x) = f(t, x) (-\partial_t \phi(T-t, u) - \langle D_t \psi(T-t, u), x \rangle_{H_C}).$$

Hence,

$$dM_t^u = M_t^u (I_t dt + \langle \psi(T-t, u), \sigma(X_t) dW_t \rangle_{H_C}), \quad (11)$$

where the drift computes to

$$\begin{aligned} I_t &= -\partial_t \phi(T-t, u) - \langle D_t \psi(T-t, u), X_t \rangle_{H_C} + \langle \psi(T-t, u), \mu(X_t) \rangle_{H_C} \\ &\quad + \frac{1}{2} \sum_{k=1}^{\infty} \langle S(X_t) \psi(T-t, u), e_k \rangle_{H_C} \langle \psi(T-t, u), e_k \rangle_{H_C}. \end{aligned} \quad (12)$$

The infinite sum equals $\langle S(X_t) \psi(T-t, u), \overline{\psi(T-t, u)} \rangle_{H_C}$. Moreover, the process M^u is a martingale only if $I_t = 0$ $dt \otimes dP$ -almost surely. By continuity of I it follows even that $I = 0$ P -almost surely. Letting $t \rightarrow 0$, continuity of X, μ, S, ϕ , and ψ implies that

$$\partial_t \phi(t, u) + \langle D_t \psi(t, u), x \rangle_{H_C} = \langle \psi(t, u), \mu(x) \rangle_{H_C} + \frac{1}{2} \langle S(x) \psi(t, u), \overline{\psi(t, u)} \rangle_{H_C} \quad (13)$$

holds for all $x \in \mathcal{X}$ and all $t \geq 0$. The left hand side is an affine function of x , and hence the right-hand side is affine in x . Using that $\psi(0, u) = u$ we obtain that μ as well as S are affine functions of x , such that representation (6) follows. Continuity of μ yields that $M \in L(H)$. Moreover, by Lemma 2.1, $S(x)$ is a symmetric, non-negative definite and trace-class operator for all $x \in \mathcal{X}$. This gives that $n_j = N e_j = 0$ for all $j \in J$. Regarding (7), it follows

$$\text{Tr } n_0 + \sum_{i \in I} \langle x, e_i \rangle_H \text{Tr } n_i = \text{Tr } S(x) < \infty$$

for all $x \in \mathcal{X}$, because S is trace-class. Define $T_n x := \sum_{i \in I, i \leq n} \langle x, e_i \rangle_H \text{Tr } n_i$. Then $T_n \in L(H, \mathbb{R})$ and

$$\sup_{n \in \mathbb{N}} \|T_n x\| \leq \sum_{i \in I} |\langle x, e_i \rangle_H| \text{Tr } n_i < \infty \quad \text{for all } x \in H.$$

By the uniform boundedness principle it follows that $\sum_{i \in I} (\text{Tr } n_i)^2 = \sup_{n \in \mathbb{N}} \|T_n\|^2 < \infty$ such that (7) follows.

Finally, inserting (6) into (13) and separating terms gives (8)-(9) since the affine hull of \mathcal{X} is the full space H , where again Lemma 2.1 was used. \square

The converse is solved in two steps. First, we derive some admissibility conditions for the coefficients of the Riccati equations (8)-(9), which are equivalent to affinity in the canonical state space. Second, we show that these admissibility conditions are sufficient for existence and uniqueness of solutions for the Riccati equations.

Proposition 2.4. *Assume that X is a strong solution of (5), μ and S are affine in the sense of (6) and the Riccati system (8)-(9) has a solution (ϕ, ψ) such that $\phi(t, u) + \langle \psi(t, u), x \rangle_{H_C}$ has a non-negative real part for all $t \geq 0, u \in \mathcal{U}$ and $x \in \mathcal{X}$. Then X is an affine process.*

Proof. If (6) and (8)-(9) hold such that $\phi(t, u) + \langle \psi(t, u), x \rangle_{H_C}$ has a non-negative real part for all $t \geq 0, u \in \mathcal{U}$ and $x \in \mathcal{X}$, then it follows as in the proof of Theorem 2.2, that the drift I of the process M^u given in (11) vanishes, such that

$$dM_t^u = M_t^u \langle \psi(T-t, u), \sigma(X_t) dW_t \rangle_{H_C}.$$

Hence, M_t^u is a continuous local martingale. From the assumption that $\text{Re}(\phi(T-t, u) + \langle \psi(T-t, u), X_t \rangle_{H_C}) \leq 0$, it follows that

$$M_t^u = \exp(\phi(T-t, u) + \langle \psi(T-t, u), X_t \rangle_{H_C})$$

is uniformly bounded by 1 and hence M^u is even a true martingale. Consequently, for all $t \geq 0$

$$E[e^{\langle u, X_T \rangle_{H_C}} | \mathcal{F}_t] = E[M_T^u | \mathcal{F}_t] = M_t^u = \exp(\phi(T-t, u) + \langle \psi(T-t, u), X_t \rangle_{H_C}).$$

Then (2) holds and X is an affine process. \square

The next result gives a partial answer to the solvability of the system of Riccati equations (8)-(9). We start with some notation. First, define

$$H_{\bar{C}} = \{z = x + iy \in H_C \mid \langle x, e_k \rangle_H \leq 0 \text{ for all } k \in \mathbb{N}\}.$$

As previously, $(H_{\bar{C}})_I$ denotes the projection to the coordinates from set I , i.e. $(H_{\bar{C}})_I = \{\sum_{i \in I} \langle z, e_i \rangle_{H_C} e_i \mid z \in H_{\bar{C}}\}$. Second, for the two index sets $K, L \subset \mathbb{N}$ and generic $A \in L(H)$ we denote $A_{KL} = \pi_K A|_{H_L}$. Then $A_{K \cup L}$ may be uniquely represented by the 2×2 block operator matrix

$$\begin{pmatrix} A_{KK} & A_{KL} \\ A_{LK} & A_{LL} \end{pmatrix}.$$

If the index sets are singletons, we write A_{kl} for $A_{\{k\}\{l\}}$. Finally, for $x \in H$ we understand $x \leq 0$ as $\langle x, e_k \rangle \leq 0$ for all $k \in \mathbb{N}$ and $x \geq 0, x < 0, x > 0, x \not\leq 0, x \not\geq 0, x = 0$ in the same manner.

Proposition 2.5. *Assume that (3) and the following admissibility conditions hold:*

$$m_0 \in \mathcal{X}, m_i \in H_{I \setminus \{i\}}^+ \oplus H_{J \cup \{i\}} \text{ for } i \in I, \text{ and } m_j \in H_J \text{ for } j \in J \quad (14)$$

$$\left\| \sum_{k \geq 1} m_k \langle \cdot, e_k \rangle \right\| < \infty \quad (15)$$

$$n_k \in L(H) \text{ is symmetric, non-negative definite and of trace class, } k \in \mathbb{N}, \quad (16)$$

$$n_j = 0 \text{ for } j \in J, \quad (17)$$

$$n_{0,II} = 0,$$

$$n_{0,IJ} = n_{0,JI}^* = 0,$$

$$n_{0,JJ} \text{ is symmetric, non-negative definite and of trace class,}$$

$$n_{i,\{kl\}} = \begin{cases} \geq 0 & \text{if } i = k = l, \\ = 0 & \text{otherwise,} \end{cases} \quad \text{for } i, k, l \in I, \quad (18)$$

$$n_{i,IJ} = n_{i,JI}^*$$

$$n_{i,JJ} \text{ is symmetric, non-negative definite and of trace class,}$$

$$\sum_{i \in I} \|n_i\|^2 < \infty. \quad (19)$$

Then the general Riccati system (8)-(9) has a unique solution $(\phi(\cdot, u), \psi(\cdot, u)) : \mathbb{R}_+ \rightarrow \mathbb{C}_- \times (H_{\bar{C}})_I \oplus iH_J$ for each $u \in (H_{\bar{C}})_I \oplus iH_J$.

These conditions directly correspond to the well-known conditions in the finite-dimensional case, see [17], with additional assumptions on summability of certain coefficients, (15), and (19). Regarding (14), this can be seen as follows: note that for $i \in I$ and $j \in J$, $M_{II}e_i = \pi_I M|_{H_I} e_i = \pi_I m_i \in H_{I \setminus \{i\}}^+ \oplus H_{\{i\}}$ as well as $M_{IJ}e_j = \pi_I M|_{H_J} e_j = \pi_I m_j = 0$ because $m_j \in H_J$. This corresponds to the condition of \mathcal{B}_{II} having nonnegative off-diagonal elements and $\mathcal{B}_{IJ} = 0$ of Theorem 10.2 in [17] (in the notation used there).

Condition (19) is always satisfied in the finite-dimensional case and appears here for the first time in literature. Denote the eigenvalues of the trace-class operator n by $\lambda_i, i \geq 1$. If n is also symmetric and non-negative definite, then

$$\|n\| \leq \sum_{i \geq 1} |\lambda_i| = \text{Tr } n. \quad (20)$$

Hence, a sufficient criterion for $\sum_{i \in I} \|n_k\|^2 < \infty$ is $\sum_{i \in I} (\text{Tr } n_i)^2 < \infty$.

The proof is separated in a number of smaller results. Set $f(\xi) = \frac{1}{2} \sum_{i \in I} \langle n_i \xi, \bar{\xi} \rangle_{H_{\mathbb{C}}} e_i$. Then the Riccati equations in (9) are equivalent to the following semilinear evolution equation

$$\partial_t \psi(t, u) = M^\top \psi(t, u) + f(\psi(t, u)), \quad u \in \mathcal{U}, \quad t \geq 0 \quad (21)$$

with initial condition $\psi(0, u) = u$. Such equations have been studied in [39] and Theorem 1 therein yields the following result.

Lemma 2.6. *Assume that (15) and (19) are satisfied. Then, for each $u \in \mathcal{U}$, (21) has a unique solution $\psi(t, u)$ on some interval $[0, T_u)$ with existence time $T_u \in (0, \infty]$.*

Proof. We will first show that $K_t := e^{tM^\top} f$, $t \geq 0$ is locally Lipschitz-continuous. To this end, note that for ξ and η in the domain of f ,

$$\begin{aligned} \|e^{tM^\top} f(\xi) - e^{tM^\top} f(\eta)\|_{H_{\mathbb{C}}}^2 &\leq \frac{1}{4} e^{2\|M\|t} \sum_{i \in I} |\langle n_i \xi, \bar{\xi} \rangle_{H_{\mathbb{C}}} - \langle n_i \eta, \bar{\eta} \rangle_{H_{\mathbb{C}}}|^2 \\ &= \frac{1}{4} e^{2\|M\|t} \sum_{i \in I} |\langle n_i \xi, \bar{\xi} \rangle_{H_{\mathbb{C}}} - \langle n_i \eta, \bar{\xi} \rangle_{H_{\mathbb{C}}} + \langle n_i \eta, \bar{\xi} \rangle_{H_{\mathbb{C}}} - \langle n_i \eta, \bar{\eta} \rangle_{H_{\mathbb{C}}}|^2 \\ &= \frac{1}{4} e^{2\|M\|t} \sum_{i \in I} |\langle n_i (\xi + \eta), \overline{(\xi - \eta)} \rangle_{H_{\mathbb{C}}}|^2 \\ &\leq \frac{1}{4} e^{2\|M\|t} \sum_{i \in I} \|n_i\|^2 \|\xi + \eta\|_{H_{\mathbb{C}}}^2 \|\xi - \eta\|_{H_{\mathbb{C}}}^2. \end{aligned}$$

Hence, for each $t \geq 0$, K_t is locally Lipschitz-continuous by (19) and its Lipschitz constant on $U_\alpha = \{x \in H : \|x\| \leq \alpha\}$, $\alpha > 0$ is bounded by $\alpha e^{\|M\|t} (\sum_{i \in I} \|n_i\|^2)^{1/2}$.

Theorem 1 in [39] now yields that, for each $u \in \mathcal{U}$, equation (21) possesses a unique solution $\psi(t, u)$ on some interval $[0, T_u)$ with $0 < T_u \leq \infty$ and the proof is finished. \square

Lemma 2.7. *Assume that the admissibility conditions (15) - (19) are satisfied. Then, for all $t \in [0, T_u)$ and $u \in \mathcal{U}$ it holds that the unique solution of (21), $\psi(t, u)$, satisfies that $\psi(t, u) \in \mathcal{U}$.*

Proof. To begin with, we note that by Lemma 2.6, (21), has, for $u \in \mathcal{U}$ a unique solution $\psi(t, u)$ on $[0, T_u)$. To show that $\psi(t, u) \in \mathcal{U}$, we utilize (4), hence, have to show that $\text{Re}(\psi_J(t, u)) = 0$ and $\text{Re}(\psi_I(t, u)) \leq 0$.

First, for $j \in J$, we obtain by (17), that the projection $\psi_J(t, u) = \pi_J \psi(t, u)$ satisfies the autonomous equation

$$\psi_J(t, u) = M_{JJ}^* \psi_J(t, u), \quad t \geq 0, \quad u \in \mathcal{U},$$

with $\psi_J(0, u) = u_J$. The unique solution of this equation is given by

$$\psi_J(t, u) = e^{tM_{JJ}^*} u_J, \quad t \geq 0.$$

From (4) it follows $\text{Re}(u_J) = 0$ and hence $\text{Re}(\psi_J(t, u)) = 0$.

As a second step we show that $\text{Re}(\psi_I(t, u)) \leq 0$ which requires more work. We start with the observation that, for $i \in I$,

$$\begin{aligned} \partial_t \text{Re}(\psi_i(t, u)) &= \langle m_i, \text{Re}(\psi(t, u)) \rangle \\ &+ \frac{1}{2} \langle n_i \text{Re}(\psi(t, u)), \text{Re}(\psi(t, u)) \rangle - \frac{1}{2} \langle n_i \text{Im}(\psi(t, u)), \text{Im}(\psi(t, u)) \rangle \\ &\leq \langle m_i, \text{Re}(\psi(t, u)) \rangle + \frac{1}{2} n_{i, \{ii\}} (\text{Re}(\psi_i(t, u)))^2, \end{aligned} \quad (22)$$

using (17) and (18).

Next, consider $\varepsilon > 0$ and $u \in \mathcal{U}$ such that $\text{Re}(u_i) < -\varepsilon$ for all $i \in I$. Let

$$T'_u = \inf\{t \in [0, T_u) : \exists i \in I \text{ s.t. } \text{Re}(\psi_i(t, u)) \geq 0\} \quad (23)$$

with the convention that $\inf \emptyset = \infty$. Since $\text{Re}(\psi_i(t, u))$ is continuous at $t = 0$ and $\text{Re}(\psi_i(0, u))$ bounded away from zero, $T'_u > 0$. Hence, for $t \in [0, T'_u)$, it follows that $\text{Re}(\psi_I(t, u)) < 0$ for any subset $I' \subset I$.

By Assumption (14), $m_i \in H_{I \setminus \{i\}}^+ \oplus H_{J \cup \{i\}}$. Hence, there exist $m'_i \in H_{I \setminus \{i\}}^+$ and $m''_i \in H_J$ such that $m_i = m'_i + m_{i,i} + m''_i$. Then $\langle m'_i, \text{Re}(\psi_{I \setminus \{i\}}(t, u)) \rangle \leq 0$, $\langle m''_i, \text{Re}(\psi_J(t, u)) \rangle = 0$ and therefore

$$\langle m_i, \text{Re}(\psi(t, u)) \rangle \leq \langle m_{i,i}, \text{Re}(\psi_i(t, u)) \rangle.$$

Together with (22) and $C_i = \frac{1}{2} \max\{n_{i, \{ii\}}, |m_{i,i}|\}$, we are able to achieve the following estimate,

$$\begin{aligned} \partial_t \text{Re}(\psi_i(t, u)) &\leq \langle m_{i,i}, \text{Re}(\psi_i(t, u)) \rangle + \frac{1}{2} n_{i, \{ii\}} (\text{Re}(\psi_i(t, u)))^2 \\ &\leq C_i ((\text{Re}(\psi_i(t, u)))^2 - 2 \text{Re}(\psi_i(t, u))), \end{aligned}$$

where we used $\text{Re}(\psi_i(t, u)) < 0$. By the comparison theorem, [4](Chapter 1, Theorem 7), $\text{Re}(\psi_i(t, u)) \leq g(t, u_i, C_i)$ for all $t \in [0, T'_u)$, where $g(t, u, C) =: g(t)$ solves

$$\begin{aligned} \partial_t g(t) &= C(g(t)^2 - 2g(t)), \\ g(0) &= \text{Re}(u), \end{aligned}$$

with $C \geq 0$ and $\text{Re}(u) < 0$. The unique solution of this Riccati equation is given by $g(t) = 2u(2e^{2Ct} - u(e^{2Ct} - 1))^{-1}$. The function $g(\cdot, u, C)$ stays negative on the whole real line when $\text{Re}(u) < 0$. Moreover, g is increasing in u and C such that we obtain that

$$\text{Re}(\psi_I(t, u)) \leq \sup_{i \in I} g(t, u_i, C_i) \leq g(t, -\varepsilon, C^*) < 0$$

for $t \in [0, T'_u)$ where $C^* = \sup_{i \in I} C_i \leq \frac{1}{2} \sup_{i \in I} (\|n_i\| + \|m_i\|) \leq \frac{1}{2} ((\sum_{i \in I} \|n_i\|^2)^{1/2} + \|M\|) < \infty$ by (15) and (19). Using continuity of $\text{Re}(\psi_I(t, u))$, we obtain that at $t = T'_u$, $\text{Re}(\psi_I(t, u)) < 0$, if $T'_u < \infty$. By the very definition of T'_u in (23), this implies that $T'_u = \infty$.

Summarizing, we obtained up to now that for $u \in \mathcal{U}$ with $\text{Re}(u_I) < -\varepsilon$ it follows that $\text{Re}(\psi_I(t, u)) \leq 0$ for all $t \in [0, T_u)$. The next step is to extend this result to all $u \in \mathcal{U}$.

In this regard, consider arbitrary $u \in \mathcal{U}$, a sequence $(\varepsilon_n) \downarrow 0$ and a sequence $u_n \rightarrow u$ satisfying $\operatorname{Re}(u_n) < -\varepsilon_n$ for all $n \geq 1$. By part (v) of Theorem 1 in [39], $\psi(t, u)$ is Lipschitz continuous on some neighborhood of u , uniformly on each compact interval $[0, T]$, $T < T_u$. Therefore, $\psi(t, u_n) \rightarrow \psi(t, u)$ for each $t \in [0, T_u)$. Hence,

$$\operatorname{Re}(\psi_I(t, u)) = \lim_{n \rightarrow \infty} \operatorname{Re}(\psi_I(t, u_n)) \leq 0, \quad t \in [0, T_u)$$

and the claim is proved. \square

Lemma 2.8. *Assume that the admissibility conditions (15) - (19) are satisfied. Then, for all $t \in [0, T_u)$ and $u \in \mathcal{U}$ it holds that the unique solution of (21), $\psi(t, u)$, satisfies the following inequality*

$$\|\psi_I(t, u)\|_{H_C}^2 \leq \|u_I\|_{H_C}^2 + C(1 + \|u_I\|_{H_C}^2) \int_0^t h_u(s) e^{C \int_s^t h_u(r) dr} ds, \quad (24)$$

where $h_u(t) = (1 + \|\psi_J(t, u)\|_{H_C}^2 + \|\psi_J(t, u)\|_{H_C}^4)$ with $\psi_J(t, u) = e^{tM^*} u_J$ and $C = \sum_{i \in I} \|A_i\|^2 + \|M\|^2 + \frac{7}{2}$.

The proof of this lemma is relegated to the appendix. Finally, we show that the unique solution exists on the whole real line, thus completing the proof of Proposition 2.5.

Proof of Proposition 2.5. First, we show that $T_u = \infty$. The proof bases on result (iv) of Theorem 1 in [39], saying that $\lim_{t \rightarrow T_u} \|\psi_I(t, u)\|_{H_C} = \infty$ if $T_u < \infty$. In this regard, note that the right hand side of Equation (24) is finite for all $t \geq 0$. Hence, the existence time T_u of $\psi_I(t, u)$ for $u \in \mathcal{U}$ must be infinite, i.e. $T_u = \infty$.

This shows existence and uniqueness regarding ψ . Existence and uniqueness for ϕ directly follow by integration. At last, we show $\operatorname{Re} \phi(t, u) \leq 0$ for all $t \geq 0$ and $u \in \mathcal{U}$: integrate the real part of (8) and consider the admissibility conditions to get

$$\begin{aligned} & \operatorname{Re} \phi(t, u) \\ &= \int_0^t \langle m_0, \operatorname{Re} \psi(s, u) \rangle + \frac{1}{2} \langle n_0 \operatorname{Re} \psi(s, u), \operatorname{Re} \psi(s, u) \rangle - \frac{1}{2} \langle n_0 \operatorname{Im} \psi(s, u), \operatorname{Im} \psi(s, u) \rangle ds \\ &= \int_0^t \langle m_{0,I}, \operatorname{Re} \psi_I(s, u) \rangle - \frac{1}{2} \langle n_{0,JJ} \operatorname{Im} \psi_J(s, u), \operatorname{Im} \psi_J(s, u) \rangle ds \leq 0. \end{aligned}$$

for all $t \geq 0$ and $u \in \mathcal{U}$. \square

Proposition 2.9. *Assume that X is a strong solution of (5), μ and S are affine as in (6), and that the Riccati system (8)-(9) has a solution $(\phi(t, u), \psi(t, u)) \in \mathbb{C}_- \times \mathcal{U}$ for all $t \geq 0, u \in \mathcal{U}$ and $x \in \mathcal{X}$. Then the admissibility conditions in Proposition 3.5 hold.*

Proof. First of all, by Lemma 2.1, $S(x)$ is a symmetric, non-negative definite and trace-class operator for all $x \in \mathcal{X}$. This gives that $n_j = Ne_j = 0$ for all $j \in J$. Moreover, from the Riccati equations (8)-(9) we obtain that

$$\partial_t \operatorname{Re} \phi(0, u) = \langle m_0, v \rangle + \frac{1}{2} \langle n_0 v, v \rangle - \frac{1}{2} \langle n_0 w, w \rangle, \quad (25)$$

$$\partial_t \operatorname{Re} \psi_i(0, u) = \langle m_i, v \rangle + \frac{1}{2} \langle n_i v, v \rangle - \frac{1}{2} \langle n_i w, w \rangle, \quad (26)$$

$$\partial_t \operatorname{Re} \psi_j(0, u) = \langle m_j, v \rangle, \quad (27)$$

where $u \in \mathcal{U}$, and we set $v = \operatorname{Re} u, w = \operatorname{Im} u$. From (4), together with (9), we obtain from (27) that $\operatorname{Re} \psi_j(\cdot, u) \equiv 0$ for all $j \in J$. This implies that $\langle m_j, v \rangle = 0$. Again from (4) we obtain that $v_J = 0$ while $v_I \leq 0$. Hence, $m_j \in H_J$ for all $j \in J$.

Next, we consider (26). As already noted in Remark 2.1, $\operatorname{Re} \psi_i(\cdot, u) \leq 0$ for all $i \in I$, such that $\partial_t \operatorname{Re} \psi_i(0, u) \leq 0$ whenever $\operatorname{Re} \psi_i(0, u) = \operatorname{Re} u_i = v_i = 0$. Choose $u = v + iw$ such that $v_{I \setminus \{i\}} < 0$, $v_{J \cup \{i\}} = 0$ and $w = 0$. Substituting such u 's into (26) leads to

$$0 \geq \partial_t \operatorname{Re} \psi_i(0, u) = \langle m_i, v_{I \setminus \{i\}} \rangle + \frac{1}{2} \langle n_i v_{I \setminus \{i\}}, v_{I \setminus \{i\}} \rangle. \quad (28)$$

This implies that $\langle n_i v_{I \setminus \{i\}}, v_{I \setminus \{i\}} \rangle = 0$: indeed, if $\langle n_i v_{I \setminus \{i\}}, v_{I \setminus \{i\}} \rangle \neq 0$, there would exist a $v_{I \setminus \{i\}}$, such that

$$\langle n_i v_{I \setminus \{i\}}, v_{I \setminus \{i\}} \rangle > 0,$$

and for $\gamma > 0$ large enough, (28) would lead to a contradiction that

$$0 \geq \partial_t \operatorname{Re} \psi_i(0, \gamma u) = \langle m_i, v_{I \setminus \{i\}} \rangle \gamma + \frac{1}{2} \langle n_i v_{I \setminus \{i\}}, v_{I \setminus \{i\}} \rangle \gamma^2 > 0.$$

Now that we have shown $n_{i, I \setminus \{i\} I \setminus \{i\}} = 0$, it follows from the non-negative definiteness of n_i that

$$n_{i, \{kl\}} = \begin{cases} \geq 0 & \text{if } i = k = l, \\ = 0 & \text{otherwise,} \end{cases} \quad \text{for } i, k, l \in I.$$

The rest conditions on n_i , such as $n_{i, II} = n_{i, II}^*$ and $n_{i, JJ}$ is symmetric, non-negative definite and of trace class, can be easily seen from its non-negative definiteness as well. Furthermore, because of $n_{i, I \setminus \{i\} I \setminus \{i\}} = 0$, (28) gives

$$\langle m_i, v_{I \setminus \{i\}} \rangle \leq 0.$$

Then we conclude that $m_i \in H_{I \setminus \{i\}}^+ \oplus H_{J \cup \{i\}}$, since $v_{I \setminus \{i\}}$ is chosen to be arbitrarily negative. Finally, we look at (25). Since $\operatorname{Re} \phi(\cdot, u) \leq 0$ and $\operatorname{Re} \phi(0, u) = 0$, we may employ the same reason as for (26) to detect $\partial_t \operatorname{Re} \phi_i(0, u) \leq 0$ for all $u \in \mathcal{U}$. Especially, we choose $u = v + iw$ with $v_I < 0$, $v_J = 0$ and $w = 0$ and get

$$0 \geq \partial_t \operatorname{Re} \phi_i(0, u) = \langle m_0, v_I \rangle + \frac{1}{2} \langle n_0 v_I, v_I \rangle. \quad (29)$$

An analogous argument applied to (28) shows that $n_{0, II} = 0$. Besides, the affine form condition tells that n_0 is a symmetric, non-negative definite, trace-class operator, which implies that $n_{0, JJ}$ must be such one as well and $n_{0, IJ} = n_{0, IJ}^* = 0$ due to $n_{0, II} = 0$. Moreover, such an n_0 turns (29) to

$$\langle m_0, v_I \rangle = \partial_t \operatorname{Re} \phi_i(0, u) \leq 0.$$

Then m_0 must be an element in \mathcal{X} , because $v_I < 0$ is arbitrary. □

Remark 2.2. Consider the canonical state space \mathcal{X} and assume X to be a strong solution of (5). Then the affinity property of X is equivalent to the admissibility conditions. The sufficiency is deduced by Theorem 2.2 and Lemma 2.6 and the necessity results from Proposition 2.5 and Proposition 2.4. Both Theorem 2.2 and Proposition 2.5 indicate that the both equivalent statements imply the existence and uniqueness of solutions of the Riccati equations (8)-(9).

Remark 2.3. By Theorem 2.2 the parameters m_0, M, n_0, N in (6) determine the law of the process X . Indeed, these parameters determine the functions $\phi(\cdot, u) : \mathbb{R}_+ \rightarrow H_{\mathbb{C}}$ and

$\psi_k(\cdot, u) : \mathbb{R}_+ \rightarrow H_{\mathbb{C}}$ for $k \in \mathbb{N}$ as solutions of the Riccati equations (8) and (9) for all $u \in \mathcal{U}$, and hence by (2) for all $0 \leq s < t$ and $u, v \in \mathcal{U}$ we have

$$\begin{aligned} \mathbb{E} \left[e^{\langle u, X_s \rangle_{H_{\mathbb{C}}} + \langle v, X_t \rangle_{H_{\mathbb{C}}}} \right] &= \int_H \int_H e^{\langle u, y \rangle_{H_{\mathbb{C}}} + \langle v, z \rangle_{H_{\mathbb{C}}}} p_{t-s}(y, dz) p_s(x, dy) \\ &= \int_H \left(\int_H e^{\langle v, z \rangle_{H_{\mathbb{C}}}} p_{t-s}(y, dz) \right) e^{\langle u, y \rangle_{H_{\mathbb{C}}}} p_s(x, dy) \\ &= \int_H \left(\exp(\phi(t-s, v) + \langle \psi(t-s, v), y \rangle_{H_{\mathbb{C}}}) \right) e^{\langle u, y \rangle_{H_{\mathbb{C}}}} p_s(x, dy) \\ &= \exp(\phi(t-s, v)) \int_H e^{\langle \psi(t-s, v) + u, y \rangle_{H_{\mathbb{C}}}} p_s(x, dy) \\ &= \exp(\phi(t-s, v)) \exp(\phi(s, u + \psi(t-s, v)) + \langle \psi(s, u + \psi(t-s, v)), x \rangle_{H_{\mathbb{C}}}), \end{aligned}$$

and analogously for every finite dimensional family $(X_{t_1}, \dots, X_{t_n})$. In particular, the law of X stays invariant under transformations of the volatility σ which provide the same dispersion operator S .

3. EXISTENCE OF AFFINE PROCESSES

The goal of this section is to provide an existence result for affine processes on Hilbert spaces. More precisely, we will prove the existence of affine processes in the spirit of strong solutions to infinite dimensional SDEs. In Subsection 3.1 we will introduce the general framework and formulate the existence result; see Theorem 3.1 below. Afterwards, Subsection 3.2 is devoted to its proof.

3.1. Formulation of the existence result. Recall that H is a separable Hilbert space with orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$, and that the state space \mathcal{X} is the direct sum $\mathcal{X} = H_I^+ \oplus H_J$, where $I, J \subset \mathbb{N}$ are disjoint index sets such that $I \cup J = \mathbb{N}$. Let $\mu : \mathcal{X} \rightarrow H$ and $\sigma : \mathcal{X} \rightarrow L_2(U_0, H)$ be measurable mappings. Again, starting point is the SDE (see (5))

$$\begin{cases} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0, \end{cases} \quad (30)$$

where W is an U -valued Wiener process on a separable Hilbert space U with some covariance operator $\Sigma_W \in L_1^{++}(U)$, and the space $U_0 := \Sigma_W^{1/2}(U)$ is the separable Hilbert space defined according to Lemma C.2. We define the measurable mapping $S : \mathcal{X} \rightarrow L_1^+(H)$ as

$$S(x) := \sigma(x)\Sigma_W^{1/2}(\sigma(x)\Sigma_W^{1/2})^* \quad \text{for all } x \in \mathcal{X}. \quad (31)$$

In the light of Theorem 2.2, we assume that $\mu : \mathcal{X} \rightarrow H$ and $S : \mathcal{X} \rightarrow L_1^+(H)$ are affine functions: let $m_0 \in H$ and $M \in L(H)$ be such that

$$\mu(x) = m_0 + Mx \quad \text{for all } x \in \mathcal{X}, \quad (32)$$

and let $n_0 \in L_1^+(H)$ and $N \in L(H, L_1(H))$ be such that

$$S(x) = n_0 + Nx \quad \text{for all } x \in \mathcal{X}. \quad (33)$$

We assume that n_0 is self-adjoint, and that for every $x \in \mathcal{X}$ the operator Nx is self-adjoint with $Nx \in L_1^+(H)$. Furthermore, we will assume that μ is inward pointing at boundary points of \mathcal{X} , and that σ is parallel to the boundary at boundary points of \mathcal{X} . Here are the formal definitions. The mapping μ is called *inward pointing* at boundary points of \mathcal{X} if

$$\langle \mu(x), \eta \rangle_H \geq 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } \eta \in H_I^+ \text{ with } \langle x, \eta \rangle_H = 0,$$

and the mapping σ is called *parallel to the boundary* at boundary points of \mathcal{X} if

$$\langle \sigma(x), \eta \rangle_H = 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } \eta \in H_I^+ \text{ with } \langle x, \eta \rangle_H = 0, \quad (34)$$

where we note that $\langle \sigma(x), \eta \rangle_H$ is an operator from $L_2(U_0, \mathbb{R})$. For short, we will call such a mapping μ inward pointing and such a mapping σ parallel. We require these two properties in order to ensure that the closed convex cone \mathcal{X} is invariant for the SDE (30).

For a linear operator $T \in L(H)$ we introduce the notations $T_I := \pi_I T$, $T_J := \pi_J T$ and $T_{II} := T_I|_{H_I}$, $T_{IJ} := T_J|_{H_I}$, $T_{JI} := T_I|_{H_J}$, $T_{JJ} := T_J|_{H_J}$. We define the sequences $\lambda = (\lambda_i)_{i \in I} \subset \mathbb{R}_+$ and $\kappa = (\kappa_i)_{i \in I} \subset \mathbb{R}_+$ as

$$\begin{aligned} \lambda_i &:= \|S(e_i)_{II} e_i\|_H, \quad i \in I, \\ \kappa_i &:= \|S(e_i)_{IJ} e_i\|_H, \quad i \in I. \end{aligned}$$

As we will show, we have $\lambda \in \ell^2(I)$, and there exists a sequence $\nu = (\nu_i)_{i \in I} \subset (0, \infty)$ such that $\nu_i \rightarrow 0$ and

$$\left(\frac{\lambda_i}{\nu_i} \right)_{i \in I} \in \ell^2(I). \quad (35)$$

Let $T \in K^{++}(H_I)$ be the compact linear operator with representation

$$Tx = \sum_{i \in I} \nu_i \langle x, e_i \rangle_H e_i \quad \text{for each } x \in H_I, \quad (36)$$

and let $H_{I,0} := T(H_I)$ be the retracted subspace with compact embedding defined according to Lemma C.2. Furthermore, we set $H_{I,0}^+ := T(H_I^+)$ and $\mathcal{X}_0 := H_{I,0}^+ \oplus H_J$. Our principle assumption for the existence result is as follows.

Assumption 3.1. We suppose that the following conditions are fulfilled:

- (i) We have $U = H$, and the operator Σ_W has a diagonal structure along the orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$.
- (ii) For each $x \in \mathcal{X}$ the operator $\sigma(x)\Sigma_W^{1/2}$ is self-adjoint.
- (iii) The mapping μ has the affine structure (32), and S has the affine structure (33).
- (iv) The mapping μ is inward pointing at boundary points of \mathcal{X} , and the mapping σ is parallel to the boundary at boundary points of \mathcal{X} .
- (v) We have

$$\left(\frac{\kappa_i}{\lambda_i} \right)_{i \in I_0} \in \ell^2(I_0), \quad (37)$$

where $I_0 := \{i \in I : \lambda_i > 0\}$.

- (vi) We have

$$m_{0,I} \in H_{I,0}^+ \quad \text{and} \quad M_{II}T = TM_{II}. \quad (38)$$

- (vii) We have

$$\sum_{i \in I} \|M_{II}^i\|_{H_I'} < \infty, \quad (39)$$

where for each $i \in I$ the continuous linear functional $M_{II}^i \in H_I'$ is given by

$$M_{II}^i x := \langle M_{II} x, e_i \rangle_H, \quad x \in H_I.$$

Conditions (i)–(iv) do not mean severe restrictions. Indeed, the first condition means that the state space of the Wiener process is the same as the state space of the SDE (30), and that its covariance operator has a diagonal form with respect to the given orthonormal basis. This is also typically assumed in finite dimension. The second condition means that for each $x \in \mathcal{X}$ we have $S(x)^{1/2} = \sigma(x)\Sigma_W^{1/2}$, and hence

$$\sigma(x) = S(x)^{1/2}\Sigma_W^{-1/2} \quad \text{for all } x \in \mathcal{X}. \quad (40)$$

As mentioned in Remark 2.3, other choices of the volatility σ with the same dispersion operator S do not change the law of the solution. The condition that μ and S have the affine structures (32) and (33) are natural in the present affine setting, and, as already pointed out, we require that μ is inward pointing and that σ is parallel in order to ensure that the state space \mathcal{X} is invariant for the SDE (30). Condition (37) ensures that we can find a linear transformation $\Lambda \in L(H)$ with $\Lambda(\mathcal{X}) = \mathcal{X}$ such that for the transformed SDE

$$\begin{cases} dY_t &= \bar{\mu}(Y_t)dt + \bar{\sigma}(Y_t)dW_t \\ Y_0 &= y_0, \end{cases} \quad (41)$$

corresponding to $Y = \Lambda X$, the drift $\bar{\mu} : \mathcal{X} \rightarrow H$ has a decomposition

$$\bar{\mu}(y) = \bar{\mu}_{II}(y_I) + \bar{\mu}_J(y), \quad y \in \mathcal{X}$$

with affine mappings $\bar{\mu}_{II} : H_I^+ \rightarrow H_I$ and $\bar{\mu}_J : \mathcal{X} \rightarrow H_J$, and the volatility $\bar{\sigma} : \mathcal{X} \rightarrow L_2(U_0, H)$ has a block diagonal structure

$$\bar{\sigma}(y)u = \bar{\sigma}_{II}(y_I)u_I + \bar{\sigma}_{JJ}(y_I)u_J, \quad y \in \mathcal{X} \text{ and } u \in U_0$$

with mappings $\bar{\sigma}_{II} : H_I^+ \rightarrow L_2(U_{I,0}, H_I)$ and $\bar{\sigma}_{JJ} : H_I^+ \rightarrow L_2(U_{J,0}, H_J)$. This allows us to express the transformed SDE (41) by the two coupled SDEs

$$\begin{cases} dY_{I,t} &= \bar{\mu}_{II}(Y_{I,t})dt + \bar{\sigma}_{II}(Y_{I,t})dW_t \\ Y_{I,0} &= y_{0,I} \end{cases} \quad (42)$$

and

$$\begin{cases} dY_{J,t} &= \bar{\mu}_J(Y_t)dt + \bar{\sigma}_{JJ}(Y_{I,t})dW_t \\ Y_{J,0} &= y_{0,J}, \end{cases} \quad (43)$$

and then our task is essentially reduced to solving the SDE (42), which is feasible by virtue of condition (38). The condition (39) ensures pathwise uniqueness. Now, our main result of this section reads as follows. Concerning the notion of a unique strong solution starting in \mathcal{X}_0 , we refer to Definition B.5.

Theorem 3.1. *Suppose that Assumption 3.1 is fulfilled. Then the affine SDE (30) has a unique strong solution starting in \mathcal{X}_0 .*

If condition (39) is skipped, then we still obtain the existence of a weak solution to the affine SDE (30), but pathwise uniqueness might not be satisfied.

3.2. Proof of the existence result. The goal of this subsection is to provide the proof of Theorem 3.1. The main idea is to apply our version of the Yamada-Watanabe theorem (see Theorem B.1). As already mentioned, after a suitable transformation we may consider the two coupled SDEs (42) and (43). This transformation procedure is similar to that in [19], where existence of affine processes has been proven in finite dimension. After this step, we obtain the existence of weak solutions by using a refined version of a result from [22] (see Theorem B.2), where $H_{I,0}$ serves as the retracted subspace with compact embedding, and pathwise uniqueness follows from a version of the uniqueness result from [42] in infinite

dimension (see Theorem B.3). For what follows, let $\mu : \mathcal{X} \rightarrow H$ and $\sigma : \mathcal{X} \rightarrow L_2(U_0, H)$ be measurable mappings, and let $S : \mathcal{X} \rightarrow L_1^+(H)$ be given by (31).

Proposition 3.2. *Suppose that μ has the affine structure (32). Then the following statements are equivalent:*

- (i) *The mapping μ is inward pointing at boundary points of \mathcal{X} .*
- (ii) *We have*

$$\langle \mu(x), e_i \rangle_H \geq 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } i \in I \text{ with } \langle x, e_i \rangle_H = 0.$$

- (iii) *We have*

$$m_0 \in \mathcal{X}, \tag{44}$$

$$Mx \in (H_I^+ + \text{lin}\{e_i\}) \oplus H_J \quad \text{for all } i \in I \text{ and } x \in \text{lin}^+\{e_i\}, \tag{45}$$

$$M(H_J) \subset H_J. \tag{46}$$

Suppose that μ has the affine structure (32) and is inward pointing at boundary points of \mathcal{X} . Then (44) immediately yields that $m_{0,I} \in H_I^+$, and therefore, the condition $m_{0,I} \in H_{I,0}^+$ appearing in (38) is equivalent to $m_{0,I} \in H_{I,0}$.

Proof of Proposition 3.2. (i) \Rightarrow (ii): This implication is obvious.

(ii) \Rightarrow (i): Let $x \in \mathcal{X}$ and $\eta \in H_I^+$ be such that $\langle x, \eta \rangle_H = 0$. Note that

$$\langle x, \eta \rangle_H = \sum_{i \in I_0} \langle x, e_i \rangle_H \langle e_i, \eta \rangle_H,$$

where $I_0 = \{i \in I : \langle e_i, \eta \rangle_H > 0\}$. Therefore, we have $\langle x, e_i \rangle_H = 0$ for all $i \in I_0$, and hence

$$\langle \mu(x), \eta \rangle_H = \sum_{i \in I_0} \langle \mu(x), e_i \rangle_H \langle e_i, \eta \rangle_H \geq 0.$$

(ii) \Leftrightarrow (iii): The proof of this equivalence is analogous to that of [37, Prop. A.10]. \square

Proposition 3.3. *Suppose that S has the affine structure (33). Then the following statements are equivalent:*

- (i) *The mapping σ is parallel to the boundary at boundary points of \mathcal{X} .*
- (ii) *We have*

$$\langle \sigma(x), e_i \rangle_H = 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } i \in I \text{ with } \langle x, e_i \rangle_H = 0.$$

- (iii) *We have*

$$n_0 \xi = 0 \quad \text{for all } \xi \in H_I, \tag{47}$$

$$N(x) = 0 \quad \text{for all } x \in H_J, \tag{48}$$

$$N(x)\xi = 0 \quad \text{for all } i, j \in I \text{ with } i \neq j \text{ and all } x \in \text{lin}^+\{e_i\} \text{ and } \xi \in \text{lin}^+\{e_j\}. \tag{49}$$

Proof. (i) \Leftrightarrow (ii): The proof of this equivalence is analogous to that in the proof of Proposition 3.2.

(ii) \Leftrightarrow (iii): Note that (34) is satisfied if and only if

$$\sigma(x)^* \eta = 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } \eta \in H_I^+ \text{ with } \langle x, \eta \rangle_H = 0.$$

Since $(\Sigma_W^{1/2})^*$ is one-to-one, by (31) this is equivalent to

$$\langle S(x)\eta, \eta \rangle_H = 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } \eta \in H_I^+ \text{ with } \langle x, \eta \rangle_H = 0.$$

Therefore, the proof of this equivalence is analogous to that of [37, Prop. A.20]. \square

Consequently, if the mappings μ and σ are affine, then the inward pointing property and the parallel property mean that the parameters m_0, M, n_0, N satisfy the admissibility conditions from Proposition 2.5. Hence, in this case the general Riccati system (8)-(9) has a unique solution $(\phi(\cdot, u), \psi(\cdot, u)) : \mathbb{R}_+ \rightarrow \mathbb{C}_- \times (H_{\mathbb{C}}^-)_I \oplus iH_J$ for each $u \in (H_{\mathbb{C}}^-)_I \oplus iH_J$.

Proposition 3.4. *Suppose that S has the affine structure (33), and that σ is parallel to the boundary at boundary points of \mathcal{X} . Then the following statements are true:*

- (i) *We have $S(x) = S(x_I)$ for all $x \in \mathcal{X}$.*
- (ii) *We have $S(x)\xi = N(x)\xi$ for all $x \in H_I^+$ and $\xi \in H_I$.*
- (iii) *We have $S(e_i)e_j = 0$ for all $i, j \in I$ with $i \neq j$.*
- (iv) *We have $\lambda \in \ell^2(I)$ and the representation*

$$\lambda_i = \langle S(e_i)_{II} e_i, e_i \rangle_H, \quad i \in I.$$

- (v) *We have the representation*

$$S(x)_{II} \xi = \sum_{i \in I} \lambda_i \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H e_i \quad \text{for all } x \in H_I^+ \text{ and } \xi \in H_I.$$

Proof. By condition (48) from Proposition 3.3 we have $S(x) = S(x_I)$ for all $x \in \mathcal{X}$, and by condition (47) from Proposition 3.3 we have $S(x)\xi = N(x)\xi$ for all $x \in H_I^+$ and $\xi \in H_I$. Therefore, by condition (49) from Proposition 3.3 we have $S(e_i)e_j = 0$ for all $i, j \in I$ with $i \neq j$. Now, the remaining statements follow from Lemma C.7. \square

Now, we will deal with linear transformations which leave the state space \mathcal{X} invariant. The next result provides a characterization of such transformations. Note that the closed convex cone \mathcal{X} has the representations

$$\mathcal{X} = \bigcap_{i \in I} \{x \in H : \langle x, e_i \rangle_H \geq 0\} = \bigcap_{y \in H_I^+} \{x \in H : \langle x, y \rangle_H \geq 0\}. \quad (50)$$

Lemma 3.5. *For a bounded linear operator $\Lambda \in L(H)$ the following statements are equivalent:*

- (i) *We have $\Lambda(\mathcal{X}) \subset \mathcal{X}$.*
- (ii) *We have $\Lambda^*(H_I^+) \subset H_I^+$ and $\Lambda(H_J) \subset H_J$.*

Proof. (i) \Rightarrow (ii): Suppose there exists $x \in H_J$ with $\Lambda x \in \mathcal{X} \setminus H_J$. Then, by (50) and since $\Lambda x \notin H_J$ there exists $i \in I$ with $\langle \Lambda x, e_i \rangle_H > 0$. We have $-x \in H_J \subset \mathcal{X}$, and hence

$$\langle \Lambda(-x), e_i \rangle_H = -\langle \Lambda x, e_i \rangle_H < 0,$$

which provides the contradiction $\Lambda(-x) \notin \mathcal{X}$. Therefore, we have $\Lambda(H_J) \subset H_J$, and hence

$$\langle \Lambda^* x, y \rangle_H = \langle x, \Lambda y \rangle_H = 0 \quad \text{for all } x \in H_I \text{ and all } y \in H_J,$$

which shows $\Lambda^*(H_I) \subset H_I$. Furthermore, we have

$$\langle \Lambda^* x, y \rangle_H = \langle x, \Lambda y \rangle_H \geq 0 \quad \text{for all } x, y \in H_I^+,$$

showing that $\Lambda^*(H_I^+) \subset H_I^+$.

(ii) \Rightarrow (i): For all $x, y \in H_I^+$ we have

$$\langle \Lambda x, y \rangle_H = \langle x, \Lambda^* y \rangle_H \geq 0,$$

and hence by (50) we deduce $\Lambda(H_I^+) \subset \mathcal{X}$. Therefore, for each $x \in \mathcal{X}$ we obtain

$$\Lambda x = \Lambda x_I + \Lambda x_J \in \mathcal{X},$$

completing the proof. \square

Now, let $\Lambda \in L(H)$ be an isomorphism such that $\Lambda(\mathcal{X}) = \mathcal{X}$; that is, the conditions from Lemma 3.5 are fulfilled. We introduce the new mappings $\bar{\mu} : \mathcal{X} \rightarrow H$ and $\bar{\sigma} : \mathcal{X} \rightarrow L_2^+(U_0, H)$ as

$$\bar{\mu}(y) := \Lambda\mu(x), \quad y \in \mathcal{X}, \quad (51)$$

$$\bar{\sigma}(y) := \Lambda\sigma(x), \quad y \in \mathcal{X}, \quad (52)$$

where $x = \Lambda^{-1}y \in \mathcal{X}$, and we define the new mapping $\bar{S} : \mathcal{X} \rightarrow L_1^+(H)$ as

$$\bar{S}(y) := \bar{\sigma}(y)\Sigma_W^{1/2}(\bar{\sigma}(y)\Sigma_W^{1/2})^*, \quad y \in \mathcal{X}. \quad (53)$$

Then we have

$$\bar{S}(y) = \Lambda S(x)\Lambda^* = \Lambda S(x)^{1/2}(\Lambda S(x)^{1/2})^* \quad \text{for all } y \in \mathcal{X}, \quad (54)$$

where $x = \Lambda^{-1}y \in \mathcal{X}$. Indeed, taking into account (31) we obtain

$$\begin{aligned} \bar{S}(y) &= \bar{\sigma}(y)\Sigma_W^{1/2}(\bar{\sigma}(y)\Sigma_W^{1/2})^* = \Lambda\sigma(x)\Sigma_W^{1/2}(\Lambda\sigma(x)\Sigma_W^{1/2})^* \\ &= \Lambda\sigma(x)\Sigma_W^{1/2}(\sigma(x)\Sigma_W^{1/2})^*\Lambda^* = \Lambda S(x)\Lambda^* = \Lambda S(x)^{1/2}(\Lambda S(x)^{1/2})^*. \end{aligned}$$

Lemma 3.6. *Let X be a solution to the SDE (30). Then $Y := \Lambda X$ is a solution to the SDE (41) with $y_0 = \Lambda x_0$.*

Proof. Since Λ is a continuous linear operator, this is straightforward. \square

Lemma 3.7. *The following statements are true:*

- (i) *If μ is inward pointing at boundary points of \mathcal{X} , then $\bar{\mu}$ is also inward pointing at boundary points of \mathcal{X} .*
- (ii) *If σ is parallel to the boundary at boundary points of \mathcal{X} , then $\bar{\sigma}$ is also parallel to the boundary at boundary points of \mathcal{X} .*

Proof. Let $y \in \mathcal{X}$ and $\eta \in H_I^+$ with $\langle y, \eta \rangle_H = 0$ be arbitrary. We set $x := \Lambda^{-1}y \in \mathcal{X}$. By Lemma 3.5 we have $\Lambda^*\eta \in H_I^+$. Furthermore, we have

$$\langle x, \Lambda^*\eta \rangle_H = \langle \Lambda x, \eta \rangle_H = \langle y, \eta \rangle_H = 0.$$

Therefore, if μ is inward pointing, then we obtain

$$\langle \bar{\mu}(y), \eta \rangle_H = \langle \Lambda(\mu(x)), \eta \rangle_H = \langle \mu(x), \Lambda^*\eta \rangle_H \geq 0.$$

Similarly, if σ is parallel, then we obtain

$$\langle \bar{\sigma}(y), \eta \rangle_H = \langle \Lambda(\sigma(x)), \eta \rangle_H = \langle \sigma(x), \Lambda^*\eta \rangle_H = 0,$$

finishing the proof. \square

Now, assume that μ has the affine structure (32). We define $\bar{m}_0 \in H$ and $\bar{M} \in L(H)$ as

$$\bar{m}_0 := \Lambda m_0 \quad \text{and} \quad \bar{M} := \Lambda M \Lambda^{-1}. \quad (55)$$

Then $\bar{\mu}$ has the affine structure

$$\bar{\mu}(y) = \bar{m}_0 + \bar{M}y \quad \text{for all } y \in \mathcal{X}. \quad (56)$$

Indeed, let $y \in \mathcal{X}$ be arbitrary. By (51), (32) and (55) we have

$$\bar{\mu}(y) = \Lambda\mu(x) = \Lambda(m_0 + Mx) = \Lambda m_0 + \Lambda M \Lambda^{-1}y = \bar{m}_0 + \bar{M}y,$$

where $x = \Lambda^{-1}y \in \mathcal{X}$. Let us decompose $\bar{\mu}$ with respect to $H = H_I \oplus H_J$. For this purpose, we additionally assume that μ is inward pointing. We define the affine mappings $\bar{\mu}_{II} : H_I^+ \rightarrow H_I$ and $\bar{\mu}_J : \mathcal{X} \rightarrow H_J$ as

$$\bar{\mu}_{II}(y) := \bar{m}_{0,I} + \bar{M}_{II}y, \quad y \in H_I^+, \quad (57)$$

$$\bar{\mu}_J(y) := \bar{m}_{0,J} + \bar{M}_Jy, \quad y \in \mathcal{X}. \quad (58)$$

Then we have the decomposition

$$\bar{\mu}(y) = \bar{\mu}_{II}(y_I) + \bar{\mu}_J(y) \quad \text{for all } y \in \mathcal{X}. \quad (59)$$

Indeed, let $y \in \mathcal{X}$ be arbitrary. Taking into account Lemma 3.7, by condition (46) from Proposition 3.2 we have $\pi_I \bar{M}y_J = 0$, and hence

$$\begin{aligned} \bar{\mu}(y) &= \bar{m}_0 + \bar{M}y = \bar{m}_{0,I} + \bar{m}_{0,J} + \pi_I \bar{M}y + \pi_J \bar{M}y \\ &= \bar{m}_{0,I} + \bar{m}_{0,J} + \pi_I \bar{M}y_I + \pi_J \bar{M}y \\ &= \bar{m}_{0,I} + \bar{M}_{II}y_I + \bar{m}_{0,J} + \bar{M}_Jy = \bar{\mu}_{II}(y_I) + \bar{\mu}_J(y). \end{aligned}$$

Lemma 3.8. *Suppose that $\bar{\mu}$ has the affine structure (56). If $\bar{\mu}$ is inward pointing at boundary points of \mathcal{X} , then $\bar{\mu}_{II}$ is inward pointing at boundary points of H_I^+ .*

Proof. Taking into account (56), by Proposition 3.2 we have

$$\begin{aligned} \bar{m}_0 &\in \mathcal{X}, \\ \bar{M}x &\in (H_I^+ + \text{lin}\{e_i\}) \oplus H_J \quad \text{for all } i \in I \text{ and } x \in \text{lin}^+\{e_i\}, \\ \bar{M}(H_J) &\subset H_J. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \bar{m}_{0,I} &\in H_I^+, \\ \bar{M}_{II}x &\in (H_I^+ + \text{lin}\{e_i\}) \quad \text{for all } i \in I \text{ and } x \in \text{lin}^+\{e_i\}. \end{aligned}$$

Hence, taking into account (57), by Proposition 3.2 we deduce that $\bar{\mu}_{II}$ is inward pointing at boundary points of H_I^+ . \square

Now, we will consider a concrete choice for the transformation Λ , which will provide the announced block diagonal structure of the volatility $\bar{\sigma}$. From now on, suppose that condition (37) is fulfilled. By Lemma C.8 the mapping

$$Dx := - \sum_{i \in I_0} \langle x, e_i \rangle_H \frac{S(e_i)_{IJ} e_i}{\lambda_i}, \quad x \in H \quad (60)$$

is a well-defined continuous linear operator $D \in L(H)$. We define $\Lambda \in L(H)$ as

$$\Lambda := \text{Id} + D. \quad (61)$$

Lemma 3.9. *The following statements are true:*

- (i) *We have $\text{ran}(D) \subset H_J \subset \ker(D)$.*
- (ii) *We have $\text{ran}(D^*) \subset H_I \subset \ker(D^*)$.*
- (iii) *Λ is an isomorphism with $\Lambda^{-1} = \text{Id} - D$.*
- (iv) *We have $\Lambda(\mathcal{X}) = \mathcal{X}$.*
- (v) *We have $\Lambda x = x$ and $\Lambda^{-1}x = x$ for all $x \in H_J$.*

Proof. The first statement immediately follows from (60), and since $H_J^\perp = H_I$, we obtain

$$\text{ran}(D^*) \subset \overline{\text{ran}(D^*)} = (\text{ran}(D^*)^\perp)^\perp = \ker(D)^\perp \subset H_I \subset \text{ran}(D)^\perp = \ker(D)^*.$$

Since $\text{ran}(D) \subset \ker(D)$, we also have $D^2 = 0$, which gives us

$$(\text{Id} + D)(\text{Id} - D) = \text{Id} - D^2 = \text{Id},$$

showing that Λ is an isomorphism with $\Lambda^{-1} = \text{Id} - D$. Furthermore, taking into account $\text{ran}(D) \subset H_J$, we obtain $\Lambda(\mathcal{X}) \subset \mathcal{X}$ and $\Lambda^{-1}(\mathcal{X}) \subset \mathcal{X}$, and hence $\Lambda(\mathcal{X}) = \mathcal{X}$. Finally, since $H_J \subset \ker(D)$, we have $Dx = 0$ for all $x \in H_J$, and hence $\Lambda x = x$ and $\Lambda^{-1}x = x$ for all $x \in H_J$. \square

The following auxiliary result concerns the quantities appearing in the affine structures (32) and (56) of μ and $\bar{\mu}$.

Lemma 3.10. *Suppose that μ has the affine structure (32) and that μ is inward pointing at boundary points of \mathcal{X} . Then we have $m_{0,I} = \bar{m}_{0,I}$ and $M_{II} = \bar{M}_{II}$.*

Proof. By (55) and Lemma 3.9 we have

$$\bar{m}_{0,I} = \pi_I \bar{m}_0 = \pi_I \Lambda m_0 = \pi_I (\text{Id} + D)m_0 = \pi_I m_0 + \pi_I Dm_0 = m_{0,I}.$$

Furthermore, by (55) and Lemma 3.9 we have

$$\bar{M} = \Lambda M \Lambda^{-1} = (\text{Id} + D)M(\text{Id} - D) = M + DM - MD - DMD.$$

By Proposition 3.2 we have $M(H_J) \subset H_J$. Therefore, by Lemma 3.9 we obtain

$$(DM)_{II} = (MD)_{II} = (DMD)_{II} = 0,$$

and hence $M_{II} = \bar{M}_{II}$. \square

From now on, we assume that S has the affine structure (33) and that σ is parallel. Then we have

$$S(e_i)e_i = 0 \quad \text{for each } i \in I \setminus I_0. \quad (62)$$

Indeed, by Proposition 3.4 we have $\langle S(e_i)e_i, e_i \rangle_H = 0$, and hence applying Lemma C.6 provides (62).

Lemma 3.11. *The following statements are true:*

- (i) *We have $DS(x)_I \pi_I = -S(x)_J \pi_I$ for all $x \in H_I^+$.*
- (ii) *We have $S(x)_I D^* = -S(x)_I \pi_J$ for all $x \in H_I^+$.*

Proof. By (60) we have

$$De_i = -\frac{S(e_i)_J e_i}{\lambda_i} \quad \text{for all } i \in I_0.$$

Therefore, by Proposition 3.4 and (62), for all $x \in H_I^+$ and $\xi \in H_I$ we obtain

$$\begin{aligned} DS(x)_I \xi &= D \left(\sum_{i \in I_0} \lambda_i \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H e_i \right) = - \sum_{i \in I_0} \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H S(e_i)_J e_i \\ &= - \sum_{i \in I} \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H S(e_i)_J e_i = -S \left(\sum_{i \in I} \langle x, e_i \rangle_H e_i \right)_J \sum_{i \in I} \langle \xi, e_i \rangle_H e_i \\ &= -S(x)_J \xi. \end{aligned}$$

Therefore, we have

$$D\pi_I S(x)\pi_I = -\pi_J S(x)\pi_I.$$

By Lemma 3.9 we have $\text{ran}(D^*) \subset H_I$. Therefore, taking adjoints we obtain

$$S(x)_I D^* = \pi_I S(x)\pi_I D^* = (D\pi_I S(x)\pi_I)^* = -\pi_I S(x)\pi_J = -S(x)_I \pi_J,$$

completing the proof. \square

Now, we introduce $\bar{n}_0 \in L_1^+(H)$ and $\bar{N} \in L(H, L_1(H))$ as

$$\bar{n}_0 := \Lambda n_0 \Lambda^* \quad \text{and} \quad \bar{N}y := \Lambda N x \Lambda^* \quad \text{for all } y \in H, \quad (63)$$

where $x = \Lambda^{-1}y \in H$. Then \bar{S} has the affine structure

$$\bar{S}(y) = \bar{n}_0 + \bar{N}y \quad \text{for all } y \in \mathcal{X}. \quad (64)$$

Indeed, let $y \in \mathcal{X}$ be arbitrary. By (54), (33) and (63) we have

$$\bar{S}(y) = \Lambda S(x)\Lambda^* = \Lambda(n_0 + Nx)\Lambda^* = \Lambda n_0 \Lambda^* + \Lambda N x \Lambda^* = \bar{n}_0 + \bar{N}y,$$

where $x = \Lambda^{-1}y \in \mathcal{X}$.

Lemma 3.12. *For each $y \in \mathcal{X}$ the following statements are true:*

- (i) *We have $\bar{S}(y)\pi_I = S(x)_I \pi_I$, where $x = \Lambda^{-1}y \in \mathcal{X}$, and hence $\bar{S}(y)(H_I) \subset H_I$.*
- (ii) *We have $\bar{S}(y)(H_J) \subset H_J$.*

Proof. Let $y \in \mathcal{X}$ be arbitrary, and set $x := \Lambda^{-1}y \in \mathcal{X}$. Furthermore, let $\eta \in H_I$ be arbitrary. By Lemma 3.9 we have $D^*\eta = 0$. Therefore, by (54) and Lemma 3.11 we obtain

$$\begin{aligned} \bar{S}(y)\eta &= \Lambda S(x)\Lambda^*\eta = (\text{Id} + D)S(x)(\eta + D^*\eta) \\ &= (\text{Id} + D)S(x)\eta = (\text{Id} + D)(S(x)_I \eta + S(x)_J \eta) \\ &= (\text{Id} + D)S(x)_I \eta + S(x)_J \eta \\ &= S(x)_I \eta + DS(x)_I \eta + S(x)_J \eta = S(x)_I \eta. \end{aligned}$$

Now, let $\eta \in H_J$ be arbitrary. Then by (54) we have

$$\begin{aligned} \bar{S}(y)\eta &= \Lambda S(x)\Lambda^*\eta = (\text{Id} + D)S(x)(\eta + D^*\eta) \\ &= (\text{Id} + D)S(x)_I(\eta + D^*\eta) + S(x)_J(\eta + D^*\eta) \\ &= S(x)_I \eta + S(x)_I D^*\eta + DS(x)_I(\eta + D^*\eta) + S(x)_J(\eta + D^*\eta). \end{aligned}$$

Note that by Lemma 3.9 we have

$$DS(x)_I(\eta + D^*\eta) + S(x)_J(\eta + D^*\eta) \in H_J.$$

Furthermore, by Lemma 3.11 we obtain

$$S(x)_I \eta + S(x)_I D^*\eta = 0,$$

and hence $\bar{S}(y)\eta \in H_J$, completing the proof. \square

Now, we are ready to analyze the structure of \bar{S} and its square root.

Proposition 3.13. *The following statements are true:*

- (i) *For all $y \in \mathcal{X}$ and $\eta \in H$ we have*

$$\bar{S}(y)\eta = \bar{S}(y_I)_{II} \eta_I + \bar{S}(y_I)_{JJ} \eta_J. \quad (65)$$

(ii) For all $y \in H_I^+$ and $\eta \in H_I$ we have

$$\bar{S}(y)_{II} \eta = \sum_{i \in I} \lambda_i \langle y, e_i \rangle_H \langle e_i, \eta \rangle_H e_i. \quad (66)$$

Proof. The mapping \bar{S} has the affine structure (64), and by Lemma 3.7 the mapping $\bar{\sigma}$ is parallel to the boundary at boundary points of \mathcal{X} . Therefore, by Proposition 3.4 we have

$$\bar{S}(y) = \bar{S}(y_I) \quad \text{for all } y \in \mathcal{X}.$$

Now, let $y \in \mathcal{X}$ and $\eta \in H$ be arbitrary. By Lemma 3.12 we have

$$\bar{S}(y)\eta = \bar{S}(y_I)\eta = \bar{S}(y_I)\eta_I + \bar{S}(y_I)\eta_J = \bar{S}(y_I)_{II} \eta_I + \bar{S}(y_I)_{JJ} \eta_J.$$

Now, let $y \in H_I^+$ be arbitrary. By Lemma 3.9, for each $i \in I$ we have

$$\langle \Lambda^{-1}y, e_i \rangle_H = \langle y - Dy, e_i \rangle_H = \langle y, e_i \rangle_H - \langle y, D^*e_i \rangle = \langle y, e_i \rangle_H.$$

Therefore, by Lemma 3.12 and Proposition 3.4, for all $y \in H_I^+$ and $\eta \in H_I$ we obtain

$$\bar{S}(y)_{II} \eta = S(\Lambda^{-1}y)_{II} \eta = \sum_{i \in I} \lambda_i \langle \Lambda^{-1}y, e_i \rangle_H \langle e_i, \eta \rangle_H e_i = \sum_{i \in I} \lambda_i \langle y, e_i \rangle_H \langle e_i, \eta \rangle_H e_i,$$

completing the proof. \square

Lemma 3.14. *The following statements are true:*

(i) We have

$$\bar{S}(y)^{1/2} \eta = \bar{S}(y_I)_{II}^{1/2} \eta_I + \bar{S}(y_I)_{JJ}^{1/2} \eta_J \quad \text{for all } y \in \mathcal{X} \text{ and } \eta \in H.$$

(ii) We have

$$\bar{S}(y)_{II}^{1/2} \eta = \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle e_i, \eta \rangle_H e_i \quad \text{for all } y \in H_I^+ \text{ and } \eta \in H_I.$$

Proof. By equation (65) from Proposition 3.13 we have

$$(\bar{S}(y_I)_{II}^{1/2} \pi_I + \bar{S}(y_I)_{JJ}^{1/2} \pi_J)^2 = \bar{S}(y_I)_{II} \pi_I + \bar{S}(y_I)_{JJ} \pi_J = \bar{S}(y),$$

proving the first statement. Furthermore, the second statement is an immediate consequence of equation (66) from Proposition 3.13. \square

Now, let us analyze the structure of the volatility $\bar{\sigma}$. From now on, we assume that $U = H$, that the covariance operator $\Sigma_W \in L_1^{++}(U)$ has a diagonal structure along $\{e_k\}_{k \in \mathbb{N}}$, and that for each $x \in \mathcal{X}$ the operator $\sigma(x)\Sigma_W^{1/2}$ is self-adjoint. Then we have

$$\bar{\sigma}(y) = \bar{S}(y)^{1/2} \Sigma_W^{-1/2} \quad \text{for all } y \in \mathcal{X}. \quad (67)$$

Indeed, let $y \in \mathcal{X}$ be arbitrary. By (54) we have

$$\bar{S}(y)^{1/2} = \Lambda S(x)^{1/2},$$

where $x = \Lambda^{-1}y \in \mathcal{X}$, and hence by (40) we obtain

$$\bar{\sigma}(y) = \Lambda \sigma(x) = \Lambda S(x)^{1/2} \Sigma_W^{-1/2} = \bar{S}(y)^{1/2} \Sigma_W^{-1/2}.$$

Recall that $U_0 = \Sigma_W^{1/2}(U)$ is a separable Hilbert space with inner product

$$\langle u, v \rangle_{U_0} := \langle \Sigma_W^{-1/2} u, \Sigma_W^{-1/2} v \rangle_U, \quad u, v \in U_0. \quad (68)$$

By Lemma C.2 the system $\{g_k\}_{k \in \mathbb{N}}$ given by

$$g_k = \Sigma_W^{1/2} e_k, \quad k \in \mathbb{N} \quad (69)$$

is an orthonormal basis of U_0 . Now, we define

$$U_{I,0} := \Sigma_W^{1/2}(U_I) \subset U_I \quad \text{and} \quad U_{J,0} := \Sigma_W^{1/2}(U_J) \subset U_J.$$

We define the mappings $\bar{\sigma}_{II} : H_I^+ \rightarrow L_2(U_{I,0}, H_I)$ and $\bar{\sigma}_{JJ} : H_J^+ \rightarrow L_2(U_{J,0}, H_J)$ as

$$\bar{\sigma}_{II}(y) := \bar{S}(y)_{II}^{1/2} \Sigma_{W,II}^{-1/2}, \quad y \in H_I^+, \quad (70)$$

$$\bar{\sigma}_{JJ}(y) := \bar{S}(y)_{JJ}^{1/2} \Sigma_{W,JJ}^{-1/2}, \quad y \in H_J^+. \quad (71)$$

Proposition 3.15. *The following statements are true:*

(i) *We have the representation*

$$\bar{\sigma}(y)u = \bar{\sigma}_{II}(y_I)u_I + \bar{\sigma}_{JJ}(y_J)u_J \quad \text{for all } y \in \mathcal{X} \text{ and } u \in U_0. \quad (72)$$

(ii) *We have the representation*

$$\bar{\sigma}_{II}(y)u = \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle g_i, u \rangle_{U_0} e_i \quad \text{for all } y \in H_I^+ \text{ and } u \in U_{I,0}. \quad (73)$$

(iii) *If $\lambda \in \ell^1(I)$ and $g_i = \sqrt{\lambda_i} e_i$ for all $i \in I$, then we have the representation*

$$\bar{\sigma}_{II}(y)u = \sum_{i \in I} \sqrt{\langle y, e_i \rangle_H} \langle e_i, u \rangle_U e_i \quad \text{for all } y \in H_I^+ \text{ and } u \in U_{I,0}.$$

Proof. Let $u \in U_0$ be arbitrary. Note that

$$\Sigma_W^{-1/2} u = \Sigma_{W,II}^{-1/2} u_I + \Sigma_{W,JJ}^{-1/2} u_J$$

is the decomposition of $\Sigma_W^{-1/2} u$ according to $U = U_I \oplus U_J$. Therefore, by (67) and Lemma 3.14, for each $y \in \mathcal{X}$ we have

$$\begin{aligned} \bar{\sigma}(y)u &= \bar{S}(y)^{1/2} \Sigma_W^{-1/2} u = \bar{S}(y_I)_{II}^{1/2} \Sigma_{W,II}^{-1/2} u_I + \bar{S}(y_J)_{JJ}^{1/2} \Sigma_{W,JJ}^{-1/2} u_J \\ &= \bar{\sigma}_{II}(y_I)u_I + \bar{\sigma}_{JJ}(y_J)u_J. \end{aligned}$$

Furthermore, by Lemma 3.14 and (68), for all $y \in H_I^+$ and $u \in U_{I,0}$ we have

$$\begin{aligned} \bar{\sigma}_{II}(y)u &= \bar{S}(y)_{II}^{1/2} \Sigma_{W,II}^{-1/2} u = \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle e_i, \Sigma_{W,II}^{-1/2} u \rangle_H e_i \\ &= \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle \Sigma_{W,II}^{1/2} e_i, u \rangle_{U_0} e_i = \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle g_i, u \rangle_{U_0} e_i. \end{aligned}$$

If $\lambda \in \ell^1(I)$ and $g_i = \sqrt{\lambda_i} e_i$ for all $i \in I$, then by (68) we obtain

$$\begin{aligned} \bar{\sigma}_{II}(y)u &= \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle g_i, u \rangle_{U_0} e_i = \sum_{i \in I} \lambda_i \sqrt{\langle y, e_i \rangle_H} \langle e_i, u \rangle_{U_0} e_i \\ &= \sum_{i \in I} \lambda_i \sqrt{\langle y, e_i \rangle_H} \langle \Sigma_{W,II}^{1/2} e_i, \Sigma_{W,II}^{-1/2} u \rangle_U e_i = \sum_{i \in I} \sqrt{\langle y, e_i \rangle_H} \langle e_i, u \rangle_U e_i, \end{aligned}$$

completing the proof. \square

Lemma 3.16. *The mapping $\bar{\sigma}_{II}$ is parallel to the boundary at boundary points of H_I^+ .*

Proof. By the representation (73) from Proposition 3.15 we have

$$\langle \bar{\sigma}_{II}(y), e_i \rangle_H = 0 \quad \text{for all } y \in H_I^+ \text{ and all } i \in I \text{ with } \langle y, e_i \rangle_H = 0.$$

Therefore, using Proposition 3.3 finishes the proof. \square

From now on, we assume that μ has the affine structure (32) and is inward pointing. By virtue of the decompositions (59) and (72) of $\bar{\mu}$ and $\bar{\sigma}$ we can express the transformed SDE (41) by the two coupled SDEs (42) and (43). The following two auxiliary results show that the drift and the volatility appearing in the SDE (42) are continuous and satisfy the linear growth condition.

Lemma 3.17. *The mapping $\bar{\mu}_{II} : H_I^+ \rightarrow H_I$ satisfies the linear growth condition, and for all $y, z \in H_I^+$ we have*

$$\|\bar{\mu}_{II}(y) - \bar{\mu}_{II}(z)\|_H \leq \|\bar{M}_{II}\|_{L(H_I)} \|y - z\|_H.$$

Proof. By (57), for each $y \in H_I^+$ we have

$$\|\bar{\mu}_{II}(y)\|_H = \|\bar{m}_{0,I} + \bar{M}_{II}y\|_H \leq \|\bar{m}_{0,I}\|_H + \|\bar{M}_{II}\|_{L(H_I)} \|y\|_H,$$

and for all $y, z \in H_I^+$ we have

$$\|\bar{\mu}_{II}(y) - \bar{\mu}_{II}(z)\|_H = \|\bar{M}_{II}(y - z)\|_H \leq \|\bar{M}_{II}\|_{L(H_I)} \|y - z\|_H,$$

completing the proof \square

Lemma 3.18. *The mapping $\bar{\sigma}_{II} : H_I^+ \rightarrow L_2(U_{I,0}, H_I)$ satisfies the linear growth condition, and for all $y, z \in H_I^+$ we have*

$$\|\bar{\sigma}_{II}(y) - \bar{\sigma}_{II}(z)\|_{L_2(U_{I,0}, H_I)}^2 \leq \|\lambda\|_{\ell^2(I)} \|y - z\|_H.$$

Proof. The mapping \bar{S} has the affine structure (64), and by Lemma 3.7 the mapping $\bar{\sigma}$ is parallel to the boundary at boundary points of \mathcal{X} . Therefore, by Proposition 3.4 we have

$$\bar{S}(y)\eta = \bar{N}(y)\eta \quad \text{for all } y \in H_I^+ \text{ and } \eta \in H_I.$$

Since $\{g_i\}_{i \in I}$ is an orthonormal basis of $U_{I,0}$, by (70) and Lemma C.5, for each $y \in H_I^+$ we obtain

$$\begin{aligned} \|\bar{\sigma}_{II}(y)\|_{L_2(U_{I,0}, H_I)}^2 &= \sum_{i \in I} \|\bar{\sigma}_{II}(y)g_i\|_H^2 = \sum_{i \in I} \|\bar{\sigma}_{II}(y)\Sigma_{W,II}^{1/2}e_i\|_H^2 \\ &= \|\bar{\sigma}_{II}(y)\Sigma_{W,II}^{1/2}\|_{L_2(H_I)}^2 = \|\bar{S}(y)_{II}^{1/2}\|_{L_2(H_I)}^2 = \|\bar{S}(y)_{II}\|_{L_1(H_I)} \\ &= \|\bar{N}_{II}(y)\|_{L_1(H_I)} \leq \|\bar{N}_{II}\|_{L(H_I, L_1(H_I))} \|y\|_H, \end{aligned}$$

where the operator $\bar{N}_{II} \in L(H_I, L_1(H_I))$ is given by $\bar{N}_{II}\eta = (\bar{N}\eta)_{II}$ for $\eta \in H_I$. This proves that $\bar{\sigma}_{II}$ satisfies the linear growth condition. Furthermore, by Lemma 3.14 and the elementary inequality

$$(\sqrt{a} - \sqrt{b})^2 \leq |a - b|, \quad a, b \in \mathbb{R}_+,$$

for all $y, z \in H_I^+$ we have

$$\begin{aligned} \|\bar{\sigma}_{II}(y) - \bar{\sigma}_{II}(z)\|_{L_2(U_{I,0}, H_I)}^2 &= \|(\bar{\sigma}_{II}(y) - \bar{\sigma}_{II}(z))\Sigma_{W,II}^{1/2}\|_{L_2(H_I)}^2 \\ &= \|\bar{S}(y)_{II}^{1/2} - \bar{S}(z)_{II}^{1/2}\|_{L_2(H_I)}^2 = \sum_{i \in I} \left(\sqrt{\lambda_i \langle y, e_i \rangle_H} - \sqrt{\lambda_i \langle z, e_i \rangle_H} \right)^2 \\ &\leq \sum_{i \in I} |\lambda_i \langle y - z, e_i \rangle_H| \leq \left(\sum_{i \in I} \lambda_i^2 \right)^{1/2} \left(\sum_{i \in I} |\langle y - z, e_i \rangle_H|^2 \right)^{1/2} = \|\lambda\|_{\ell^2(I)} \|y - z\|_H, \end{aligned}$$

finishing the proof. \square

Now, we define the retracted subspace with compact embedding. By Lemma A.1 there exists a sequence $\nu = (\nu_i)_{i \in I} \subset (0, \infty)$ with $\nu_i \rightarrow 0$ such that we have (35), which gives rise to the compact linear operator $T \in K^{++}(H_I)$ with representation (36) and the retracted subspace $H_{I,0} := T(H_I)$ according to Lemma C.2. We also recall the notations $H_{I,0}^+ := T(H_I^+)$ and $\mathcal{X}_0 := H_{I,0}^+ \oplus H_J$.

Remark 3.1. If I is finite, then we simply take $\nu_i := 1$ for all $i \in I$. In this case, the operator T is the identity operator, and we have $H_{I,0} = H_I$, $H_{I,0}^+ = H_I^+$ and $\mathcal{X}_0 = \mathcal{X}$.

Lemma 3.19. *We have $\Lambda(\mathcal{X}_0) = \mathcal{X}_0$.*

Proof. Let $x \in \mathcal{X}_0$ be arbitrary. By Lemma 3.9 we have $Dx \in H_J$, and hence $\Lambda x = x + Dx \in \mathcal{X}_0$ as well as $\Lambda^{-1}x = x - Dx \in \mathcal{X}_0$. \square

The following two results show that the drift and the volatility appearing in the SDE (42) satisfy the linear growth condition with respect to the norm $\|\cdot\|_{H_{I,0}}$.

Proposition 3.20. *Suppose that condition (38) is fulfilled. Then we have $\bar{\mu}_{II}(H_{I,0}^+) \subset H_{I,0}$, and $\bar{\mu}_{II}|_{H_{I,0}^+} : H_{I,0}^+ \rightarrow H_{I,0}$ satisfies the linear growth condition with respect to $\|\cdot\|_{H_{I,0}}$.*

Proof. By Lemma 3.10 we have $\bar{m}_{0,I} \in H_{I,0}^+$ and $\bar{M}_{II}T = T\bar{M}_{II}$. Hence, by Lemma C.3 we have $\bar{M}_{II}(H_{I,0}) \subset H_{I,0}$, and $\bar{M}_{II}|_{H_{I,0}} \in L(H_{I,0})$ with respect to $\|\cdot\|_{H_{I,0}}$. Therefore, taking into account (57) we have $\bar{\mu}_{II}(H_{I,0}^+) \subset H_{I,0}$, and for each $y \in H_{I,0}^+$ we obtain

$$\|\bar{\mu}_{II}(y)\|_{H_{I,0}} = \|\bar{m}_{0,I} + \bar{M}_{II}y\|_{H_{I,0}} \leq \|\bar{m}_{0,I}\|_{H_{I,0}} + \|\bar{M}_{II}\|_{L(H_{I,0})}\|y\|_{H_{I,0}},$$

proving the linear growth condition. \square

Recalling the representation (36), by Lemma C.2 the system $\{f_i\}_{i \in I}$ given by

$$f_i = Te_i = \nu_i e_i, \quad i \in I$$

is an orthonormal basis of $H_{I,0}$. Also recall that system $\{g_k\}_{k \in \mathbb{N}}$ given by (69) is an orthonormal basis of U_0 . In view of the upcoming result, we emphasize that the spaces $H_{I,0}$ and $U_{I,0}$ have to be distinguished, although we have $H = U$. Indeed, by definition we have $H_{I,0} = T(H_I)$, where T is given by (36), and we have $U_{I,0} = \Sigma_W^{1/2}(H_I)$.

Proposition 3.21. *The following statements are true:*

(i) *For all $y \in H_{I,0}^+$ and all $u \in U_{I,0}$ we have $\bar{\sigma}_{II}(y)u \in H_{I,0}$.*

(ii) *For all $y \in H_{I,0}^+$ we have $\bar{\sigma}_{II}(y) \in L_2(U_{I,0}, H_{I,0})$ with representation*

$$\bar{\sigma}_{II}(y)u = \sum_{i \in I} \sqrt{\frac{\lambda_i}{\nu_i}} \sqrt{\langle y, f_i \rangle_{H_0}} \langle g_i, u \rangle_{U_0} f_i, \quad u \in U_{I,0}.$$

(iii) *The mapping $\bar{\sigma}_{II}|_{H_{I,0}^+} : H_{I,0}^+ \rightarrow L_2(U_{I,0}, H_{I,0})$ satisfies the linear growth condition with respect to $\|\cdot\|_{H_{I,0}}$.*

Proof. Let $y \in H_{I,0}^+$ and $u \in U_{I,0}$ be arbitrary. By Proposition 3.15 and Lemma C.2 we have

$$\begin{aligned} \bar{\sigma}_{II}(y)u &= \sum_{i \in I} \sqrt{\lambda_i \langle y, e_i \rangle_H} \langle g_i, u \rangle_{U_0} e_i \\ &= \sum_{i \in I} \sqrt{\lambda_i \nu_i \langle y, f_i \rangle_{H_{I,0}}} \langle g_i, u \rangle_{U_0} e_i = \sum_{i \in I} \sqrt{\frac{\lambda_i}{\nu_i}} \sqrt{\langle y, f_i \rangle_{H_{I,0}}} \langle g_i, u \rangle_{U_0} f_i. \end{aligned}$$

Therefore, by the Cauchy Schwarz inequality and (35) we obtain

$$\begin{aligned} \sum_{i \in I} \frac{1}{\nu_i^2} |\langle \bar{\sigma}_{II}(y)u, e_i \rangle_H|^2 &= \sum_{i \in I} \frac{1}{\nu_i^2} \lambda_i \nu_i \langle y, f_i \rangle_{H_{I,0}} |\langle g_i, u \rangle_{U_0}|^2 = \sum_{i \in I} \frac{\lambda_i}{\nu_i} \langle y, f_i \rangle_{H_{I,0}} |\langle g_i, u \rangle_{U_0}|^2 \\ &\leq \left(\sum_{i \in I} \left(\frac{\lambda_i}{\nu_i} \right)^2 \right)^{1/2} \left(\sum_{i \in I} |\langle y, f_i \rangle_{H_{I,0}}|^2 \right)^{1/2} \|u\|_{U_0}^2 = \left\| \left(\frac{\lambda_i}{\nu_i} \right)_{i \in I} \right\|_{\ell^2(I)} \|y\|_{H_{I,0}} \|u\|_{U_0}^2 < \infty, \end{aligned}$$

and by Lemma C.2 it follows that $\bar{\sigma}_{II}(y)u \in H_{I,0}$. Recall that $\{g_i\}_{i \in I}$ is an orthonormal basis of $U_{I,0}$. Hence, by Lemma C.2 and the Cauchy Schwarz inequality we obtain

$$\begin{aligned} \|\bar{\sigma}_{II}(y)\|_{L_2(U_{I,0}, H_{I,0})}^2 &= \sum_{i \in I} \|\bar{\sigma}_{II}(y)g_i\|_{H_{I,0}}^2 = \sum_{i \in I} \frac{\lambda_i}{\nu_i} \langle y, f_i \rangle_{H_{I,0}} \\ &\leq \left(\sum_{i \in I} \left(\frac{\lambda_i}{\nu_i} \right)^2 \right)^{1/2} \left(\sum_{i \in I} |\langle y, f_i \rangle_{H_{I,0}}|^2 \right)^{1/2} = \left\| \left(\frac{\lambda_i}{\nu_i} \right)_{i \in I} \right\|_{\ell^2(I)} \|y\|_{H_{I,0}}, \end{aligned}$$

proving that $\bar{\sigma}_{II}(y) \in L_2(U_{I,0}, H_{I,0})$. Moreover, it follows that $\bar{\sigma}_{II}|_{H_{I,0}^+}$ satisfies the linear growth condition with respect to $\|\cdot\|_{H_{I,0}}$. \square

The following auxiliary result implies that the drift appearing in the SDE (43) is Lipschitz continuous. Note that the volatility in the SDE (43) is constant by definition.

Lemma 3.22. *Let $y_I \in H_I^+$ be arbitrary. Then for all $y_J, z_J \in H_J$ we have*

$$\|\bar{\mu}_J(y_I + y_J) - \bar{\mu}_J(y_I + z_J)\|_H \leq \|\bar{M}_J\|_{L(H)} \|y_J - z_J\|_H.$$

Proof. Taking into account (58), for all $y_J, z_J \in H_J$ we have

$$\|\bar{\mu}_J(y_I + y_J) - \bar{\mu}_J(y_I + z_J)\|_H = \|\bar{M}_J(y_J - z_J)\|_H \leq \|\bar{M}_J\|_{L(H)} \|y_J - z_J\|_H,$$

completing the proof. \square

The following two results provide the existence of weak solutions and pathwise uniqueness of solutions for the affine SDE (30), which we require in order to apply the Yamada-Watanabe theorem.

Proposition 3.23. *Suppose that condition (38) is fulfilled. Then for each probability measure ν on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$ there exists a weak solution to the affine SDE (30) such that $\nu = \mathbb{P} \circ X_0$.*

Proof. Let ν be a probability measure on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$. Taking into account Lemma 3.19, setting $\bar{\nu} := \nu \circ \Lambda$, that is $\bar{\nu}(B) = \nu(\Lambda^{-1}B)$ for all $B \in \mathcal{B}(\mathcal{X}_0)$, defines another probability measure on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$. Let $\bar{\nu}_I$ be the probability measure on $(H_{I,0}^+, \mathcal{B}(H_{I,0}^+))$ given by $\bar{\nu}_I(B) = \bar{\nu}(B \times H_J)$ for all $B \in \mathcal{B}(H_{I,0})$. In view of Lemmas 3.17, 3.18, Propositions 3.20, 3.21 and Lemmas 3.7, 3.8, 3.16 we may apply Theorem B.2, which provides a weak solution (Y_I, W) to the affine SDE (42) on some stochastic basis \mathbb{B} such that $\bar{\nu}_I = \mathbb{P} \circ Y_{I,0}$. By desintegration, there exists a stochastic kernel K from $(H_{I,0}^+, \mathcal{B}(H_{I,0}^+))$ into $(H_J, \mathcal{B}(H_J))$ such that $\bar{\nu} = K \otimes \bar{\nu}_I$. We define the probability kernel \bar{K} from (Ω, \mathcal{F}_0) into $(H_J, \mathcal{B}(H_J))$ as

$$\bar{K}(\omega, B) := K(Y_{I,0}(\omega), B) \quad \text{for all } \omega \in \Omega \text{ and } B \in \mathcal{B}(H_J).$$

Furthermore, we define the new stochastic basis

$$\begin{aligned}\tilde{\mathbb{B}} &:= (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in \mathbb{R}_+}, \tilde{\mathbb{P}}) \\ &:= (\Omega \times H_J, \mathcal{F} \otimes \mathcal{B}(H_J), (\mathcal{F}_t \otimes \mathcal{B}(H_J))_{t \in \mathbb{R}_+}, \bar{K} \otimes \mathbb{P}),\end{aligned}$$

and we define the $\tilde{\mathcal{F}}_0$ -measurable random variable $Y_0 : \tilde{\Omega} \rightarrow \mathcal{X}_0$ as

$$Y_0(\omega) := (Y_{I,0}(\omega_1), Y_{J,0}(\omega_2)), \quad \omega \in \tilde{\Omega},$$

where $Y_{J,0}(\omega_2) := \omega_2$. For all $A \in \mathcal{B}(H_{0,I}^+)$ and $B \in \mathcal{B}(H_J)$ we have

$$\begin{aligned}\tilde{\mathbb{P}}(Y_0 \in A \times B) &= \int_{\Omega} \int_{H_J} \mathbf{1}_{\{Y_0 \in A \times B\}}(\omega_1, \omega_2) \bar{K}(\omega_1, d\omega_2) \mathbb{P}(d\omega_1) \\ &= \int_{\Omega} \int_{H_J} \mathbf{1}_{A \times B}(Y_{I,0}(\omega_1), x_2) K(Y_{I,0}(\omega_1), dx_2) \mathbb{P}(d\omega_1) \\ &= \int_{H_{0,I}^+} \int_{H_J} \mathbf{1}_{A \times B}(x_1, x_2) K(x_1, dx_2) \bar{\nu}_I(dx_1) = \bar{\nu}(A \times B),\end{aligned}$$

showing that $\tilde{\mathbb{P}} \circ Y_0 = \bar{\nu}$. In view of Lemma 3.22, there exists a strong solution Y_J to the affine SDE (43) on the stochastic basis $\tilde{\mathbb{B}}$ with initial condition $Y_{J,0}$. Setting $Y := (Y_I, Y_J)$, this gives us a weak solution (Y, W) to the affine SDE (41) such that $\tilde{\mathbb{P}} \circ Y_0 = \bar{\nu}$. Now, we set $X := \Lambda^{-1}Y$. Taking into account Lemma 3.6 we deduce (X, W) is a weak solution to the affine SDE (30). Moreover, for each $B \in \mathcal{B}(\mathcal{X}_0)$ we have

$$\tilde{\mathbb{P}}(X_0 \in B) = \tilde{\mathbb{P}}(\Lambda^{-1}Y_0 \in B) = \tilde{\mathbb{P}}(Y_0 \in \Lambda(B)) = \bar{\nu}(\Lambda(B)) = \nu(B),$$

showing that $\tilde{\mathbb{P}} \circ X_0 = \nu$. □

Proposition 3.24. *Suppose that condition (39) is fulfilled. Then we have pathwise uniqueness with starting points in \mathcal{X} for the affine SDE (30).*

Proof. Note that $\bar{\mu}_{II}$ has the affine structure (57), and that $\bar{\sigma}_{II}$ has the diagonal structure (73) from Proposition 3.15. Also noting (39), we may apply Theorem B.3, which provides pathwise uniqueness with starting points in H_I^+ for the affine SDE (42). Moreover, in view of Lemma 3.22, for every weak solution Y_I to (42) we have pathwise uniqueness with starting points in H_J for the affine SDE (43). Consequently, we have pathwise uniqueness with starting points in \mathcal{X} for the affine SDE (41). Since $\Lambda \in L(H)$ is an isomorphism, taking into account Lemma 3.6 we deduce that pathwise uniqueness with starting points in \mathcal{X} for the affine SDE (30) holds. □

Remark 3.2. Note that we cannot apply Thm. 2.1 from [41] in order to derive pathwise uniqueness. Indeed, defining $\psi_1, \varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as $\psi_1(\theta) := \psi(\theta) := \theta$ and $\varphi(\theta) := \sqrt{\theta}$ for $\theta \in \mathbb{R}_+$, the integral divergence condition from (A4) in [41] is not fulfilled, because

$$\int_0^1 \frac{1}{\psi(\theta) + \varphi(\theta)} d\theta = \int_0^1 \frac{1}{\theta + \theta^{1/2}} d\theta = \int_0^1 \frac{1}{\theta^{1/2}(\theta^{1/2} + 1)} d\theta = \int_0^1 \left(\frac{1}{\theta^{1/2}} - \frac{1}{\theta^{1/2} + 1} \right) d\theta < \infty.$$

Noting that $\mathcal{B}(\mathcal{X}_0) = \mathcal{B}(\mathcal{X})_{\mathcal{X}_0}$ by (83), the proof of Theorem 3.1 is now an immediate consequence of Propositions 3.23 and 3.24, combined with our version of the Yamada-Watanabe theorem (Theorem B.1).

4. EXAMPLES

In this section, we present examples where Theorem 3.1 applies. In Subsections 4.1 and 4.2 the application is straightforward. More care is required in Subsections 4.3 and 4.4, where we consider infinite dimensional processes of Cox-Ingersoll-Ross type and of Heston type. In both of these two examples, we will first specify the volatility structure, then we define the retracted subspace with compact embedding, and in the last step we introduce the drift.

4.1. Particular situations. As a consequence of the following result, Theorem 3.1 in particular applies when the subspace H_I is finite dimensional.

Corollary 4.1. *If the index set I is finite, then the affine SDE (30) has a unique strong solution starting in \mathcal{X} .*

Proof. Since I is finite, condition (37) is fulfilled. Furthermore, by Remark 3.1 the compact linear operator $T \in K(H_I)$ given by (36) is the identity operator, and we have $H_{I,0} = H_I$, $H_{I,0}^+ = H_I^+$ and $\mathcal{X}_0 = \mathcal{X}$. Therefore, Assumption 3.1 is satisfied. Consequently, applying Theorem 3.1 completes the proof. \square

In particular, Corollary 4.1 applies when the Hilbert space H is finite dimensional. Therefore, we have generalized [19, Thm. 8.1], which provides the existence of affine processes in finite dimension.

4.2. Infinite dimensional processes of Ornstein-Uhlenbeck type. Note that Corollary 4.1 in particular applies when $I = \emptyset$, which provides the existence of affine processes with state space being the whole Hilbert space H . In this situation, where the process X is a so-called Ornstein-Uhlenbeck process, we can say even more. By condition (48) from Proposition 3.3 the mapping S is constant, and hence the volatility σ given by (40) is constant as well. Therefore, the affine SDE (30) has the explicit solution given by the variation of constants formula

$$X_t = x_0 + \int_0^t S_{t-s} m_0 ds + \int_0^t S_{t-s} \sigma dW_s, \quad t \in \mathbb{R}_+,$$

where $(S_t)_{t \geq 0}$ denotes the uniformly continuous semigroup generated by the linear operator $M \in L(H)$ appearing in (32). Ornstein-Uhlenbeck processes on Hilbert spaces have recently been studied in [2] and [3], and they provide a link to the theory of semilinear stochastic partial differential equations (SPDEs) in the spirit of the semigroup approach; see for example [12]. More precisely, in the general situation, where I and J are arbitrary disjoint index sets, we could also regard the affine SDE (30) as a SPDE and look for (mild) martingale solutions, which means that the variation of constants formula

$$X_t = S_t x_0 + \int_0^t S_{t-s} m_0 ds + \int_0^t S_{t-s} \sigma(X_s) dW_s, \quad t \in \mathbb{R}_+$$

is satisfied. According to [12, Thm. 8.1], a sufficient condition for the existence of martingale solutions is that the semigroup $(S_t)_{t \geq 0}$ is compact. However, we have the following negative result.

Proposition 4.2. *Suppose that $\dim H = \infty$. Then no uniformly continuous, compact semigroup exists.*

Proof. Suppose that $A \in L(H)$ is the infinitesimal generator of a compact semigroup. According to [33, Prop. 3.1.4], for every $B \in L(H)$ the operator $A+B$ is also the generator of a compact semigroup. In particular, choosing $B = -A$ we obtain that the semigroup $(S_t)_{t \geq 0}$ given by $S_t = \text{Id}$ for each $t \geq 0$ is compact. Since $\dim H = \infty$, this is a contradiction. \square

Consequently, apart from the particular case of Ornstein-Uhlenbeck processes, the SPDE approach is not appropriate in order to establish the existence of affine processes, and this is why we regard affine processes as solutions to infinite dimensional SDEs in this paper.

4.3. Infinite dimensional processes of Cox-Ingersoll-Ross type. In this subsection, we establish the existence of infinite dimensional processes of Cox-Ingersoll-Ross type. Here we have the index sets $I = \mathbb{N}$ and $J = \emptyset$, and hence the state space is given by $\mathcal{X} = H^+$. We define the volatility structure $S : H^+ \rightarrow L_1^+(H)$, the retracted subspace H_0 and the drift $\mu : H^+ \rightarrow H$ in three steps:

Volatility structure. Let $\lambda = (\lambda_i)_{i \in \mathbb{N}} \subset (0, \infty)$ be a sequence such that $\lambda \in \ell^2(\mathbb{N})$. By Lemma C.9 the mapping

$$N(x)\xi := \sum_{i \in \mathbb{N}} \lambda_i \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H e_i, \quad x, \xi \in H$$

is a well-defined continuous linear operator $N \in L(H, L_1(H))$, and for every $x \in H^+$ the operator $N(x)$ is self-adjoint with $N(x) \in L_1^+(H)$. We define the affine mapping $S : H^+ \rightarrow L_1^+(H)$ as

$$S(x) := N(x) \quad \text{for all } x \in H^+.$$

Then S has the affine form (33) with $n_0 = 0$. Note that conditions (47)–(49) from Proposition 3.3 are fulfilled, ensuring that the associated volatility $\sigma : H^+ \rightarrow L^2(U_0, H)$ given by (40) is parallel. Furthermore, condition (37) is fulfilled, because $\kappa_i = 0$ for each $i \in \mathbb{N}$, and the transformation $\Lambda \in L(H)$ given by (61) is simply the identity operator. By Proposition 3.15 the volatility is given by

$$\sigma(x)u = \sum_{i \in \mathbb{N}} \sqrt{\lambda_i \langle x, e_i \rangle_H} \langle g_i, u \rangle_{U_0} e_i \quad \text{for all } x \in H^+ \text{ and } u \in U_0,$$

where $g_i = \Sigma_W^{1/2} e_i$ for all $i \in \mathbb{N}$. If we even have $\lambda \in \ell^1(\mathbb{N})$, then we can take the covariance operator $\Sigma_W \in L_1^{++}(U)$ defined as

$$\Sigma_W u := \sum_{i \in \mathbb{N}} \lambda_i \langle e_i, u \rangle_U e_i \quad \text{for all } u \in U,$$

and then the volatility admits the representation

$$\sigma(x)u = \sum_{i \in \mathbb{N}} \sqrt{\langle x, e_i \rangle_H} \langle e_i, u \rangle_U e_i \quad \text{for all } x \in H^+ \text{ and } u \in U_0.$$

Retracted subspace. By Lemma A.1 there exists a sequence $(\nu_i)_{i \in \mathbb{N}} \subset (0, \infty)$ such that $\nu_i \rightarrow 0$ and

$$\left(\frac{\lambda_i}{\nu_i} \right)_{i \in \mathbb{N}} \in \ell^2(\mathbb{N}).$$

Let $T \in K^{++}(H)$ be the compact linear operator with representation

$$Tx = \sum_{i \in \mathbb{N}} \nu_i \langle x, e_i \rangle_H e_i, \quad x \in H, \tag{74}$$

and let $H_0 := T(H)$ be the retracted subspace defined according to Lemma C.2. Moreover, we set $H_0^+ := T(H^+)$.

Drift. We define the affine mapping $\mu : H^+ \rightarrow H$ as

$$\mu(x) := m_0 + Mx, \quad x \in H^+,$$

where $m_0 \in H_0^+$, and $M \in L(H)$ is of the form

$$Mx = \sum_{i \in \mathbb{N}} \rho_i \langle x, e_i \rangle_H e_i, \quad x \in H \quad (75)$$

with a sequence $\rho = (\rho_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ such that $\rho \in \ell^1(\mathbb{N})$. Note that condition (39) and conditions (44)–(46) from Proposition 3.2 are fulfilled, which shows that μ is inward pointing. By the representations (74) and (75) we have $MT = TM$, showing that condition (38) is fulfilled. Consequently, by Theorem 3.1 the affine Cox-Ingersoll-Ross type SDE (30) with parameters specified above has a unique strong solution starting in H_0^+ .

4.4. Infinite dimensional processes of Heston type. In this subsection, we establish the existence of infinite dimensional processes of Heston type. We assume that the disjoint index sets I and J are both infinite. Let $\tau : I \rightarrow J$ be a bijection such that $\tau(i_1) \leq \tau(i_2)$ for all $i_1, i_2 \in I$ with $i_1 \leq i_2$. We define the volatility structure $S : \mathcal{X} \rightarrow L_1^+(H)$, the retracted subspace $H_{I,0}$ and the drift $\mu : \mathcal{X} \rightarrow H$ in three steps:

Volatility structure. Let $n_0 \in L_1^+(H)$ be such that $n_0\xi = 0$ for all $\xi \in H_I$. Furthermore, let $\lambda = (\lambda_i)_{i \in I} \subset (0, \infty)$ and $\kappa = (\kappa_i)_{i \in I} \subset (0, \infty)$ be sequences such that $\lambda \in \ell^2(I)$ and $\kappa_i \leq \lambda_i$ for all $i \in I$. Moreover, we assume that

$$\left(\frac{\kappa_i}{\lambda_i} \right)_{i \in I} \in \ell^2(I).$$

By Lemma C.10 the mapping

$$N(x)\xi := \sum_{i \in I} \langle x, e_i \rangle_H \left(\langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \xi \rangle_H e_i + \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \xi \rangle_H e_{\tau(i)} \right), \quad x, \xi \in H$$

is a well-defined continuous linear operator $N \in L(H, L_1(H))$, and for every $x \in \mathcal{X}$ the operator Nx is self-adjoint with $Nx \in L_1^+(H)$. We define the affine mapping $S : \mathcal{X} \rightarrow L_1^+(H)$ as

$$S(x) := n_0 + Nx, \quad x \in \mathcal{X}.$$

By Lemma C.10 we have the identities

$$\begin{aligned} \lambda_i &= \|S(e_i)_{II} e_i\|_H, \quad i \in I, \\ \kappa_i &= \|S(e_i)_{IJ} e_i\|_H, \quad i \in I. \end{aligned}$$

Note that conditions (47)–(49) from Proposition 3.3 are fulfilled, ensuring that the associated volatility $\sigma : \mathcal{X} \rightarrow L^2(U_0, H)$ given by (40) is parallel. Furthermore, condition (37) is fulfilled, and the transformation $\Lambda \in L(H)$ specified by (61) is given by $\Lambda = \text{Id} + D$, where $D \in L(H)$ denotes the linear operator

$$Dx = - \sum_{i \in I} \frac{\kappa_i}{\lambda_i} \langle x, e_i \rangle_H e_{\tau(i)}, \quad x \in H.$$

Retracted subspace. By Lemma A.1 there exists a sequence $(\nu_i)_{i \in I} \subset (0, \infty)$ such that $\nu_i \rightarrow 0$ and

$$\left(\frac{\lambda_i}{\nu_i} \right)_{i \in I} \in \ell^2(I).$$

Let $T \in K(H_I)$ be the compact linear operator with representation

$$Tx = \sum_{i \in I} \nu_i \langle x, e_i \rangle_H e_i, \quad x \in H_I, \quad (76)$$

and let $H_{I,0} := T(H_I)$ be the retracted subspace defined according to Lemma C.2. Moreover, we set $H_{I,0}^+ := T(H_I^+)$ and $\mathcal{X}_0 := H_{I,0}^+ \oplus H_J$.

Drift. Let $m_0 \in \mathcal{X}$ be such that $m_{0,I} \in H_{I,0}^+$. Furthermore, let $M_{II} \in L(H_I)$ be a linear operator of the form

$$M_{II}x = \sum_{i \in I} \rho_i \langle x, e_i \rangle_H e_i, \quad x \in H_I \quad (77)$$

with a bounded sequence $\rho = (\rho_i)_{i \in I} \subset \mathbb{R}$ such that $\rho \in \ell^1(I)$, and let $M_J \in L(H, H_J)$ be arbitrary. We define $M \in L(H)$ as

$$Mx := M_{II}x_I + M_Jx, \quad x \in H,$$

and the affine mapping $\mu : \mathcal{X} \rightarrow H$ as

$$\mu(x) = m_0 + Mx, \quad x \in \mathcal{X}.$$

Note that condition (39) and conditions (44)–(46) from Proposition 3.2 are fulfilled, which shows that μ is inward pointing. By the representations (76) and (77) we have $M_{II}T = TM_{II}$, showing that condition (38) is fulfilled. Consequently, by Theorem 3.1 the affine Heston type SDE (30) has a unique strong solution starting in \mathcal{X}_0 .

APPENDIX A. AUXILIARY RESULTS

We begin the proof of Lemma 2.8 with the following remark: an $A \in L(H; L(H))$ is considered to be a sesquilinear map from $H_{\mathbb{C}} \times H_{\mathbb{C}}$ into $H_{\mathbb{C}}$ by defining $\eta^* A \xi = \sum_{i=1}^{\infty} \langle A_i \xi, \bar{\eta} \rangle_{H_{\mathbb{C}}} e_i$ for all $\xi, \eta \in H_{\mathbb{C}}$. It is well-defined, if $(A_i)_{i \in \mathbb{N}}$ are trace class operators such that $\sum_{i \in I} (\text{tr } A_i)^2 < \infty$ and $A_j = 0$ for all $j \in J$. Actually, we have

$$\begin{aligned} \|\eta^* A \xi\|_{H_{\mathbb{C}}}^2 &= \sum_{i=1}^{\infty} |\langle A_i \xi, \bar{\eta} \rangle_{H_{\mathbb{C}}}|^2 \leq \sum_{i=1}^{\infty} (\text{tr } A_i)^2 \|\xi\|_{H_{\mathbb{C}}}^2 \|\eta\|_{H_{\mathbb{C}}}^2 \\ &= \sum_{i \in I} (\text{tr } A_i)^2 \|\xi\|_{H_{\mathbb{C}}}^2 \|\eta\|_{H_{\mathbb{C}}}^2 < \infty. \end{aligned}$$

Note that, for all $i \in \mathbb{N}$, $\langle \psi(t, u), M_i \rangle_{H_{\mathbb{C}}} = \langle M^* \psi(t, u), e_i \rangle_{H_{\mathbb{C}}}$ and $\langle n_i \psi(t, u), \overline{\psi(t, u)} \rangle_{H_{\mathbb{C}}} = \langle \psi(t, u)^* N \psi(t, u), e_i \rangle_{H_{\mathbb{C}}}$. Here, we use M and N from Equation (6).

Since $\psi(t, u)$ is Fréchet differentiable and $\langle D_t \psi(t, u), e_i \rangle_{H_{\mathbb{C}}} = \partial_t \psi_i(t, u)$, Equation (9) is equivalent to

$$\begin{aligned} D_t \psi(t, u) &= M^* \psi(t, u) + \frac{1}{2} \psi(t, u)^* N \psi(t, u), \quad t \geq 0, \\ \psi(0, u) &= u \in \mathcal{U}. \end{aligned} \quad (78)$$

Proof of Lemma 2.8. We truncate I at n by considering $I_n = I \cap \{1, \dots, n\}$, $n \in \mathbb{N}$. Denote $J_n = \mathbb{N} \setminus I_n$. By the very definition of $\langle \cdot, \cdot \rangle_{H_{\mathbb{C}}}$ we directly obtain that

$$\begin{aligned} \partial_t \|\psi_{I_n}(t, u)\|_{H_{\mathbb{C}}}^2 &= \sum_{i \in I_n} \partial_t \left(\operatorname{Re} \psi_i(t, u)^2 + \operatorname{Im} \psi_i(t, u)^2 \right) \\ &= 2 \sum_{i \in I_n} \left(\operatorname{Re} \psi_i(t, u) \partial_t (\operatorname{Re} \psi_i(t, u)) + \operatorname{Im} \psi_i(t, u) \partial_t (\operatorname{Im} \psi_i(t, u)) \right) \\ &= 2 \sum_{i \in I_n} \operatorname{Re} \left(\overline{\psi_i(t, u)} \partial_t \psi_i(t, u) \right). \end{aligned} \quad (79)$$

It follows that for a bounded linear operator $L \in L(H)$ and $\psi \in H$, $\langle L\psi, \overline{\psi} \rangle_H + \langle L\overline{\psi}, \psi \rangle_H = 2 \operatorname{Re} \langle L\psi, \overline{\psi} \rangle_H$. Then, (78) together with (18) yields

$$\begin{aligned} \partial_t \psi_i(t, u) &= \langle \psi_{I_n}(t, u) + \psi_{J_n}(t, u), m_i \rangle_{H_{\mathbb{C}}} \\ &\quad + \frac{1}{2} \langle n_i (\psi_{I_n}(t, u) + \psi_{J_n}(t, u)), \psi_{I_n}(t, u) + \psi_{J_n}(t, u) \rangle_{H_{\mathbb{C}}} \\ &= \langle \psi_{I_n}(t, u), m_{i, I_n} \rangle_{H_{\mathbb{C}}} + \langle \psi_{J_n}(t, u), m_{i, J_n} \rangle_{H_{\mathbb{C}}} + \frac{1}{2} n_{i, ii} |\psi_i(t, u)|^2 \\ &\quad + \operatorname{Re} \langle n_{i, I_n J_n} \psi_{J_n}(t, u), \overline{\psi_{I_n}(t, u)} \rangle_{H_{\mathbb{C}}} + \frac{1}{2} \langle n_{i, J_n J_n} \psi_{J_n}(t, u), \overline{\psi_{J_n}(t, u)} \rangle_{H_{\mathbb{C}}}, \end{aligned}$$

for all $i \in I_n$. Hence, for $t \in [0, T_u)$ and $i \in I_n$,

$$\begin{aligned} 2 \operatorname{Re} \left(\overline{\psi_i(t, u)} \partial_t \psi_i(t, u) \right) &= 2 \operatorname{Re} \left(\overline{\psi_i(t, u)} \left(\langle \psi_{I_n}(t, u), m_{i, I_n} \rangle_{H_{\mathbb{C}}} + \langle \psi_{J_n}(t, u), m_{i, J_n} \rangle_{H_{\mathbb{C}}} \right) \right) \\ &\quad + \operatorname{Re} \psi_i(t, u) n_{i, ii} |\psi_i(t, u)|^2 \\ &\quad + 2 \operatorname{Re} \psi_i(t, u) \operatorname{Re} \langle n_{i, I_n J_n} \psi_{J_n}(t, u), \overline{\psi_{I_n}(t, u)} \rangle_{H_{\mathbb{C}}} \\ &\quad + \operatorname{Re} \left(\overline{\psi_i(t, u)} \langle n_{i, J_n J_n} \psi_{J_n}(t, u), \overline{\psi_{J_n}(t, u)} \rangle_{H_{\mathbb{C}}} \right). \end{aligned} \quad (80)$$

Since $\operatorname{Re} \psi_i(t, u) \leq 0$ for $i \in I_n \subset I$ and $t \in [0, T_u)$ and $n_{i, ii} \geq 0$, we obtain that $\operatorname{Re} \psi_i(t, u) n_{i, ii} |\psi_i(t, u)|^2 \leq 0$ and it follows that

$$\begin{aligned} (80) &\leq 2 \operatorname{Re} \left(\overline{\psi_i(t, u)} \left(\langle \psi_{I_n}(t, u), m_{i, I_n} \rangle_{H_{\mathbb{C}}} + \langle \psi_{J_n}(t, u), m_{i, J_n} \rangle_{H_{\mathbb{C}}} \right) \right) \\ &\quad + 2 \operatorname{Re} \psi_i(t, u) \operatorname{Re} \langle n_{i, I_n J_n} \psi_{J_n}(t, u), \overline{\psi_{I_n}(t, u)} \rangle_{H_{\mathbb{C}}} \\ &\quad + \operatorname{Re} \left(\overline{\psi_i(t, u)} \langle n_{i, J_n J_n} \psi_{J_n}(t, u), \overline{\psi_{J_n}(t, u)} \rangle_{H_{\mathbb{C}}} \right). \end{aligned}$$

Using that $2 \operatorname{Re}(\alpha\beta) \leq |\alpha|^2 + |\beta|^2$ and $2 \operatorname{Re} \alpha \operatorname{Re} \beta \leq |\alpha|^2 + |\beta|^2$ for $\alpha, \beta \in \mathbb{C}$, we get

$$\begin{aligned} (80) &\leq |\psi_i(t, u)|^2 + |\langle \psi_{I_n}(t, u), m_{i, I_n} \rangle_{H_{\mathbb{C}}}|^2 + |\psi_i(t, u)|^2 + |\langle \psi_{J_n}(t, u), m_{i, J_n} \rangle_{H_{\mathbb{C}}}|^2 \\ &\quad + |\psi_i(t, u)|^2 + |\langle n_{i, I_n J_n} \psi_{J_n}(t, u), \overline{\psi_{I_n}(t, u)} \rangle_{H_{\mathbb{C}}}|^2 \\ &\quad + \frac{1}{2} |\psi_i(t, u)|^2 + \frac{1}{2} |\langle n_{i, J_n J_n} \psi_{J_n}(t, u), \overline{\psi_{J_n}(t, u)} \rangle_{H_{\mathbb{C}}}|^2. \end{aligned}$$

Note that for $i \in I_n$, $\langle \psi_{I_n}(t, u), m_{i, I_n} \rangle_{H_{\mathbb{C}}} = \langle \psi_{I_n}(t, u), \pi_{I_n} M e_i \rangle_{H_{\mathbb{C}}} = \langle M^* \psi_{I_n}(t, u), e_i \rangle_{H_{\mathbb{C}}}$ and that this is also true when I_n is replaced by J_n . Hence,

$$\begin{aligned} (80) &\leq \frac{7}{2} |\psi_i(t, u)|^2 + |\langle M^* \psi_{I_n}(t, u), e_i \rangle_{H_{\mathbb{C}}}|^2 + |\langle M^* \psi_{J_n}(t, u), e_i \rangle_{H_{\mathbb{C}}}|^2 \\ &\quad + |\langle n_{i, I_n J_n} \psi_{J_n}(t, u), \overline{\psi_{I_n}(t, u)} \rangle_{H_{\mathbb{C}}}|^2 + \frac{1}{2} |\langle n_{i, J_n J_n} \psi_{J_n}(t, u), \overline{\psi_{J_n}(t, u)} \rangle_{H_{\mathbb{C}}}|^2. \end{aligned}$$

Moreover, $\|n_{i,KL}\| \leq \|n_i\|$ for any subset $K, L \subset \mathbb{N}$, such that

$$\begin{aligned} 2 \operatorname{Re} (\overline{\psi_i(t, u)} \partial_t \psi_i(t, u)) &\leq \frac{7}{2} |\psi_i(t, u)|^2 + |\langle M^* \psi_{I_n}(t, u), e_i \rangle_{H_C}|^2 + |\langle M^* \psi_{J_n}(t, u), e_i \rangle_{H_C}|^2 \\ &\quad + \|n_i\|^2 (\|\psi_{J_n}(t, u)\|_{H_C}^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 + \frac{1}{2} \|\psi_{J_n}(t, u)\|_{H_C}^4). \end{aligned}$$

It follows from (79) that

$$\begin{aligned} \partial_t \|\psi_{I_n}(t, u)\|_{H_C}^2 &\leq \sum_{i \in I_n} \left(\frac{7}{2} |\psi_i(t, u)|^2 + |\langle M^* \psi_{I_n}(t, u), e_i \rangle_{H_C}|^2 + |\langle M^* \psi_{J_n}(t, u), e_i \rangle_{H_C}|^2 \right) \\ &\quad + \sum_{i \in I_n} \|n_i\|^2 (\|\psi_{J_n}(t, u)\|_{H_C}^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^4) \\ &\leq \frac{7}{2} \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|M^* \psi_{I_n}(t, u)\|_{H_C}^2 + \|M^* \psi_{J_n}(t, u)\|_{H_C}^2 \\ &\quad + \sum_{i \in I_n} \|n_i\|^2 (\|\psi_{J_n}(t, u)\|_{H_C}^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^4) \\ &\leq \frac{7}{2} \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|M\|^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|M\|^2 \|\psi_{J_n}(t, u)\|_{H_C}^2 \\ &\quad + \sum_{i \in I_n} \|n_i\|^2 (\|\psi_{J_n}(t, u)\|_{H_C}^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^4) \\ &\leq C (\|\psi_{I_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^2 \|\psi_{I_n}(t, u)\|_{H_C}^2 \\ &\quad + \|\psi_{J_n}(t, u)\|_{H_C}^4) \\ &\leq C (1 + \|\psi_{I_n}(t, u)\|_{H_C}^2) (1 + \|\psi_{J_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^4) \end{aligned}$$

holds for all $t \in [0, T_u)$, where $C = \sum_{i \in I} \|n_i\|^2 + \|M\|^2 + \frac{7}{2}$. We set $h_{n,u}(t) := 1 + \|\psi_{J_n}(t, u)\|_{H_C}^2 + \|\psi_{J_n}(t, u)\|_{H_C}^4$. Applying Gronwall's inequality yields that for $t \in [0, T_u)$,

$$1 + \|\psi_{I_n}(t, u)\|_{H_C}^2 \leq 1 + \|u_{I_n}\|_{H_C}^2 + C(1 + \|u_{I_n}\|_{H_C}^2) \int_0^t h_{n,u}(s) e^{C \int_s^t h_{n,u}(r) dr} ds. \quad (81)$$

Let $t \in [0, T_u)$. By the definitions of I_n and J_n , it holds that $\lim_{n \rightarrow \infty} \|\psi_{I_n}(t, u)\|_{H_C} = \|\psi_I(t, u)\|_{H_C}$, $\lim_{n \rightarrow \infty} u_{I_n} = u_I$ increasingly and $\lim_{n \rightarrow \infty} \|\psi_{J_n}(t, u)\|_{H_C} = \|\psi_J(t, u)\|_{H_C}$ decreasingly. It follows that $\lim_{n \rightarrow \infty} h_{n,u}(r) = h_u(r)$ decreasingly for all $r \in [s, t]$. By Dini's theorem this convergence is uniform over $[s, t]$, thus, $\lim_{n \rightarrow \infty} \int_s^t h_{n,u}(r) dr = \int_s^t h_u(r) dr$ decreasingly for all $s \in [0, t]$.

Consider the sequence of functions $(h_{n,u}(s) \exp(\int_s^t C h_{n,u}(r) dr))_{n \in \mathbb{N}}$. Similar to the discussion above, this sequence converges to $h_u(s) \exp(C \int_s^t h_u(r) dr)$ decreasingly for all $s \in [0, t]$. Again, by Dini's Theorem, this convergence is uniform on $[0, t]$ and therefore it holds that

$$\lim_{n \rightarrow \infty} \int_0^t h_{n,u}(s) e^{C \int_s^t h_{n,u}(r) dr} ds = \int_0^t h_u(s) e^{C \int_s^t h_u(r) dr} ds, \quad t \in [0, T_u).$$

Taking limits on both sides of (81), we finally get

$$1 + \|\psi_I(t, u)\|_{H_C}^2 \leq 1 + \|u_I\|_{H_C}^2 + C(1 + \|u_I\|_{H_C}^2) \int_0^t h_u(s) e^{C \int_s^t h_u(r) dr} ds, \quad t \in [0, T_u).$$

Subtracting 1 from both sides, the required inequality is proved. \square

We proceed with further auxiliary results, which we require in this paper.

Lemma A.1. *Let $\lambda = (\lambda_k)_{k \in \mathbb{N}} \subset (0, \infty)$ be a sequence such that $\lambda \in \ell^1(\mathbb{N})$. Then there exists a sequence $(\nu_k)_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\nu_k \downarrow 0$ and*

$$\left(\frac{\lambda_k}{\nu_k} \right)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N}).$$

Proof. We define the decreasing sequences $(\rho_k)_{k \in \mathbb{N}} \subset (0, \infty)$ and $(\nu_k)_{k \in \mathbb{N}} \subset (0, \infty)$ as

$$\rho_k := \sum_{n=k}^{\infty} \lambda_n \quad \text{and} \quad \nu_k := \sqrt{\rho_k}.$$

Then we have $\nu_k \downarrow 0$. Furthermore, we have

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \sum_{k=1}^{\infty} \frac{\rho_k - \rho_{k+1}}{\sqrt{\rho_k}} \leq \sum_{k=1}^{\infty} \int_{\rho_{k+1}}^{\rho_k} \frac{1}{\sqrt{x}} dx = \int_0^{\rho_1} \frac{1}{\sqrt{x}} dx = 2\sqrt{\rho_1},$$

completing the proof. \square

Definition A.1. Let X, Y be normed spaces, and let $A \subset X$ be a subset. A function $f : A \rightarrow Y$ satisfies the *linear growth condition* if there is a constant $K \in \mathbb{R}_+$ such that

$$\|f(x)\|_Y \leq K(1 + \|x\|_X) \quad \text{for all } x \in A.$$

Lemma A.2. *Let X, Y, Z be normed spaces, let $A \subset X$ and $B \subset Y$ be subsets, and let $f : A \rightarrow Y$ with $f(A) \subset B$ and $g : B \rightarrow Z$ be functions satisfying the linear growth condition. Then $g \circ f : A \rightarrow Z$ satisfies the linear growth condition, too.*

Proof. There are constants $K, L \in \mathbb{R}_+$ such that

$$\begin{aligned} \|f(x)\|_Y &\leq K(1 + \|x\|_X) \quad \text{for all } x \in A, \\ \|g(y)\|_Z &\leq L(1 + \|y\|_Y) \quad \text{for all } y \in B. \end{aligned}$$

Therefore, for each $x \in A$ we obtain

$$\begin{aligned} \|g(f(x))\|_Z &\leq L(1 + \|f(x)\|_Y) \leq L(1 + K(1 + \|x\|_X)) \\ &= L + LK(1 + \|x\|_X) \leq L(K + 1)(1 + \|x\|_X), \end{aligned}$$

completing the proof. \square

APPENDIX B. INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

The goal of this appendix is to provide the required results about the existence of solutions to infinite dimensional SDEs. In particular, we will present a version of the Yamada-Watanabe theorem for starting points from a subspace which is equipped with a finer topology. This version of the Yamada-Watanabe theorem is inspired by [22], where an existence result for starting points from a retracted subspace with compact embedding is presented. We will also provide a refined version of this existence result as well as a result for pathwise uniqueness, which is a version of the uniqueness result from [42] in infinite dimension.

Let H be a separable Hilbert space, and let $\mathcal{X} \subset H$ be a subset. Let U be a separable Hilbert space and let $\Sigma_W \in L_1^{++}(U)$ be a self-adjoint, positive trace class operator. By Lemma C.2 the set $U_0 := \Sigma_W^{1/2}U$, equipped with the inner product

$$\langle u, v \rangle_{U_0} := \langle \Sigma_W^{-1/2}u, \Sigma_W^{-1/2}v \rangle_U, \quad u, v \in U_0$$

is a separable Hilbert space. Let $\mu : \mathcal{X} \rightarrow H$ and $\sigma : \mathcal{X} \rightarrow L_2(U_0, H)$ be measurable mappings. We consider the SDE

$$\begin{cases} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0, \end{cases} \quad (82)$$

where W is a U -valued Wiener process with covariance operator Σ_W . Let $(H_0, \|\cdot\|_{H_0})$ be a separable Hilbert space as in Lemma C.2. Then we have $H_0 \subset H$ as a set, and by Lemma C.2 we have $H_0 \in \mathcal{B}(H)$ and $\mathcal{B}(H_0) = \mathcal{B}(H)_{H_0}$. We define $\mathcal{X}_0 := \mathcal{X} \cap H_0$, and denote by $\mathcal{B}(\mathcal{X})$ the Borel σ -algebra of \mathcal{X} with respect to $\|\cdot\|_H$, and by $\mathcal{B}(\mathcal{X}_0)$ the Borel σ -algebra of \mathcal{X}_0 with respect to $\|\cdot\|_{H_0}$. Then we have

$$\mathcal{B}(\mathcal{X}_0) = \mathcal{B}(H)_{\mathcal{X}_0} = \mathcal{B}(\mathcal{X})_{\mathcal{X}_0}, \quad (83)$$

because $\mathcal{B}(\mathcal{X}_0) = \mathcal{B}(H_0)_{\mathcal{X}} = (\mathcal{B}(H)_{H_0})_{\mathcal{X}} = \mathcal{B}(H)_{\mathcal{X}_0} = \mathcal{B}(\mathcal{X})_{\mathcal{X}_0}$. Furthermore, we set

$$\mathbb{W}(\mathcal{X}) := \{w \in C(\mathbb{R}_+; H) : w(t) \in \mathcal{X} \text{ for all } t \in \mathbb{R}_+\}$$

as well as

$$\mathbb{W}_0(U) := \{w \in C(\mathbb{R}_+; U) : w(0) = 0\}.$$

For what follows, the letter \mathbb{B} will denote a stochastic basis $\mathbb{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$. We briefly recall the relevant solution concepts, which are adjusted to the present setting, where the solutions are \mathcal{X} -valued, and where starting points from H_0 are considered.

Definition B.1. A pair (X, W) , where X is an adapted process with $X_0 \in \mathcal{X}_0$ and paths in $\mathbb{W}(\mathcal{X})$, and where W is a U -valued Wiener process with covariance operator Σ_W on a stochastic basis \mathbb{B} , is called a *weak solution to (82) starting in \mathcal{X}_0* if we have \mathbb{P} -almost surely

$$\int_0^t \left(\|\mu(X_s)\|_H + \|\sigma(X_s)\|_{L_2(U_0, H)}^2 \right) ds < \infty \quad \text{for all } t \in \mathbb{R}_+$$

as well as

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad \text{for all } t \in \mathbb{R}_+.$$

Definition B.2. We say that *pathwise uniqueness with starting points in \mathcal{X}_0* holds for (82) if for two weak solutions (X, W) and (X', W) starting in \mathcal{X}_0 on the same stochastic basis \mathbb{B} and with the same U -valued Wiener process W such that $\mathbb{P}(X_0 = X'_0) = 1$ we have $X = X'$ up to indistinguishability.

In the sequel $\mathbb{P} \circ \Sigma_W$ denotes the distribution of an U -valued Wiener process W with covariance operator Σ_W on the space $(\mathbb{W}_0(U), \mathcal{B}(\mathbb{W}_0(U)))$.

Definition B.3. Let $\hat{\mathcal{E}}_{\mathcal{X}_0}(\mathcal{X})$ be the set of maps $F : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathbb{W}(\mathcal{X})$ such that for every probability measure ν on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$ there exists a map

$$F_\nu : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathbb{W}(\mathcal{X})$$

such that the following conditions are fulfilled:

- (i) F_ν is $\overline{\mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathbb{W}_0(U))}^{\nu \otimes (\mathbb{P} \circ \Sigma_W)} / \mathcal{B}(\mathbb{W}(\mathcal{X}))$ -measurable.
- (ii) For ν -almost all $x \in \mathcal{X}_0$ we have

$$F(x, w) = F_\nu(x, w) \quad \text{for } \mathbb{P} \circ \Sigma_W\text{-almost all } w \in \mathbb{W}_0(U).$$

Definition B.4. A weak solution (X, W) to (82) starting in \mathcal{X}_0 on a stochastic basis \mathbb{B} is called a *strong solution starting in \mathcal{X}_0* if there exists a mapping $F \in \hat{\mathcal{E}}_{\mathcal{X}_0}(\mathcal{X})$ such that the following conditions are satisfied:

(i) For all $x \in \mathcal{X}_0$ and $t \in \mathbb{R}_+$ the mapping

$$\mathbb{W}_0(U) \rightarrow \mathbb{W}(\mathcal{X}), \quad w \mapsto F(x, w)$$

is $\overline{\mathcal{B}_t(\mathbb{W}_0(U))}^{\mathbb{P} \circ \Sigma_W} / \mathcal{B}_t(\mathbb{W}(\mathcal{X}))$ -measurable.

(ii) We have up to indistinguishability

$$X = F_{\mathbb{P} \circ X_0}(X_0, W).$$

Definition B.5. We say that the SDE (82) has a *unique strong solution starting in \mathcal{X}_0* if there exists a mapping $F \in \hat{\mathcal{E}}_{\mathcal{X}_0}(\mathcal{X})$ such that:

(i) Condition (i) from Definition B.4 is satisfied.

(ii) For every U -valued Wiener process W on a stochastic basis \mathbb{B} and any $\mathcal{F}_0 / \mathcal{B}(\mathcal{X}_0)$ -measurable random variable $x_0 : \Omega \rightarrow \mathcal{X}_0$ the pair (X, W) , where $X := F(x_0, W)$, is a weak solution to (82) with $\mathbb{P}(X_0 = x_0) = 1$.

(iii) For any martingale solution (X, W) to (82) we have up to indistinguishability

$$X = F_{\mathbb{P} \circ X_0}(X_0, W).$$

Now, we are ready to present our version of the Yamada-Watanabe theorem. For related work about the Yamada-Watanabe theorem in infinite dimension, we refer to [31], [34] and [36].

Theorem B.1. *The SDE (82) has a unique strong solution starting in \mathcal{X}_0 if and only if both of the following two conditions are satisfied:*

(i) *For each probability measure ν on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$ there exists a weak solution to (82) such that $\nu = \mathbb{P} \circ X_0$.*

(ii) *Pathwise uniqueness for solutions to (82) starting in \mathcal{X}_0 holds.*

Proof. The proof is analogous to the proof of [34, Thm. 2.1], and we provide an outline. First, suppose the SDE (82) has a unique strong solution starting in \mathcal{X}_0 . Then pathwise uniqueness for solutions to (82) starting in \mathcal{X}_0 holds. For a probability measure ν on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$ we consider the probability space

$$\left(\mathcal{X}_0 \times \mathbb{W}_0(U), \overline{\mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathbb{W}_0(U))}^{\nu \otimes (\mathbb{P} \circ \Sigma_W)}, \nu \otimes (\mathbb{P} \circ \Sigma_W) \right)$$

with corresponding completed filtration, and let $x_0 : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathcal{X}_0$ and $W : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathbb{W}_0(U)$ be the canonical projections. Then $X := F_{\mathbb{P} \circ x_0}(x_0, W)$ is the desired weak solution to (82). Conversely, suppose that conditions (i) and (ii) are fulfilled. Let ν be a probability measure on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$, and let (X, W) be a weak solution to (82) with $\mathbb{P} \circ X_0 = \nu$. We define the probability measure \mathbb{P}_ν on

$$(\mathcal{X}_0 \times \mathbb{W}(\mathcal{X}) \times \mathbb{W}_0(U), \mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathbb{W}(\mathcal{X})) \otimes \mathcal{B}(\mathbb{W}_0(U)))$$

as $\mathbb{P}_\nu := \mathbb{P} \circ (X_0, X, W)$. By desintegration, there exists a stochastic kernel K_ν from $(\mathcal{X}_0 \times \mathbb{W}_0(U), \mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathbb{W}_0(U)))$ into $(\mathbb{W}(\mathcal{X}), \mathcal{B}(\mathbb{W}(\mathcal{X})))$ such that

$$\mathbb{P}_\nu = K_\nu \otimes (\nu \otimes (\mathbb{P} \circ \Sigma_W)).$$

Furthermore, there exists a map $F_\nu : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathbb{W}(\mathcal{X})$ satisfying condition (i) from Definition B.4 such that

$$K_\nu((x, w), \cdot) = \delta_{F_\nu(x, w)} \quad \text{for } \nu \otimes (\mathbb{P} \circ \Sigma_W)\text{-almost all } (x, w) \in \mathcal{X}_0 \times \mathbb{W}_0(U).$$

Now, we define $F : \mathcal{X}_0 \times \mathbb{W}_0(U) \rightarrow \mathbb{W}(\mathcal{X})$ as

$$F(x, w) := F_{\delta_x}(x, w) \quad \text{for all } x \in \mathcal{X}_0 \text{ and all } w \in \mathbb{W}_0(U).$$

Then we have $F \in \hat{\mathcal{E}}_{\mathcal{X}_0}(\mathcal{X})$, and this mapping provides the desired unique strong solution starting in \mathcal{X}_0 . \square

Now, we present sufficient conditions for an application of our version of the Yamada-Watanabe theorem (Theorem B.1). We start with sufficient conditions for the existence of weak solutions. Here we present a refined version of a result from [22], where the essential idea is to consider starting points from a retracted subspace with compact embedding. For the rest of this section, we assume that the set \mathcal{X} is of the form

$$\mathcal{X} = \text{lin}^+ \{e_k : k \in \mathbb{N}\}$$

for some orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of H .

Theorem B.2. *We suppose that the following conditions are fulfilled:*

- (i) *The compact linear operator $T \in K(H)$ from Lemma C.2 has the representation (91) with respect to the given orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$.*
- (ii) *The mappings $\mu : \mathcal{X} \rightarrow H$ and $\sigma : \mathcal{X} \rightarrow L_2(U_0, H)$ are continuous and satisfy the linear growth condition.*
- (iii) *We have $\mu(\mathcal{X}_0) \subset H_0$ and $\sigma(\mathcal{X}_0) \subset L_2(U_0, H_0)$, and the mappings $\mu|_{\mathcal{X}_0} : \mathcal{X}_0 \rightarrow H_0$ and $\sigma|_{\mathcal{X}_0} : \mathcal{X}_0 \rightarrow L_2(U_0, H_0)$ satisfy the linear growth condition with respect to $\|\cdot\|_{H_0}$.*
- (iv) *The mapping μ is inward pointing at boundary points of \mathcal{X} , and the mapping σ is parallel to the boundary at boundary points of \mathcal{X} .*

Then for each probability measure ν on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$ there exists a weak solution (X, W) to the SDE (82) such that $\nu = \mathbb{P} \circ X_0$.

Proof. Let $\Pi : H \rightarrow \mathcal{X}$ be the metric projection on the closed convex cone \mathcal{X} . Then we have

$$\Pi x = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle_H^+ e_i \quad \text{for each } x \in H,$$

and Π is continuous and satisfies $\|\Pi x\|_H \leq \|x\|_H$ for all $x \in H$. Furthermore, we have $\Pi T = T \Pi$. By Lemma C.4 we have $\Pi(H_0) \subset H_0$, and $\Pi|_{H_0} : H_0 \rightarrow H_0$ satisfies the linear growth condition with respect to $\|\cdot\|_{H_0}$. Consider the H -valued SDE

$$\begin{cases} dX_t &= \bar{\mu}(X_t)dt + \bar{\sigma}(X_t)dW_t \\ X_0 &= x_0, \end{cases} \quad (84)$$

where $\bar{\mu} : H \rightarrow H$ is given by $\bar{\mu} := \mu \circ \Pi$, and $\bar{\sigma} : H \rightarrow L_2(U_0, H)$ is given by $\bar{\sigma} := \sigma \circ \Pi$. Then the functions $\bar{\mu}$ and $\bar{\sigma}$ are continuous, and by Lemma A.2 they satisfy the linear growth condition. Furthermore, we have $\bar{\mu}(H_0) \subset H_0$ and $\bar{\sigma}(H_0) \subset L_2(U_0, H_0)$, and by Lemma A.2 the mappings $\bar{\mu}|_{H_0} : H_0 \rightarrow H_0$ and $\bar{\sigma}|_{H_0} : H_0 \rightarrow L_2(U_0, H_0)$ satisfy the linear growth condition with respect to $\|\cdot\|_{H_0}$. Now, let ν be a probability measure on $(\mathcal{X}_0, \mathcal{B}(\mathcal{X}_0))$. Let $x_0 : \Omega \rightarrow \mathcal{X}_0$ be a \mathcal{F}_0 -measurable random variable with $\mathbb{P} \circ x_0 = \nu$, and let W be a U -valued Wiener process with covariance operator Σ_W , defined on some stochastic basis. Note that x_0 is also $\mathcal{F}_0/\mathcal{B}(H)_{\mathcal{X}_0}$ -measurable, because we have $\mathcal{B}(\mathcal{X}_0) = \mathcal{B}(H)_{\mathcal{X}_0}$ by (83). Now, we proceed as in the proof of [22, Thm. 2] (see also [21, Thm. 3.12]), where only deterministic starting points are considered, and provide an outline of the arguments. Note that, apart from the initial conditions, our framework is a special case of that considered in [22, Thm. 2], because for each $T \in \mathbb{R}_+$ the mapping

$$C([0, T]; H) \times [0, T] \rightarrow H, \quad (f, t) \mapsto f(t)$$

is continuous, and hence conditions (B1)–(B3) and (A3) appearing in [22, Thm. 2] are fulfilled. There exist sequences of coefficients $(\bar{\mu}_n)_{n \in \mathbb{N}}$ and $(\bar{\sigma}_n)_{n \in \mathbb{N}}$ with $\bar{\mu}_n(H_0) \subset H_0$ and

$\bar{\sigma}_n(H_0) \subset L_2(U_0, H_0)$ for each $n \in \mathbb{N}$, which are Lipschitz continuous with respect to $\|\cdot\|_{H_0}$, and we have $\bar{\mu}_n \rightarrow \bar{\mu}$ and $\bar{\sigma}_n \rightarrow \bar{\sigma}$ for $n \rightarrow \infty$. Hence, for each $n \in \mathbb{N}$ the H_0 -valued SDE

$$\begin{cases} dX_t^n &= \bar{\mu}_n(X_t^n)dt + \bar{\sigma}_n(X_t^n)dW_t \\ X_0^n &= x_0 \end{cases}$$

has unique strong solution X^n . For an arbitrary $T \in \mathbb{R}_+$ we define the solution measures $\varrho_n := \mathbb{P} \circ X^n$ on $C([0, T]; H_0) \subset C([0, T]; H)$ for each $n \in \mathbb{N}$. Since the identity operator $\text{Id} : (H_0, \|\cdot\|_{H_0}) \rightarrow (H, \|\cdot\|_H)$ is compact, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $\varrho_{n_k} \rightarrow \varrho$ on $C([0, T]; H)$, and this provides a weak solution (X, W) to the SDE (84). It remains to prove that (X, W) is also a weak solution to the original SDE (82). For this purpose, we will show that the closed convex cone \mathcal{X} is invariant; more precisely that $X \in \mathcal{X}$ up to an evanescent set. Let $i \in \mathbb{N}$ be arbitrary. For each $x \in H$ with $\langle x, e_i \rangle_H \leq 0$ we have $\langle \Pi(x), e_i \rangle_H = 0$, and hence, by Propositions 3.2 and 3.3 we obtain

$$\langle \mu(\Pi(x)), e_i \rangle_H \geq 0 \quad \text{and} \quad \langle \sigma(\Pi(x)), e_i \rangle_H = 0. \quad (85)$$

We define the stopping time

$$S := \inf\{t \in \mathbb{R}_+ : \langle X_t, e_i \rangle_H < 0\},$$

and claim that $\mathbb{P}(S = \infty) = 1$. Suppose, on the contrary, that $\mathbb{P}(S < \infty) > 0$. For each $n \in \mathbb{N}$ we define the stopping time

$$T_n := \inf\left\{t \in \mathbb{R}_+ : \langle X_t, e_i \rangle_H < -\frac{1}{n}\right\}.$$

By the continuity of sample paths of X , there exists $n \in \mathbb{N}$ such that $\mathbb{P}(B_n) > 0$, where

$$B_n := \{T_n < \infty\} \cap \{\langle X_t, e_i \rangle_H \leq 0 \text{ for all } t \in \llbracket S, T_n \rrbracket\}.$$

Note that $B_n \in \mathcal{F}$. Indeed, by [26, Props. I.1.21 and I.1.23] we have

$$\llbracket S, T_n \rrbracket \in \mathcal{O} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+),$$

and hence

$$B_n = \{T_n < \infty\} \cap \bigcap_{t \in \mathbb{Q}_+} \left(\{\langle X_t, e_i \rangle_H \leq 0\} \cap \llbracket S, T_n \rrbracket_{(\bullet, t)} \right) \in \mathcal{F}.$$

On the set B_n we have

$$S < T_n < \infty \quad \text{as well as} \quad \langle X_S, e_i \rangle_H = 0 \quad \text{and} \quad \langle X_{T_n}, e_i \rangle_H = -\frac{1}{n}.$$

Furthermore, we have

$$\langle X, e_i \rangle_H \leq 0 \quad \text{on} \quad \llbracket S, T_n \rrbracket \cap (B_n \times \mathbb{R}_+).$$

Therefore, using (85), on the set B_n we obtain

$$\begin{aligned} -\frac{1}{n} &= \langle X_{T_n}, e_i \rangle_H - \langle X_S, e_i \rangle_H = \int_S^{T_n} \langle \bar{\mu}(X_s), e_i \rangle_H ds + \int_S^{T_n} \langle \bar{\sigma}(X_s), e_i \rangle_H dW_s \\ &= \int_S^{T_n} \langle \mu(\Pi(X_s)), e_i \rangle_H ds + \int_S^{T_n} \langle \sigma(\Pi(X_s)), e_i \rangle_H dW_s \geq 0, \end{aligned}$$

which is a contradiction. Consequently, we have $X \in \mathcal{X}$ up to an evanescent set, and hence (X, W) is also a weak solution to the SDE (82). \square

We conclude this appendix with sufficient conditions for pathwise uniqueness. The following result is a version of [42, Thm. 2] in infinite dimension.

Theorem B.3. *We suppose that the following conditions are fulfilled:*

- (i) *We have $U = H$, and the operator Σ_W has a diagonal structure along the orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$.*
- (ii) *There exists a sequence $L = (L_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ with $L \in \ell^1(\mathbb{N})$ such that for each $i \in \mathbb{N}$ we have*

$$|\mu_i(x) - \mu_i(y)| \leq L_i \|x - y\|_H \quad \text{for all } x, y \in \mathcal{X}, \quad (86)$$

where $\mu_i : H \rightarrow \mathbb{R}$ is defined as

$$\mu_i(x) := \langle \mu(x), e_i \rangle_H, \quad x \in H.$$

- (iii) *For each $i \in \mathbb{N}$ there is a mapping $\sigma_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\sigma_i(0) = 0$ such that*

$$\sigma(x)u = \sum_{i \in \mathbb{N}} \sigma_i(\langle x, e_i \rangle_H) \langle g_i, u \rangle_{U_0} e_i \quad \text{for all } x \in \mathcal{X} \text{ and } u \in U_0, \quad (87)$$

where $g_i = \Sigma_W^{1/2} e_i$ for all $i \in \mathbb{N}$.

- (iv) *There exists a measurable, increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\rho(0) = 0$ and $\rho(u) \in (0, \infty)$ for all $u \in (0, \infty)$ satisfying*

$$\int_0^\epsilon \frac{1}{\rho(u)^2} du = \infty \quad \text{for all } \epsilon > 0 \quad (88)$$

such that for each $i \in \mathbb{N}$ we have

$$|\sigma_i(x) - \sigma_i(y)| \leq \rho(|x - y|) \quad \text{for all } x, y \in \mathbb{R}_+. \quad (89)$$

Then we have pathwise uniqueness with starting points in \mathcal{X} for the SDE (82).

Proof. The proof is similar to that of [42, Thm. 2], and we only sketch the most relevant arguments. The sequence $(\lambda_k)_{k \in \mathbb{N}} \subset (0, \infty)$ given by $\lambda_k := \langle \Sigma_W e_k, e_k \rangle_U$ for $k \in \mathbb{N}$ satisfies $\sum_{k \in \mathbb{N}} \lambda_k < \infty$ and we have $\Sigma_W e_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. Furthermore, the system $\{g_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of U_0 . Let (X, W) and (Y, W) be two weak solutions to the SDE (82) with $\mathbb{P}(X_0 = Y_0) = 1$. We set $Z := X - Y$. Let $i \in \mathbb{N}$ be arbitrary. By the diagonal structure (87) of σ we have \mathbb{P} -almost surely

$$\langle Z_t, e_i \rangle_H = \int_0^t (\mu_i(X_s) - \mu_i(Y_s)) ds + \int_0^t (\sigma_i(\langle X_s, e_i \rangle_H) - \sigma_i(\langle Y_s, e_i \rangle_H)) d\beta_s^i, \quad t \in \mathbb{R}_+,$$

where the process

$$\beta^i := \frac{1}{\sqrt{\lambda_i}} \langle W, e_i \rangle_U$$

is a real-valued standard Wiener process; see [12, Prop. 4.3.ii]. Using (88), we can choose a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of functions $\varphi_n \in C^2(\mathbb{R})$ precisely as in the proof of [42, Thm. 2]. Now, let $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ be arbitrary. By Itô's formula we obtain \mathbb{P} -almost surely

$$\begin{aligned} \varphi_n(\langle Z_t, e_i \rangle_H) &= \int_0^t \varphi_n'(\langle Z_s, e_i \rangle_H) (\mu_i(X_s) - \mu_i(Y_s)) ds \\ &\quad + \frac{1}{2} \int_0^t \varphi_n''(\langle Z_s, e_i \rangle_H) (\sigma_i(\langle X_s, e_i \rangle_H) - \sigma_i(\langle Y_s, e_i \rangle_H))^2 ds \\ &\quad + \int_0^t \varphi_n'(\langle Z_s, e_i \rangle_H) (\sigma_i(\langle X_s, e_i \rangle_H) - \sigma_i(\langle X'_s, e_i \rangle_H)) d\beta_s^i. \end{aligned}$$

Taking expectations, the last term vanishes, and by (86) for the first term we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[\int_0^t \varphi'_n(\langle Z_s, e_i \rangle_H) (\mu_i(X_s) - \mu_i(Y_s)) ds \right] \right| \leq \int_0^t \mathbb{E} [|\mu_i(X_s) - \mu_i(Y_s)|] ds \\ & \leq L_i \int_0^t \mathbb{E} [\|X_s - Y_s\|_H] ds = L_i \int_0^t \mathbb{E} [\|Z_s\|_H] ds. \end{aligned}$$

Furthermore, using (89) the second term is estimated as

$$\begin{aligned} & \frac{1}{2} \int_0^t \varphi''_n(\langle Z_s, e_i \rangle_H) (\sigma_i(\langle X_s, e_i \rangle_H) - \sigma_i(\langle Y_s, e_i \rangle_H))^2 ds \\ & \leq \frac{1}{2} \int_0^t \varphi''_n(\langle Z_s, e_i \rangle_H) \rho(|\langle Z_s, e_i \rangle_H|)^2 ds \leq \frac{t}{n}. \end{aligned}$$

Therefore, letting $n \rightarrow \infty$ we obtain

$$\mathbb{E} [|\langle Z_t, e_i \rangle_H|] \leq L_i \int_0^t \mathbb{E} [\|Z_s\|_H] ds.$$

Using the monotone convergence theorem, we deduce

$$\mathbb{E} [\|Z_t\|_H] \leq \mathbb{E} \left[\sum_{i \in \mathbb{N}} |\langle Z_t, e_i \rangle_H| \right] = \sum_{i \in \mathbb{N}} \mathbb{E} [|\langle Z_t, e_i \rangle_H|] \leq \|L\|_{\ell^1(\mathbb{N})} \int_0^t \mathbb{E} [\|Z_s\|_H] ds.$$

Since $L \in \ell^1(\mathbb{N})$, by Gronwall's inequality we obtain $X = Y$ up to indistinguishability, which concludes the proof. \square

APPENDIX C. LINEAR OPERATORS IN HILBERT SPACES

In this appendix we provide the required results about linear operators in Hilbert spaces. For two Hilbert spaces H and G the notations $L(H, G)$, $K(H, G)$, $L_1(H, G)$, $L_2(H, G)$, $L^+(H, G)$ and $L^{++}(H, G)$ mean the sets of all bounded linear operators, compact linear operators, nuclear operators, Hilbert-Schmidt operators, nonnegative operators and strictly positive operators. The following is a well-known result about compact operators.

Proposition C.1. *Let H_0 and H be separable Hilbert spaces. For every compact linear operator $T \in K(H_0, H)$ there exist orthonormal bases $\{f_k\}_{k \in \mathbb{N}}$ of H_0 and $\{e_k\}_{k \in \mathbb{N}}$ of H , and a decreasing sequence $(s_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ with $s_k \downarrow 0$ such that*

$$Tx = \sum_{k=1}^{\infty} s_k \langle x, f_k \rangle_{H_0} e_k \quad \text{for each } x \in H_0. \quad (90)$$

Proof. See, for example [40, Satz VI.3.6]. \square

The numbers $(s_k^2)_{k \in \mathbb{N}}$ are the eigenvalues of T^*T , and the $(s_k)_{k \in \mathbb{N}}$ are called the *singular values* of T . We say that a compact linear operator $T \in K(H_0, H)$ with representation (90) has *positive singular values* if $s_k > 0$ for all $k \in \mathbb{N}$. For what follows, let H be a separable Hilbert space.

Lemma C.2. *Let $T \in K^{++}(H)$ be a compact, self-adjoint, positive linear operator with representation*

$$Tx = \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H e_k \quad \text{for each } x \in H, \quad (91)$$

where $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of H , and $(\lambda_k)_{k \in \mathbb{N}} \subset (0, \infty)$ is a decreasing sequence with $\lambda_k \downarrow 0$. Then the following statements are true:

(i) The space $H_0 := T(H)$ equipped with the inner product

$$\langle x, y \rangle_{H_0} := \langle T^{-1}x, T^{-1}y \rangle_H, \quad x, y \in H_0 \quad (92)$$

is a separable Hilbert space, which is dense in H .

(ii) The operator $T : (H, \|\cdot\|_H) \rightarrow (H_0, \|\cdot\|_{H_0})$ is an isometric isomorphism.

(iii) The system $\{f_k\}_{k \in \mathbb{N}}$ given by

$$f_k := Te_k = \lambda_k e_k, \quad k \in \mathbb{N}$$

is an orthonormal basis of H_0 .

(iv) For all $x \in H_0$ and $k \in \mathbb{N}$ we have

$$\langle x, e_k \rangle_H = \lambda_k \langle x, f_k \rangle_{H_0}. \quad (93)$$

(v) We have the representation

$$H_0 = \left\{ x \in H : \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k^2} |\langle x, e_k \rangle_H|^2 < \infty \right\}. \quad (94)$$

(vi) The identity operator $\text{Id} : (H_0, \|\cdot\|_{H_0}) \rightarrow (H, \|\cdot\|_H)$ is a compact linear operator with positive singular values.

(vii) We have $H_0 \in \mathcal{B}(H)$ and $\mathcal{B}(H_0) = \mathcal{B}(H)_{H_0}$.

Proof. By (92), we have

$$\langle Tx, Ty \rangle_{H_0} = \langle x, y \rangle_H, \quad x, y \in H.$$

Hence $T : (H, \|\cdot\|_H) \rightarrow (H_0, \|\cdot\|_{H_0})$ is an isometric isomorphism, and it follows that H_0 is a separable Hilbert space. Furthermore, for all $k, l \in \mathbb{N}$ we have

$$\langle f_k, f_l \rangle_{H_0} = \langle Te_k, Te_l \rangle_{H_0} = \langle e_k, e_l \rangle_H,$$

showing that $\{f_k\}_{k \in \mathbb{N}}$ is an orthonormal system of H_0 . For all $x, y \in H_0$ we have

$$\begin{aligned} \langle x, y \rangle_{H_0} &= \langle T^{-1}x, T^{-1}y \rangle_H = \sum_{k \in \mathbb{N}} \langle T^{-1}x, e_k \rangle_H \langle e_k, T^{-1}y \rangle_H \\ &= \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k^2} \langle x, e_k \rangle_H \langle e_k, y \rangle_H. \end{aligned}$$

In particular, for all $x \in H_0$ and $k \in \mathbb{N}$ we have

$$\langle x, e_k \rangle_{H_0} = \frac{1}{\lambda_k^2} \langle x, e_k \rangle_H,$$

and hence

$$\langle x, e_k \rangle_H = \lambda_k^2 \langle x, e_k \rangle_{H_0} = \lambda_k \langle x, \lambda_k e_k \rangle_{H_0} = \lambda_k \langle x, f_k \rangle_{H_0},$$

showing (93). Therefore, we obtain

$$x = \sum_{k \in \mathbb{N}} \langle x, e_k \rangle_H e_k = \sum_{k \in \mathbb{N}} \lambda_k \langle x, f_k \rangle_{H_0} e_k = \sum_{k \in \mathbb{N}} \langle x, f_k \rangle_{H_0} f_k \quad \text{for each } x \in H_0.$$

Consequently, the system $\{f_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of H_0 , and the identity operator $\text{Id} : (H_0, \|\cdot\|_{H_0}) \rightarrow (H, \|\cdot\|_H)$ is a compact linear operator with positive singular values. In particular, we have $\mathcal{B}(H)_{H_0} \subset \mathcal{B}(H_0)$, and by Kuratowski's theorem (see, for example [32, Thm. I.3.9]) we obtain $H_0 \in \mathcal{B}(H)$ and $\mathcal{B}(H_0) = \mathcal{B}(H)_{H_0}$. Furthermore, by (93) we have

$$\sum_{k \in \mathbb{N}} \frac{1}{\lambda_k^2} |\langle x, e_k \rangle_H|^2 = \sum_{k \in \mathbb{N}} |\langle x, f_k \rangle_{H_0}|^2 \quad \text{for each } x \in H_0,$$

proving the representation (94), which also shows that H_0 is dense in H . \square

Informally, we will call a space of the form $H_0 = T(H)$, as provided in Lemma C.2, a retracted subspace with compact embedding.

Lemma C.3. *Let $T \in K^{++}(H)$ be a compact linear operator as in Lemma C.2, and let $S \in L(H)$ be a linear operator such that $ST = TS$. Then the following statements are true:*

- (i) *We have $S(H_0) \subset H_0$ and $S|_{H_0} \in L(H_0)$ with respect to $\|\cdot\|_{H_0}$.*
- (ii) *If S is self-adjoint, then $S|_{H_0}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{H_0}$.*
- (iii) *If $S \in L^+(H)$, then we have $S|_{H_0} \in L^+(H_0)$ with respect to $\langle \cdot, \cdot \rangle_{H_0}$.*
- (iv) *If $S \in L^{++}(H)$, then we have $S|_{H_0} \in L^{++}(H_0)$ with respect to $\langle \cdot, \cdot \rangle_{H_0}$.*
- (v) *If $S \in L_1(H)$, then we have $S|_{H_0} \in L_1(H_0)$ with respect to $\langle \cdot, \cdot \rangle_{H_0}$.*
- (vi) *If $S \in L_2(H)$, then we have $S|_{H_0} \in L_2(H_0)$ with respect to $\langle \cdot, \cdot \rangle_{H_0}$.*

Proof. Recall that $H_0 = T(H)$. Since $ST = TS$, we have

$$S|_{H_0} = STT^{-1} = TST^{-1},$$

showing that $S(H_0) \subset H_0$. By Lemma C.2 the operators $T : (H, \|\cdot\|_H) \rightarrow (H_0, \|\cdot\|_{H_0})$ and $T^{-1} : (H_0, \|\cdot\|_{H_0}) \rightarrow (H, \|\cdot\|_H)$ are isometric isomorphisms, showing that $S|_{H_0} \in L(H_0)$ with respect to $\|\cdot\|_{H_0}$. Furthermore, by (92) we have

$$\langle Sx, y \rangle_{H_0} = \langle TST^{-1}x, y \rangle_{H_0} = \langle ST^{-1}x, T^{-1}y \rangle_H \quad \text{for all } x, y \in H_0,$$

which, together with Lemma C.2, provides the remaining statements. \square

For the next result, we refer to Definition A.1 for the notion of the linear growth condition.

Lemma C.4. *Let $T \in K^{++}(H)$ be a compact linear operator as in Lemma C.2, and let $f : H \rightarrow H$ be a mapping such that $fT = Tf$. Then the following statements are true:*

- (i) *We have $f(H_0) \subset H_0$.*
- (ii) *If f satisfies the linear growth condition with respect to $\|\cdot\|_H$, then $f|_{H_0} : H_0 \rightarrow H_0$ satisfies the linear growth condition with respect to $\|\cdot\|_{H_0}$.*

Proof. Recall that $H_0 = T(H)$. Since $fT = Tf$, we have

$$f|_{H_0} = fTT^{-1} = TfT^{-1},$$

showing that $f(H_0) \subset H_0$. By Lemma C.2 the operators $T : (H, \|\cdot\|_H) \rightarrow (H_0, \|\cdot\|_{H_0})$ and $T^{-1} : (H_0, \|\cdot\|_{H_0}) \rightarrow (H, \|\cdot\|_H)$ are isometric isomorphisms, proving the remaining statement regarding the linear growth condition. \square

Lemma C.5. *For every self-adjoint operator $T \in L_1^+(H)$ we have*

$$\|T^{1/2}\|_{L_2(H)}^2 = \|T\|_{L_1(H)}.$$

Proof. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of H . Then we have

$$\begin{aligned} \|T^{1/2}\|_{L_2(H)}^2 &= \sum_{k \in \mathbb{N}} \|T^{1/2}e_k\|_H^2 = \sum_{k \in \mathbb{N}} \langle T^{1/2}e_k, T^{1/2}e_k \rangle_H \\ &= \sum_{k \in \mathbb{N}} \langle Te_k, e_k \rangle_H = \|T\|_{L_1(H)}, \end{aligned}$$

completing the proof. \square

Lemma C.6. *Let $T \in L^+(H)$ be a self-adjoint operator, and let $x \in H$ be such that $\langle Tx, x \rangle_H = 0$. Then we have $Tx = 0$.*

Proof. By assumption we have $\langle T\xi, \xi \rangle_H = 0$ for each $\xi \in \text{lin}\{x\}$. This gives us

$$0 \leq \langle T(\xi + y), \xi + y \rangle_H = 2\langle Ty, \xi \rangle_H + \langle Ty, y \rangle_H \quad \text{for all } \xi \in \text{lin}\{x\} \text{ and } y \in H.$$

Therefore, we have $\langle Ty, x \rangle_H = 0$ for all $y \in H$, and hence $\langle Tx, y \rangle_H = 0$ for all $y \in H$, which implies $Tx = 0$. \square

For what follows, let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of H . The closed convex cone H^+ is defined as

$$H^+ := \{x \in H : \langle x, e_k \rangle_H \geq 0 \text{ for all } k \in \mathbb{N}\}.$$

Lemma C.7. *Let $T \in L(H, L_1(H))$ be such that the following conditions are fulfilled:*

- (i) *We have $T(e_i)e_j = 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$.*
- (ii) *For each $x \in H$ the operator $T(x)$ is self-adjoint.*
- (iii) *For each $x \in H^+$ we have $T(x) \in L_1^+(H)$.*

We define the sequence $\lambda = (\lambda_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ as

$$\lambda_i := \|T(e_i)e_i\|_H \quad \text{for all } i \in \mathbb{N}.$$

Then we have $\lambda \in \ell^2(\mathbb{N})$, the representation

$$\lambda_i = \langle T(e_i)e_i, e_i \rangle_H, \quad i \in \mathbb{N} \tag{95}$$

as well as

$$T(x)\xi = \sum_{i \in \mathbb{N}} \lambda_i \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H e_i \quad \text{for all } x, \xi \in H. \tag{96}$$

Proof. Let $i \in \mathbb{N}$ be arbitrary. Since $T(e_i)$ is self-adjoint, we obtain

$$\langle T(e_i)e_i, e_j \rangle_H = \langle e_i, T(e_i)e_j \rangle_H = 0 \quad \text{for each } j \in \mathbb{N} \text{ with } j \neq i,$$

and hence $T(e_i)e_i \in \text{lin}\{e_i\}$. Therefore, we have

$$T(e_i)e_i = \langle T(e_i)e_i, e_i \rangle_H e_i,$$

and hence

$$\lambda_i = \|T(e_i)e_i\|_H = \langle T(e_i)e_i, e_i \rangle_H,$$

showing (95) and $T(e_i)e_i = \lambda_i e_i$. Therefore, for all $x, \xi \in H$ we obtain

$$\begin{aligned} T(x)\xi &= T\left(\sum_{i \in \mathbb{N}} \langle x, e_i \rangle_H e_i\right) \sum_{j \in \mathbb{N}} \langle \xi, e_j \rangle_H e_j = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H T(e_i)e_i \\ &= \sum_{i \in \mathbb{N}} \lambda_i \langle x, e_i \rangle_H \langle e_i, \xi \rangle_H e_i, \end{aligned}$$

showing (96). Now, let $x \in H^+$ be arbitrary. Since $T(x) \in L_1^+(H)$, we have

$$\sum_{i \in \mathbb{N}} \lambda_i \langle x, e_i \rangle_H = \sum_{i \in \mathbb{N}} \langle T(x)e_i, e_i \rangle_H < \infty.$$

Since this series converges absolutely, we deduce that

$$\sum_{i \in \mathbb{N}} \lambda_i |\langle x, e_i \rangle_H| < \infty.$$

For each $n \in \mathbb{N}$ let $T_n \in H'$ be the continuous linear functional given by

$$T_n x = \left\langle x, \sum_{i=1}^n \lambda_i e_i \right\rangle_H, \quad x \in H.$$

Let $x \in H$ be arbitrary. Then for each $n \in \mathbb{N}$ we have

$$|T_n x| = \left| \left\langle x, \sum_{i=1}^n \lambda_i e_i \right\rangle_H \right| = \left| \sum_{i=1}^n \lambda_i \langle x, e_i \rangle_H \right| \leq \sum_{i=1}^n \lambda_i |\langle x, e_i \rangle_H|,$$

and hence

$$\sup_{n \in \mathbb{N}} |T_n x| \leq \sum_{i \in \mathbb{N}} \lambda_i |\langle x, e_i \rangle_H| < \infty.$$

By the uniform boundedness principle (see, for example [40, Thm. IV.2.1]) it follows that $\sup_{n \in \mathbb{N}} \|T_n\|_{H'} < \infty$. Note that for each $n \in \mathbb{N}$ we have

$$\|T_n\|_{H'}^2 = \left\| \sum_{i=1}^n \lambda_i e_i \right\|_H^2 = \sum_{i=1}^n \lambda_i^2,$$

and hence

$$\sup_{n \in \mathbb{N}} \|T_n\|_{H'} = \sum_{i \in \mathbb{N}} \lambda_i^2.$$

Consequently, we have $\lambda \in \ell^2(\mathbb{N})$. □

Lemma C.8. *Let $(y_k)_{k \in \mathbb{N}} \subset H$ be a sequence such that $(\|y_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$. Then the mapping*

$$Tx := \sum_{k=1}^{\infty} \langle x, e_k \rangle_H y_k, \quad x \in H$$

is a well-defined continuous linear operator $T \in L(H)$.

Proof. By the Cauchy Schwarz inequality, for each $x \in H$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} \|\langle x, e_k \rangle_H y_k\| &= \sum_{k=1}^{\infty} |\langle x, e_k \rangle_H| \|y_k\| \leq \left(\sum_{k=1}^{\infty} |\langle x, e_k \rangle_H|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \|y_k\|_H^2 \right)^{1/2} \\ &= \|x\|_H \left(\sum_{k=1}^{\infty} \|y_k\|_H^2 \right)^{1/2} < \infty, \end{aligned}$$

because $(\|y_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$. Thus, applying the uniform boundedness principle (see, for example [40, Kor. IV.2.5]) concludes the proof. □

Lemma C.9. *Let $\lambda = (\lambda_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ be a nonnegative sequence such that $\lambda \in \ell^2(\mathbb{N})$. Then the following statements are true:*

(i) *The mapping*

$$T(x)\xi := \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H \langle e_k, \xi \rangle_H e_k, \quad x, \xi \in H \tag{97}$$

is a well-defined continuous linear operator $T \in L(H, L_1(H))$.

(ii) *For each $x \in H^+$ we have $T(x) \in L_1^+(H)$.*

(iii) *For each $x \in H$ the operator $T(x)$ is self-adjoint.*

(iv) *We have $T(e_i)e_j = 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$.*

(v) *For each $k \in \mathbb{N}$ we have*

$$\lambda_k = \|T(e_k)e_k\|_H = \langle T(e_k)e_k, e_k \rangle_H.$$

Proof. Let $x \in H$ be arbitrary. Then we have

$$T(x)\xi = \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H e_k \langle e_k, \xi \rangle_H \quad \text{for all } \xi \in H,$$

and by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_{k \in \mathbb{N}} \|\lambda_k \langle x, e_k \rangle_H e_k\|_H \|e_k\|_H &= \sum_{k \in \mathbb{N}} |\lambda_k \langle x, e_k \rangle_H| \leq \left(\sum_{k \in \mathbb{N}} \lambda_k^2 \right)^{1/2} \left(\sum_{k \in \mathbb{N}} |\langle x, e_k \rangle_H|^2 \right)^{1/2} \\ &= \|\lambda\|_{\ell^2(I)} \|x\|_H. \end{aligned}$$

Therefore, we have $T(x) \in L_1(H)$ with

$$\|T(x)\|_{L_1(H)} \leq \|\lambda\|_{\ell^2(I)} \|x\|_H,$$

and hence $T \in L(H, L_1(H))$. If $x \in H^+$, then we have $\lambda_k \langle x, e_k \rangle_H \geq 0$ for each $k \in \mathbb{N}$, and hence

$$\langle T(x)\xi, \xi \rangle_H = \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H |\langle e_k, \xi \rangle_H|^2 \geq 0 \quad \text{for all } \xi \in H,$$

showing that $T(x) \in L_1^+(H)$. For all $\xi, \eta \in H$ we have

$$\begin{aligned} \langle T(x)\xi, \eta \rangle_H &= \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H \langle e_k, \xi \rangle_H \langle e_k, \eta \rangle_H \\ &= \sum_{k \in \mathbb{N}} \lambda_k \langle x, e_k \rangle_H \langle e_k, \eta \rangle_H \langle \xi, e_k \rangle_H = \langle \xi, T(x)\eta \rangle_H, \end{aligned}$$

showing that Tx is self-adjoint. The remaining statements follow from the representation (97). \square

In accordance with the notation from Subsection 4.4, let $I, J \subset \mathbb{N}$ be infinite, disjoint index sets such that $I \cup J = \mathbb{N}$, and let $\tau : I \rightarrow J$ be a bijection such that $\tau(i_1) \leq \tau(i_2)$ for all $i_1, i_2 \in I$ with $i_1 \leq i_2$. We define

$$H_I^+ := \text{lin}^+\{e_i : i \in I\}, \quad H_I := \text{lin}\{e_i : i \in I\} \quad \text{and} \quad H_J := \text{lin}\{e_j : j \in J\}$$

as well as the closed convex cone $\mathcal{X} := H_I^+ \oplus H_J$. We also recall that $\pi_I : H \rightarrow H_I$ and $\pi_J : H \rightarrow H_J$ denote the corresponding projections, as well as the notations $T_{II} = \pi_I T|_{H_I}$ and $T_{IJ} = \pi_J T|_{H_I}$ for $T \in L(H)$.

Lemma C.10. *Let $\lambda = (\lambda_i)_{i \in I} \subset \mathbb{R}_+$ and $\kappa = (\kappa_i)_{i \in I} \subset \mathbb{R}_+$ be nonnegative sequences such that $\lambda \in \ell^2(I)$ and $\kappa_i \leq \lambda_i$ for all $i \in I$. Then the following statements are true:*

(i) *The mapping*

$$T(x)\xi := \sum_{i \in I} \langle x, e_i \rangle_H \left(\langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \xi \rangle_H e_i + \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \xi \rangle_H e_{\tau(i)} \right), \quad x, \xi \in H \quad (98)$$

is a well-defined continuous linear operator $T \in L(H, L_1(H))$.

- (ii) *For each $x \in \mathcal{X}$ we have $T(x) \in L_1^+(H)$.*
- (iii) *For each $x \in H$ the operator $T(x)$ is self-adjoint.*
- (iv) *We have $T(e_i)e_j = 0$ for all $i, j \in I$ with $i \neq j$.*
- (v) *For each $i \in I$ we have*

$$\lambda_i = \|T(e_i)_{II} e_i\|_H = \langle T(e_i)_{II} e_i, e_i \rangle_H.$$

(vi) *For each $i \in I$ we have*

$$\kappa_i = \|T(e_i)_{IJ} e_i\|_H = \langle T(e_i)_{IJ} e_i, e_{\tau(i)} \rangle_H.$$

Proof. Let $x \in H$ be arbitrary. Then we have the decomposition $T(x) = T_1(x) + T_2(x)$, where

$$\begin{aligned} T_1(x)\xi &= \sum_{i \in I} \langle x, e_i \rangle_H e_i \langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \xi \rangle_H, \quad \xi \in H, \\ T_2(x)\xi &= \sum_{i \in I} \langle x, e_i \rangle_H e_{\tau(i)} \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \xi \rangle_H, \quad \xi \in H. \end{aligned}$$

By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_{i \in I} \|\langle x, e_i \rangle_H e_i\|_H \|\lambda_i e_i + \kappa_i e_{\tau(i)}\|_H &= \sum_{i \in I} |\langle x, e_i \rangle_H| \sqrt{\lambda_i^2 + \kappa_i^2} \\ &\leq \left(\sum_{i \in I} |\langle x, e_i \rangle_H|^2 \right)^{1/2} \left(\sum_{i \in I} (\lambda_i^2 + \kappa_i^2) \right)^{1/2} \leq (\|\lambda\|_{\ell^2(I)}^2 + \|\kappa\|_{\ell^2(I)}^2)^{1/2} \|x\|_H, \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{i \in I} \|\langle x, e_i \rangle_H e_{\tau(i)}\|_H \|\kappa_i e_i + \lambda_i e_{\tau(i)}\|_H &= \sum_{i \in I} |\langle x, e_i \rangle_H| \sqrt{\lambda_i^2 + \kappa_i^2} \\ &\leq \left(\sum_{i \in I} |\langle x, e_i \rangle_H|^2 \right)^{1/2} \left(\sum_{i \in I} (\lambda_i^2 + \kappa_i^2) \right)^{1/2} \leq (\|\lambda\|_{\ell^2(I)}^2 + \|\kappa\|_{\ell^2(I)}^2)^{1/2} \|x\|_H. \end{aligned}$$

Therefore, we have $T_1(x), T_2(x) \in L_1(H)$, and hence $T(x) \in L_1(H)$ with

$$\|T(x)\|_{L_1(H)} \leq \|T_1(x)\|_{L_1(H)} + \|T_2(x)\|_{L_1(H)} \leq 2(\|\lambda\|_{\ell^2(I)}^2 + \|\kappa\|_{\ell^2(I)}^2)^{1/2} \|x\|_H,$$

showing that $T \in L(H, L_1(H))$. Now let $x \in \mathcal{X}$ and $\xi \in H$ be arbitrary. Since $\kappa_i \leq \lambda_i$ for all $i \in I$, we obtain

$$\begin{aligned} \langle T(x)\xi, \xi \rangle_H &= \sum_{i \in I} \langle x, e_i \rangle_H \left(\langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \xi \rangle_H \langle e_i, \xi \rangle_H + \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \xi \rangle_H \langle e_{\tau(i)}, \xi \rangle_H \right) \\ &= \sum_{i \in I} \langle x, e_i \rangle_H \left(\lambda_i (|\langle e_i, \xi \rangle|^2 + |\langle e_{\tau(i)}, \xi \rangle|^2) + 2\kappa_i \langle e_i, \xi \rangle_H \langle e_{\tau(i)}, \xi \rangle_H \right) \\ &= \sum_{i \in I} \langle x, e_i \rangle_H \left((\lambda_i - \kappa_i) (|\langle e_i, \xi \rangle|^2 + |\langle e_{\tau(i)}, \xi \rangle|^2) + \kappa_i (\langle e_i, \xi \rangle_H + \langle e_{\tau(i)}, \xi \rangle_H)^2 \right) \\ &\geq 0, \end{aligned}$$

showing that $T(x) \in L_1^+(H)$. Furthermore, for all $x \in H$ and all $\xi, \eta \in H$ we have

$$\begin{aligned} \langle T(x)\xi, \eta \rangle_H &= \sum_{i \in I} \langle x, e_i \rangle_H \left(\langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \xi \rangle_H \langle e_i, \eta \rangle_H + \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \xi \rangle_H \langle e_{\tau(i)}, \eta \rangle_H \right) \\ &= \sum_{i \in I} \langle x, e_i \rangle_H \left(\langle \lambda_i e_i + \kappa_i e_{\tau(i)}, \eta \rangle_H \langle e_i, \xi \rangle_H + \langle \kappa_i e_i + \lambda_i e_{\tau(i)}, \eta \rangle_H \langle e_{\tau(i)}, \xi \rangle_H \right) \\ &= \langle T(x)\eta, \xi \rangle_H, \end{aligned}$$

showing that $T(x)$ is self-adjoint. Moreover, by the representation (98) we have $T(e_i)e_j = 0$ for all $i, j \in I$ with $i \neq j$ as well as

$$T(e_i)e_i = \lambda_i e_i + \kappa_i e_{\tau(i)} \quad \text{for each } i \in I.$$

Therefore, the remaining statements follow immediately. \square

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FREIBURG INSTITUTE OF ADVANCED STUDIES (FRIAS), GERMANY. UNIVERSITY OF STRASBOURG INSTITUTE FOR ADVANCED STUDY (USIAS), FRANCE. UNIVERSITY OF FREIBURG, DEPARTMENT OF MATHEMATICAL STOCHASTICS, ERNST-ZERMELO-STR. 1, 79104 FREIBURG, GERMANY

E-mail address: `thorsten.schmidt@stochastik.uni-freiburg.de`

ALBERT LUDWIG UNIVERSITY OF FREIBURG, DEPARTMENT OF MATHEMATICAL STOCHASTICS, ERNST-ZERMELO-STRASSE 1, 79104 FREIBURG, GERMANY.

E-mail address: `stefan.tappe@math.uni-freiburg.de`

ALBERT LUDWIG UNIVERSITY OF FREIBURG, DEPARTMENT OF MATHEMATICAL STOCHASTICS, ERNST-ZERMELO-STRASSE 1, 79104 FREIBURG, GERMANY.

E-mail address: `ywj1210@gmail.com`