TERM STRUCTURE MODELS DRIVEN BY WIENER PROCESSES AND POISSON MEASURES: EXISTENCE AND POSITIVITY

DAMIR FILIPOVIĆ, STEFAN TAPPE, AND JOSEF TEICHMANN

ABSTRACT. In the spirit of [3], we investigate term structure models driven by Wiener processes and Poisson measures with forward curve dependent volatilities. This includes a full existence and uniqueness proof for the corresponding Heath-Jarrow-Morton type term structure equation. Furthermore, we characterize positivity preserving models by means of the characteristic coefficients, which was open for jump-diffusions. A key role in our investigation is played by the method of the moving frame, which allows to transform term structure equations to time-dependent SDEs.

Key Words: term structure models driven by Wiener processes and Poisson measures, Heath-Jarrow-Morton-Musiela equation, positivity preserving models.

91B28, 60H15

1. Introduction

Interest rate theory is dealing with zero-coupon bonds, which are subject to a stochastic evolution due to daily trading of related products like coupon bearing bonds, swaps, caps, floors, swaptions, etc. Zero-coupon bonds, which are financial assets paying the holder one unit of cash at maturity time T, are conceptually important products, since one can easily write all other products as derivatives on them. We always assume default-free bonds, i.e. there are no counterparty risks in the considered markets. The Heath-Jarrow-Morton methodology takes the bond market as a whole as today's aggregation of information on interest rates and one tries to model future flows of information by a stochastic evolution equation on the set of possible scenarios of bond prices. For the set of possible scenarios of bond prices the forward rate proved to be a flexible and useful parametrization, since it maps possible states of the bond market to open subsets of (Hilbert) spaces of forward rate curves. Under some regularity assumptions the price of a zero coupon bond at $t \leq T$ can be written as

$$P(t, T) = \exp\left(- \int_t^T f(t, u) du \right),$$

where $f(t, T)$ is the forward rate for date T. We usually assume the forward rate to be continuous in maturity time T. The classical continuous framework for the evolution of the forward rates goes back to Heath, Jarrow and Morton (HJM) [23]. They assume that, for every date T, the forward rates $f(t, T)$ follow an Itô process

of the form

$$(1.1) \qquad df(t,T) = \alpha(t,T)dt + \sum_{j=1}^{d} \sigma^j(t,T)dW_t^j, \quad t \in [0,T]$$

where $W = (W^1, \ldots, W^d)$ is a standard Brownian motion in $\mathbb{R}^d$.

There are several reasons for generalizing the HJM framework (1.1) by introducing jumps. Namely, this allows us to model the impact of unexpected news about the economy, such as interventions by central banks, credit events, or (natural) disasters. Indeed, there is strong statistical evidence in the finance literature that empirical features of the data cannot be captured by continuous models. We also mention that for the particular case of deterministic integrands α and σ in (1.1), as it is for the Vasiček model, the log returns for discounted zero coupon bonds are normally distributed, which, however, is not true for empirically observed log returns, see the discussion in [37, Chap. 5].

Björk et al. [3, 4], Eberlein et al. [10, 11, 12, 13, 14, 15] and others ([38, 26, 24]) thus proposed to replace the classical Brownian motion W in (1.1) by a more general driving noise, also taking into account the occurrence of jumps. Carmona and Tehranchi [6] proposed models based on infinite dimensional Wiener processes, see also [16]. In the spirit of Björk et al. [3] and Carmona and Tehranchi [6], we focus on term structure models of the type

$$(1.2) \qquad df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t + \int_E \gamma(t,x,T)(\mu(dt,dx) - F(dx)dt),$$

where W denotes a (possibly infinite dimensional) Wiener process and, in addition, μ is a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $dt \otimes F(dx)$, where E denotes the mark space.

For what follows, it will be convenient to switch to the alternative parametrization

$$r_t(\xi) := f(t, t+\xi), \quad \xi \geq 0$$

which is due to Musiela [32]. Then, we may regard $(r_t)_{t \geq 0}$ as *one* stochastic process with values in H, that is

$$r : \Omega \times \mathbb{R}_+ \to H,$$

where H denotes a Hilbert space of forward curves $h : \mathbb{R}_+ \to \mathbb{R}$ to be specified later. Recall that we always assume that forward rate curves are continuous. Denoting by $(S_t)_{t \geq 0}$ the shift semigroup on H, that is $S_t h = h(t + \cdot)$, equation (1.2) becomes in integrated form

$$(1.3) \qquad \begin{aligned} r_t(\xi) &= S_t h_0(\xi) + \int_0^t S_{t-s} \alpha(s, s+\xi)ds + \int_0^t S_{t-s} \sigma(s, s+\xi)dW_s \\ &+ \int_0^t \int_E S_{t-s} \gamma(s, x, s+\xi)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0 \end{aligned}$$

where $h_0 \in H$ denotes the initial forward curve and S_{t-s} operates on the functions $\xi \mapsto \alpha(s, s+\xi)$, $\xi \mapsto \sigma(s, s+\xi)$ and $\xi \mapsto \gamma(s, x, s+\xi)$.

From a financial modeling point of view, one would rather consider drift and volatilities to be functions of the prevailing forward curve, that is

$$\alpha : H \to H,$$
$$\sigma^j : H \to H, \quad \text{for all } j$$
$$\gamma : H \times E \to H.$$

For example, the volatilities could be of the form $\sigma^j(h) = \phi_j(\ell_1(h), \ldots, \ell_p(h))$ for some $p \in \mathbb{N}$ with $\phi_j : \mathbb{R}^p \to H$ and $\ell_i : H \to \mathbb{R}$. We may think of $\ell_i(h) = \frac{1}{\xi_i} \int_0^{\xi_i} h(\eta) d\eta$ (benchmark yields) or $\ell_i(h) = h(\xi_i)$ (benchmark forward rates).

The implied bond market

$$(1.4) \qquad P(t, T) = \exp\left(-\int_0^{T-t} r_t(\xi) d\xi\right)$$

is free of arbitrage if we can find an equivalent (local) martingale measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted bond prices

$$(1.5) \qquad \exp\left(-\int_0^t r_s(0) ds\right) P(t, T), \quad t \in [0, T]$$

are local $\mathbb{Q}$-martingales for all maturities T. In the sequel, we will directly specify the HJM equation under a martingale measure. More precisely, we will assume that the drift $\alpha = \alpha_{\mathrm{HJM}} : H \to H$ is given by

$$(1.6) \qquad \alpha_{\mathrm{HJM}}(h) := \sum_j \sigma^j(h) \Sigma^j(h) - \int_E \gamma(h, x) \left(e^{\Gamma(h,x)} - 1\right) F(dx)$$

for all $h \in H$, where we have set

$$(1.7) \qquad \Sigma^j(h)(\xi) := \int_0^\xi \sigma^j(h)(\eta) d\eta, \quad \text{for all } j$$

$$(1.8) \qquad \Gamma(h, x)(\xi) := -\int_0^\xi \gamma(h, x)(\eta) d\eta.$$

According to [3] (if the Brownian motion is infinite dimensional, see also [16]), condition (1.6) guarantees that the discounted zero coupon bond prices (1.5) are local martingales for all maturities T, whence the bond market (1.4) is free of arbitrage. In the classical situation, where the model is driven by a finite dimensional standard Brownian motion, (1.6) is the well-known HJM drift condition derived in [23].

Our requirements lead to the forward rates $(r_t)_{t \geq 0}$ in (1.3) being a solution of the stochastic equation

$$(1.9) \qquad \begin{aligned} r_t = {} & S_t h_0 + \int_0^t S_{t-s} \alpha_{\mathrm{HJM}}(r_s) ds + \int_0^t S_{t-s} \sigma(r_s) dW_s \\ & + \int_0^t \int_E S_{t-s} \gamma(r_{s-}, x)(\mu(ds, dx) - F(dx) ds), \quad t \geq 0 \end{aligned}$$

and it arises the question whether this equation possesses a solution. To our knowledge, there has not yet been an explicit proof for the *existence* of a solution to the Poisson measure driven equation (1.9). We thus provide such a proof in our paper, see Theorem 3.4. For term structure models driven by a Brownian motion, the existence proof has been provided in [16] and for the Lévy case in [19]. We also refer to the related papers [36] and [28].

In the spirit of [8] and [35], an H-valued stochastic process $(r_t)_{t \geq 0}$ satisfying (1.9) is a so-called *mild solution* for the (semi-linear) stochastic partial differential equation

$$(1.10)$$
$$\begin{cases} dr_t &= (\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t)) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx) dt) \\ r_0 &= h_0, \end{cases}$$

where $\frac{d}{d\xi}$ becomes the infinitesimal generator of the strongly continuous semigroup of shifts $(S_t)_{t \geq 0}$.

As in [20], we understand stochastic partial differential equations as time-dependent transformations of time-dependent stochastic differential equations with infinite dimensional state space. More precisely, on an enlarged space $\mathcal{H}$ of forward curves $h : \mathbb{R} \to \mathbb{R}$, which are indexed by the whole real line, equipped with the strongly continuous *group* $(U_t)_{t \in \mathbb{R}}$ of shifts, we solve the stochastic differential equation

$$(1.11) \quad \begin{cases} df_t = U_{-t}\ell\alpha_{\mathrm{HJM}}(\pi U_t f_t)dt + U_{-t}\ell\sigma(\pi U_t f_t)dW_t \\ \qquad + \int_E U_{-t}\ell\gamma(\pi U_t f_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ f_0 = \ell h_0, \end{cases}$$

where $\ell : H \to \mathcal{H}$ is an isometric embedding and $\pi : \mathcal{H} \to H$ is the orthogonal projection on H, and afterwards, we transform the solution process $(f_t)_{t \geq 0}$ by $r_t := \pi U_t f_t$ in order to obtain a mild solution for (1.10). Notice that (1.11) just corresponds to the original HJM dynamics in (1.2), where, of course, the forward rate $f_t(T)$ has no economic interpretation for $T < t$. Thus, we will henceforth refer to (1.11) as the HJM (Heath-Jarrow-Morton) equation.

We emphasize that knowledge about the HJM equation (1.11) is not necessary in order to deduce existence and uniqueness for the forward curve evolution (1.10). The only thing we require in order to apply the existence result from [20] is that we can embed the space H of forward curves into a larger Hilbert space $\mathcal{H}$, on which the shift semigroup extends to a group. The point of the "method of the moving frame" from [20] is to transform an SPDE into a time-dependent SDE on $\mathcal{H}$, but we do not need the particular structure of this extension.

Our existence result of this paper generalizes that for pure diffusion models from [16] and for term structure models driven by Lévy processes from [19] (we also mention the papers [36] and [28] concerning Lévy term structure models), see Corollary 3.6. As described above, in order to establish the proof we apply a result from [20], which contains existence and uniqueness results for general SPDEs on Hilbert spaces. Other references for SPDEs on Hilbert spaces are [1] and [29]. Since the HJM drift term α_{HJM} is of the particular form (1.6), we have to work out sufficient conditions which permit an application of this existence result.

In practice, we are interested in term structure models producing positive forward curves, since negative forward rates are very rarely observed. After establishing the existence issue, we shall therefore focus on positivity preserving term structure models, and give a characterization of such models. The HJM equation (1.11) on the enlarged function space will be the key for analyzing positivity of forward curves. Indeed, the "method of the moving frame", see [20], allows us to use standard stochastic analysis (see [25]) for our investigations. It will turn out that the conditions

$$\sigma^j(h)(\xi) = 0, \text{ for all } \xi \in (0, \infty),\, h \in \partial P_\xi \text{ and all } j$$
$$h + \gamma(h, x) \in P, \text{ for all } h \in P \text{ and } F\text{-almost all } x \in E$$
$$\gamma(h, x)(\xi) = 0, \text{ for all } \xi \in (0, \infty),\, h \in \partial P_\xi \text{ and } F\text{-almost all } x \in E$$

where P denotes the convex cone of all nonnegative forward curves, and ∂P_ξ is the "edge" consisting of all nonnegative forward curves h with $h(\xi) = 0$, are necessary and sufficient for the positivity preserving property, see Theorem 4.24. For this purpose, we provide a general positivity preserving result, see Theorem 4.20, which is of independent interest and can also be applied on other function spaces. Positivity results for the diffusion case have been worked out in [27] and [30]. In particular, we would like to mention the important and beautiful work [34], where, through an application of a general support theorem, positivity is proved. We shall also apply this general argument for our reasonings.

The remainder of this text is organized as follows. In Section 2 we introduce the space H_β of forward curves. Using this space, we prove in Section 3, under appropriate regularity assumptions, the existence of a unique solution for the HJMM equation (1.10). The positivity issue of term structure models is treated in Section 4. There, we show first the necessary conditions with a general semimartingale argument. The sufficient conditions are proved to hold true by switching on the jumps "slowly". This allows for a reduction to results from [34]. For the sake of lucidity, we postpone the proofs of some auxiliary results to Appendix A.

2. The space of forward curves

In this section, we introduce the space of forward curves, on which we will solve the HJMM equation (1.10) in Section 3.

We fix an arbitrary constant $\beta > 0$. Let H_β be the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta \xi} d\xi \right)^{1/2} < \infty.$$

Let $(S_t)_{t \geq 0}$ be the shift semigroup on H_β defined by $S_t h := h(t + \cdot)$ for $t \in \mathbb{R}_+$.

Since forward curves should flatten for large time to maturity ξ, the choice of H_β is reasonable from an economic point of view.

Moreover, let $\mathcal{H}_\beta$ be the space of all absolutely continuous functions $h : \mathbb{R} \to \mathbb{R}$ such that

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}} |h'(\xi)|^2 e^{\beta |\xi|} d\xi \right)^{1/2} < \infty.$$

Let $(U_t)_{t \in \mathbb{R}}$ be the shift group on $\mathcal{H}_\beta$ defined by $U_t h := h(t + \cdot)$ for $t \in \mathbb{R}$.

The linear operator $\ell : H_\beta \to \mathcal{H}_\beta$ defined by

$$\ell(h)(\xi) := \begin{cases} h(0), & \xi < 0 \\ h(\xi), & \xi \geq 0, \end{cases} \quad h \in H_\beta$$

is an isometric embedding with adjoint operator $\pi := \ell^* : \mathcal{H}_\beta \to H_\beta$ given by $\pi(h) = h|_{\mathbb{R}_+}$, $h \in \mathcal{H}_\beta$.

2.1. **Theorem.** *Let $\beta > 0$ be arbitrary.*

(1) *The space $(H_\beta, \| \cdot \|_\beta)$ is a separable Hilbert space.*
(2) *For each $\xi \in \mathbb{R}_+$, the point evaluation $h \mapsto h(\xi) : H_\beta \to \mathbb{R}$ is a continuous linear functional.*
(3) *$(S_t)_{t \geq 0}$ is a C_0-semigroup on H_β with infinitesimal generator $\frac{d}{d\xi} : \mathcal{D}(\frac{d}{d\xi}) \subset H_\beta \to H_\beta$, $\frac{d}{d\xi} h = h'$, and domain*

$$\mathcal{D}(\tfrac{d}{d\xi}) = \{h \in H_\beta \,|\, h' \in H_\beta\}.$$

(4) *Each $h \in H_\beta$ is continuous, bounded and the limit $h(\infty) := \lim_{\xi \to \infty} h(\xi)$ exists.*
(5) *$H_\beta^0 := \{h \in H_\beta \,|\, h(\infty) = 0\}$ is a closed subspace of H_β.*
(6) *There are universal constants $C_1, C_2, C_3, C_4 > 0$, only depending on β, such that for all $h \in H_\beta$ we have the estimates*

$$(2.1) \qquad \|h'\|_{L^1(\mathbb{R}_+)} \leq C_1 \|h\|_\beta,$$

$$(2.2) \qquad \|h\|_{L^\infty(\mathbb{R}_+)} \leq C_2 \|h\|_\beta,$$

$$(2.3) \qquad \|h - h(\infty)\|_{L^1(\mathbb{R}_+)} \leq C_3 \|h\|_\beta,$$

$$(2.4) \qquad \|(h - h(\infty))^4 e^{\beta \bullet}\|_{L^1(\mathbb{R}_+)} \leq C_4 \|h\|_\beta^4.$$

(7) *For each $\beta' > \beta$, we have $H_{\beta'} \subset H_\beta$, the relation*

(2.5)
$$\|h\|_\beta \leq \|h\|_{\beta'}, \quad h \in H_{\beta'}$$

and there is a universal constant $C_5 > 0$, only depending on β and β', such that for all $h \in H_{\beta'}$ we have the estimate

(2.6)
$$\|(h - h(\infty))^2 e^{\beta \bullet}\|_{L^1(\mathbb{R}_+)} \leq C_5 \|h\|_{\beta'}^2.$$

(8) *The space $(\mathcal{H}_\beta, \|\cdot\|_\beta)$ is a separable Hilbert space, $(U_t)_{t \in \mathbb{R}}$ is a C_0-group on $\mathcal{H}_\beta$ and, for each $\xi \in \mathbb{R}$, the point evaluation $h \mapsto h(\xi)$, $\mathcal{H}_\beta \to \mathbb{R}$ is a continuous linear functional.*

(9) *The diagram*

$$
\begin{array}{ccc}
\mathcal{H}_\beta & \xrightarrow{\;U_t\;} & \mathcal{H}_\beta \\
\Big\uparrow{\scriptstyle \ell} & & \Big\downarrow{\scriptstyle \pi} \\
H_\beta & \xrightarrow{\;S_t\;} & H_\beta
\end{array}
$$

commutes for every $t \in \mathbb{R}_+$, that is

(2.7)
$$\pi U_t \ell = S_t \quad \text{for all } t \in \mathbb{R}_+.$$

Proof. See Appendix A. $\qquad\square$

3. Existence of term structure models driven by Wiener processes and Poisson measures

In this section, we establish existence and uniqueness of the HJMM equation (1.10) with diffusive and jump components on the Hilbert spaces introduced in the last section.

Let $0 < \beta < \beta'$ be arbitrary real numbers. We denote by H_β and $H_{\beta'}$ the Hilbert spaces of the previous section, equipped with the strongly continuous semigroup $(S_t)_{t \geq 0}$ of shifts, which has the infinitesimal generator $\frac{d}{d\xi}$.

In the sequel, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ denotes a filtered probability space satisfying the usual conditions.

Let U be another separable Hilbert space and let $Q \in L(U)$ be a compact, self-adjoint, strictly positive linear operator. Then there exist an orthonormal basis $\{e_j\}$ of U and a bounded sequence λ_j of strictly positive real numbers such that

$$Qu = \sum_j \lambda_j \langle u, e_j \rangle e_j, \quad u \in U$$

namely, the λ_j are the eigenvalues of Q, and each e_j is an eigenvector corresponding to λ_j, see, e.g., [41, Thm. VI.3.2].

The space $U_0 := Q^{1/2}(U)$, equipped with the inner product

$$\langle u, v \rangle_{U_0} := \langle Q^{-1/2} u, Q^{-1/2} v \rangle_U,$$

is another separable Hilbert space and $\{\sqrt{\lambda_j} e_j\}$ is an orthonormal basis.

Let W be a Q-Wiener process [8, p. 86,87]. We assume that $\text{tr}(Q) = \sum_j \lambda_j < \infty$. Otherwise, which is the case if W is a cylindrical Wiener process, there always exists a separable Hilbert space $U_1 \supset U$ on which W has a realization as a finite trace class Wiener process, see [8, Chap. 4.3].

We denote by $L_2^0(H_\beta) := L_2(U_0, H_\beta)$ the space of Hilbert-Schmidt operators from U_0 into H_β, which, endowed with the Hilbert-Schmidt norm

$$\|\Phi\|_{L_2^0(H_\beta)} := \sqrt{\sum_j \lambda_j \|\Phi e_j\|^2}, \quad \Phi \in L_2^0(H_\beta)$$

itself is a separable Hilbert space.

According to [8, Prop. 4.1], the sequence of stochastic processes $\{\beta^j\}$ defined as $\beta^j := \frac{1}{\sqrt{\lambda_j}}\langle W, e_j \rangle$ is a sequence of real-valued independent $(\mathcal{F}_t)$-Brownian motions and we have the expansion

$$(3.1) \qquad W = \sum_j \sqrt{\lambda_j}\beta^j e_j,$$

where the series is convergent in the space $M^2(U)$ of U-valued square-integrable martingales. Let $\Phi : \Omega \times \mathbb{R}_+ \to L_2^0(H_\beta)$ be an integrable process, i.e. Φ is predictable and satisfies

$$\mathbb{P}\bigg(\int_0^T \|\Phi_t\|_{L_2^0(H_\beta)}^2 dt < \infty\bigg) = 1 \quad \text{for all } T \in \mathbb{R}_+.$$

Setting $\Phi^j := \sqrt{\lambda_j}\Phi e_j$ for each j, we have

$$(3.2) \qquad \int_0^t \Phi_s dW_s = \sum_j \int_0^t \Phi_s^j d\beta_s^j, \quad t \in \mathbb{R}_+$$

where the convergence is uniform on compact time intervals in probability, see [8, Thm. 4.3].

Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [9, 22]). We remark that every Polish space with its Borel σ-field is a Blackwell space.

Furthermore, let μ be a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [25, Def. II.1.20]. Then its compensator is of the form $dt \otimes F(dx)$, where F is a σ-finite measure on $(E, \mathcal{E})$.

Let measurable vector fields $\sigma : H_\beta \to L_2^0(H_\beta^0)$ and $\gamma : H_\beta \times E \to H_{\beta'}^0$ be given, where the subspace H_β^0 was defined in Theorem 2.1. For each j we define $\sigma^j : H_\beta \to H_\beta^0$ as $\sigma^j(h) := \sqrt{\lambda_j}\sigma(h)e_j$. We shall now focus on the HJMM equation (1.10).

3.1. **Assumption.** *We assume there exists a measurable function $\Phi : E \to \mathbb{R}_+$ satisfying*

$$(3.3) \qquad |\Gamma(h,x)(\xi)| \leq \Phi(x), \quad h \in H_\beta, \, x \in E \text{ and } \xi \in \mathbb{R}_+$$

a constant $L > 0$ such that

$$(3.4) \qquad \|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H_\beta)} \leq L\|h_1 - h_2\|_\beta$$

$$(3.5) \qquad \bigg(\int_E e^{\Phi(x)}\|\gamma(h_1,x) - \gamma(h_2,x)\|_{\beta'}^2 F(dx)\bigg)^{1/2} \leq L\|h_1 - h_2\|_\beta$$

for all $h_1, h_2 \in H_\beta$, and a constant $M > 0$ such that

$$(3.6) \qquad \|\sigma(h)\|_{L_2^0(H_\beta)} \leq M$$

$$(3.7) \qquad \int_E e^{\Phi(x)}(\|\gamma(h,x)\|_{\beta'}^2 \vee \|\gamma(h,x)\|_{\beta'}^4)F(dx) \leq M$$

for all $h \in H_\beta$. Furthermore, we assume that for each $h \in H_\beta$ the map

$$(3.8) \qquad \alpha_2(h) := -\int_E \gamma(h,x)\left(e^{\Gamma(h,x)} - 1\right)F(dx)$$

is absolutely continuous with weak derivative

$$(3.9) \qquad \frac{d}{d\xi}\alpha_2(h) = \int_E \gamma(h,x)^2 e^{\Gamma(h,x)}F(dx) - \int_E \frac{d}{d\xi}\gamma(h,x)\left(e^{\Gamma(h,x)} - 1\right)F(dx).$$

3.2. **Remark.** *The proof of Proposition 3.3 below gives rise to the following remarks concerning conditions (3.8), (3.9) from Assumption 3.1.*

- *For each $h \in H_\beta$ the map $\alpha_2(h)$ is well-defined (that is, for every $\xi \in \mathbb{R}_+$ the integral in (3.8) exists) and continuous (but not necessarily absolutely continuous) with $\alpha_2(h)(0) = 0$ and $\lim_{\xi \to \infty} \alpha_2(h)(\xi) = 0$.*
- *If $\alpha_2(h)$ is absolutely continuous for some $h \in H_\beta$, then we have $\alpha_2(h) \in H_\beta^0$.*
- *For Lévy driven term structure models we can verify conditions (3.8) and (3.9) directly, see Corollary 3.6 below.*
- *Let $h \in H_\beta$ be such that $\gamma(h, x) \in C^1(\mathbb{R}_+)$ for all $x \in E$. Suppose for each $\xi \in \mathbb{R}_+$ there exist $\delta = \delta(h, \xi) > 0$ and $C = C(h, \xi) > 0$ such that*

$$\left| \frac{d}{d\eta} \gamma(h, x)(\eta) \right| \le C \|\gamma(h, x)\|_{\beta'} \quad \text{for all } x \in E \text{ and } \eta \in (\xi - \delta, \xi + \delta) \cap \mathbb{R}_+.$$

  Then, we even have $\alpha_2(h) \in C^1(\mathbb{R}_+)$ with derivative (3.9).

3.3. **Proposition.** *Suppose Assumption 3.1 is fulfilled. Then we have $\alpha_{\mathrm{HJM}}(H_\beta) \subset H_\beta^0$ and there is a constant $K > 0$ such that*

$$(3.10) \qquad \|\alpha_{\mathrm{HJM}}(h_1) - \alpha_{\mathrm{HJM}}(h_2)\|_\beta \le K \|h_1 - h_2\|_\beta$$

for all $h_1, h_2 \in H_\beta$.

Proof. Note that $\alpha_{\mathrm{HJM}} = \alpha_1 + \alpha_2$, where

$$\alpha_1(h) := \sum_j \sigma^j(h) \Sigma^j(h), \quad h \in H_\beta$$

and α_2 is given by (3.8). By [16, Cor. 5.1.2] we have $\sigma^j(h) \Sigma^j(h) \in H_\beta^0$, $h \in H_\beta$ for all j. For an arbitrary $h \in H_\beta$ we obtain, by using [16, Cor. 5.1.2] again,

$$\sum_j \|\sigma^j(h) \Sigma^j(h)\|_\beta \le \sqrt{3(C_3^2 + 2C_4)} \sum_j \|\sigma^j(h)\|_\beta^2 = \sqrt{3(C_3^2 + 2C_4)} \|\sigma(h)\|_{L_2^0(H_\beta)}^2,$$

and hence, we deduce $\alpha_1(H_\beta) \subset H_\beta^0$.

Let $h \in H_\beta$ be arbitrary. For all $x \in E$ and $\xi \in \mathbb{R}_+$ we have by (2.2) and (2.5)

$$(3.11) \qquad |\gamma(h, x)(\xi)| \le C_2 \|\gamma(h, x)\|_\beta \le C_2 \|\gamma(h, x)\|_{\beta'},$$

and for all $x \in E$ and $\xi \in \mathbb{R}_+$ we have by (3.3), (2.3) and (2.5)

$$(3.12)$$
$$|e^{\Gamma(h,x)(\xi)} - 1| \le e^{\Phi(x)} |\Gamma(h, x)(\xi)| \le e^{\Phi(x)} \|\gamma(h, x)\|_{L^1(\mathbb{R}_+)} \le C_3 e^{\Phi(x)} \|\gamma(h, x)\|_{\beta'}.$$

Estimates (3.11), (3.12) and (3.7) show that $\lim_{\xi \to \infty} \alpha_2(h)(\xi) = 0$. From (3.3), (3.11), (3.7) and (2.6) it follows that

$$\int_{\mathbb{R}_+} \left(\int_E \gamma(h, x)(\xi)^2 e^{\Gamma(h,x)(\xi)} F(dx) \right)^2 e^{\beta\xi} d\xi$$
$$\le C_2^2 M \int_{\mathbb{R}_+} \left(\int_E \gamma(h, x)(\xi)^2 e^{\Gamma(h,x)(\xi)} F(dx) \right) e^{\beta\xi} d\xi$$
$$\le C_2^2 M \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} \gamma(h, x)(\xi)^2 e^{\beta\xi} d\xi F(dx)$$
$$\le C_2^2 M C_5 \int_E e^{\Phi(x)} \|\gamma(h, x)\|_{\beta'}^2 F(dx) \le C_2^2 M^2 C_5.$$

We obtain by (3.12), Hölder's inequality, (3.7) and (2.5)

$$\int_{\mathbb{R}_+} \left(\int_E \frac{d}{d\xi} \gamma(h,x)(\xi) \left(e^{\Gamma(h,x)(\xi)} - 1 \right) F(dx) \right)^2 e^{\beta\xi} d\xi$$

$$\leq C_3^2 \int_{\mathbb{R}_+} \left(\int_E \left| \frac{d}{d\xi} \gamma(h,x)(\xi) \right| e^{\frac{1}{2}\Phi(x)} e^{\frac{1}{2}\Phi(x)} \|\gamma(h,x)\|_{\beta'} F(dx) \right)^2 e^{\beta\xi} d\xi$$

$$\leq C_3^2 M \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} \left| \frac{d}{d\xi} \gamma(h,x)(\xi) \right|^2 e^{\beta\xi} d\xi F(dx)$$

$$\leq C_3^2 M \int_E e^{\Phi(x)} \|\gamma(h,x)\|_{\beta'}^2 F(dx) \leq C_3^2 M^2.$$

In view of (3.9), we conclude that $\alpha_2(H_\beta) \subset H_\beta^0$, and hence $\alpha_{\mathrm{HJM}}(H_\beta) \subset H_\beta^0$.

Let $h_1, h_2 \in H_\beta$ be arbitrary. By [16, Cor. 5.1.2], Hölder's inequality, (3.4) and (3.6) we have

$$\|\alpha_1(h_1) - \alpha_1(h_2)\|_\beta$$

$$\leq \sqrt{3(C_3^2 + 2C_4)} \sum_j (\|\sigma^j(h_1)\|_\beta + \|\sigma^j(h_2)\|_\beta) \|\sigma^j(h_1) - \sigma^j(h_2)\|_\beta$$

$$\leq \sqrt{3(C_3^2 + 2C_4)} \sqrt{\sum_j (\|\sigma^j(h_1)\|_\beta + \|\sigma^j(h_2)\|_\beta)^2} \sqrt{\sum_j \|\sigma^j(h_1) - \sigma^j(h_2)\|_\beta^2}$$

$$\leq \sqrt{6(C_3^2 + 2C_4)} (\|\sigma(h_1)\|_{L_2^0(H_\beta)} + \|\sigma(h_2)\|_{L_2^0(H_\beta)}) \|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H_\beta)}$$

$$\leq 2ML\sqrt{6(C_3^2 + 2C_4)} \|h_1 - h_2\|_\beta.$$

Furthermore, by (3.9),

$$\|\alpha_2(h_1) - \alpha_2(h_2)\|_\beta^2 \leq 4(I_1 + I_2 + I_3 + I_4),$$

where we have put

$$I_1 := \int_{\mathbb{R}_+} \left(\int_E \gamma(h_1,x)(\xi)^2 \left(e^{\Gamma(h_1,x)(\xi)} - e^{\Gamma(h_2,x)(\xi)} \right) F(dx) \right)^2 e^{\beta\xi} d\xi,$$

$$I_2 := \int_{\mathbb{R}_+} \left(\int_E e^{\Gamma(h_2,x)(\xi)} (\gamma(h_1,x)(\xi)^2 - \gamma(h_2,x)(\xi)^2) F(dx) \right)^2 e^{\beta\xi} d\xi,$$

$$I_3 := \int_{\mathbb{R}_+} \left(\int_E \frac{d}{d\xi} \gamma(h_1,x)(\xi) \left(e^{\Gamma(h_1,x)(\xi)} - e^{\Gamma(h_2,x)(\xi)} \right) F(dx) \right)^2 e^{\beta\xi} d\xi,$$

$$I_4 := \int_{\mathbb{R}_+} \left(\int_E \left(e^{\Gamma(h_2,x)(\xi)} - 1 \right) \left(\frac{d}{d\xi} \gamma(h_1,x)(\xi) - \frac{d}{d\xi} \gamma(h_2,x)(\xi) \right) F(dx) \right)^2 e^{\beta\xi} d\xi.$$

We get for all $x \in E$ and $\xi \in \mathbb{R}_+$ by (3.3), (2.3) and (2.5)

$$(3.13) \quad \begin{aligned} |e^{\Gamma(h_1,x)(\xi)} - e^{\Gamma(h_2,x)(\xi)}| &\leq e^{\Phi(x)} |\Gamma(h_1,x)(\xi) - \Gamma(h_2,x)(\xi)| \\ &\leq e^{\Phi(x)} \|\gamma(h_1,x) - \gamma(h_2,x)\|_{L^1(\mathbb{R}_+)} \leq C_3 e^{\Phi(x)} \|\gamma(h_1,x) - \gamma(h_2,x)\|_{\beta'}. \end{aligned}$$

Relations (3.13), Hölder's inequality, (3.5), (2.4), (2.5) and (3.7) give us

$$I_1 \leq C_3^2 \int_{\mathbb{R}_+} \left(\int_E \gamma(h_1, x)(\xi)^2 e^{\frac{1}{2}\Phi(x)} e^{\frac{1}{2}\Phi(x)} \|\gamma(h_1, x) - \gamma(h_2, x)\|_{\beta'} F(dx) \right)^2 e^{\beta\xi} d\xi$$

$$\leq C_3^2 L^2 \|h_1 - h_2\|_\beta^2 \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} \gamma(h_1, x)(\xi)^4 e^{\beta\xi} d\xi F(dx)$$

$$\leq C_3^2 L^2 C_4 \|h_1 - h_2\|_\beta^2 \int_E e^{\Phi(x)} \|\gamma(h_1, x)\|_{\beta'}^4 F(dx) \leq C_3^2 L^2 C_4 M \|h_1 - h_2\|_\beta^2.$$

For every $\xi \in \mathbb{R}_+$ we obtain by (3.11) and (3.7)
(3.14)
$$\int_E e^{\Phi(x)} (\gamma(h_1, x)(\xi) + \gamma(h_2, x)(\xi))^2 F(dx)$$

$$\leq 2 \int_E e^{\Phi(x)} (\gamma(h_1, x)(\xi)^2 + \gamma(h_2, x)(\xi)^2) F(dx)$$

$$\leq 2C_2^2 \left(\int_E e^{\Phi(x)} \|\gamma(h_1, x)\|_{\beta'}^2 F(dx) + \int_E e^{\Phi(x)} \|\gamma(h_2, x)\|_{\beta'}^2 F(dx) \right) \leq 4C_2^2 M.$$

Using (3.3), Hölder's inequality, (3.14), (2.6) and (3.5) we get

$$I_2 \leq \int_{\mathbb{R}_+} \left(\int_E (\gamma(h_1, x)(\xi) + \gamma(h_2, x)(\xi)) e^{\frac{1}{2}\Phi(x)} \right.$$

$$\left. \times e^{\frac{1}{2}\Phi(x)} (\gamma(h_1, x)(\xi) - \gamma(h_2, x)(\xi)) F(dx) \right)^2 e^{\beta\xi} d\xi$$

$$\leq 4C_2^2 M \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} (\gamma(h_1, x)(\xi) - \gamma(h_2, x)(\xi))^2 e^{\beta\xi} d\xi F(dx)$$

$$\leq 4C_2^2 M C_5 \int_E e^{\Phi(x)} \|\gamma(h_1, x)(\xi) - \gamma(h_2, x)(\xi)\|_{\beta'}^2 F(dx)$$

$$\leq 4C_2^2 M C_5 L^2 \|h_1 - h_2\|_\beta^2.$$

Using (3.13), Hölder's inequality, (3.5), (2.5) and (3.7) gives us

$$I_3 \leq C_3^2 \int_{\mathbb{R}_+} \left(\int_E \left| \frac{d}{d\xi} \gamma(h_1, x)(\xi) \right| e^{\frac{1}{2}\Phi(x)} e^{\frac{1}{2}\Phi(x)} \|\gamma(h_1, x) - \gamma(h_2, x)\|_{\beta'} F(dx) \right)^2$$

$$\times e^{\beta\xi} d\xi$$

$$\leq C_3^2 L^2 \|h_1 - h_2\|_\beta^2 \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} \left| \frac{d}{d\xi} \gamma(h_1, x)(\xi) \right|^2 e^{\beta\xi} d\xi F(dx)$$

$$\leq C_3^2 L^2 \|h_1 - h_2\|_\beta^2 \int_E e^{\Phi(x)} \|\gamma(h_1, x)\|_{\beta'}^2 F(dx) \leq C_3^2 L^2 M \|h_1 - h_2\|_\beta^2.$$

We obtain by (3.12), Hölder's inequality, (3.7), (2.5) and (3.5)

$$I_4 \leq C_3^2 \int_{\mathbb{R}_+} \left(\int_E \|\gamma(h_2, x)\|_{\beta'} e^{\frac{1}{2}\Phi(x)} e^{\frac{1}{2}\Phi(x)} \left| \frac{d}{d\xi} \gamma(h_1, x)(\xi) - \frac{d}{d\xi} \gamma(h_2, x)(\xi) \right| F(dx) \right)^2$$

$$\times e^{\beta\xi} d\xi$$

$$\leq C_3^2 M \int_E e^{\Phi(x)} \int_{\mathbb{R}_+} \left| \frac{d}{d\xi} \gamma(h_1, x)(\xi) - \frac{d}{d\xi} \gamma(h_2, x)(\xi) \right|^2 e^{\beta\xi} d\xi F(dx)$$

$$\leq C_3^2 M \int_E e^{\Phi(x)} \|\gamma(h_1, x) - \gamma(h_2, x)\|_{\beta'}^2 F(dx) \leq C_3^2 M L^2 \|h_1 - h_2\|_\beta^2.$$

Summing up, we deduce that there is a constant $K > 0$ such that (3.10) is satisfied for all $h_1, h_2 \in H_\beta$. $\qquad\square$

3.4. **Theorem.** *Suppose Assumption 3.1 is fulfilled. Then, for each initial curve $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ there exists a unique adapted, càdlàg, mean-square continuous $\mathcal{H}_\beta$-valued solution $(f_t)_{t \geq 0}$ for the HJM equation (1.11) with $f_0 = \ell h_0$ satisfying*

$$(3.15) \qquad \mathbb{E}\left[\sup_{t \in [0,T]} \|f_t\|_\beta^2 \right] < \infty \quad \text{for all } T \in \mathbb{R}_+,$$

and there exists a unique adapted, càdlàg, mean-square continuous mild and weak H_β-valued solution $(r_t)_{t \geq 0}$ for the HJMM equation (1.10) with $r_0 = h_0$ satisfying

$$(3.16) \qquad \mathbb{E}\left[\sup_{t \in [0,T]} \|r_t\|_\beta^2 \right] < \infty \quad \text{for all } T \in \mathbb{R}_+,$$

which is given by $r_t := \pi U_t f_t$, $t \geq 0$. Moreover, the implied bond market (1.4) is free of arbitrage.

Proof. By virtue of Theorem 2.1, Proposition 3.3 and (3.4), (3.5), (3.7), (2.5), all assumptions from [20, Cor. 10.9] are fulfilled, which therefore applies and establishes the claimed existence and uniqueness result.

For all $h \in H_\beta$, $x \in E$ and $\xi \in \mathbb{R}_+$ we have by (3.3), (2.3) and (2.5)

$$(3.17) \qquad \begin{aligned} |e^{\Gamma(h,x)(\xi)} - 1 - \Gamma(h,x)(\xi)| &\leq \frac{1}{2} e^{\Phi(x)} \Gamma(h,x)(\xi)^2 \\ &\leq \frac{1}{2} e^{\Phi(x)} \|\gamma(h,x)\|_{L^1(\mathbb{R}_+)}^2 \leq \frac{C_3^2}{2} e^{\Phi(x)} \|\gamma(h,x)\|_{\beta'}^2. \end{aligned}$$

Integrating (1.6) we obtain, by using [16, Lemma 4.3.2] and (3.17), (3.7)

$$\int_0^\bullet \alpha_{\mathrm{HJM}}(h)(\eta) d\eta = \frac{1}{2} \sum_j \Sigma^j(h)^2 + \int_E \left(e^{\Gamma(h,x)} - 1 - \Gamma(h,x) \right) F(dx)$$

for all $h \in H_\beta$. Combining [3, Prop. 5.3] and [16, Lemma 4.3.3] (the latter result is only required if W is infinite dimensional), the probability measure $\mathbb{P}$ is a local martingale measure, and hence the bond market (1.4) is free of arbitrage. $\qquad\square$

The case of Lévy-driven HJMM models is now a special case. We assume that the mark space is $E = \mathbb{R}^e$ for some positive integer $e \in \mathbb{N}$, equipped with its Borel σ-algebra $\mathcal{E} = \mathcal{B}(\mathbb{R}^e)$. The measure F is given by

$$(3.18) \qquad F(B) := \sum_{k=1}^e \int_\mathbb{R} \mathbb{1}_B(x e_k) F_k(dx), \quad B \in \mathcal{B}(\mathbb{R}^e)$$

where $F_1, \ldots, F_e$ are Lévy measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$(3.19) \qquad F_k(\{0\}) = 0, \qquad k = 1, \ldots, e$$

$$(3.20) \qquad \int_\mathbb{R} (|x|^2 \wedge 1) F_k(dx) < \infty, \quad k = 1, \ldots, e$$

and where the $(e_k)_{k=1,\ldots,e}$ denote the unit vectors in $\mathbb{R}^e$. Note that Definition (3.18) implies

$$(3.21) \qquad \int_{\mathbb{R}^e} g(x) F(dx) = \sum_{k=1}^e \int_\mathbb{R} g(x e_k) F_k(dx)$$

for any nonnegative measurable function $g : \mathbb{R}^e \to \mathbb{R}$. In particular, the support of F is contained in $\bigcup_{k=1}^e \mathrm{span}\{e_k\}$, the union of the coordinate axes in $\mathbb{R}^e$. For each $k = 1, \ldots, e$ let $\delta^k : H_\beta \to H_{\beta'}^0$ be a vector field. We define $\gamma : H_\beta \times \mathbb{R}^e \to H_{\beta'}^0$ as

$$(3.22) \qquad \gamma(h,x) := \sum_{k=1}^e \delta^k(h) x_k.$$

Then, equation (1.10) corresponds to the situation where the term structure model is driven by several real-valued, independent Lévy processes with Lévy measures F_k. For all $h \in H_\beta$ and $\xi \in \mathbb{R}_+$ we set

$$\Delta^k(h)(\xi) := -\int_0^\xi \delta^k(h)(\eta)d\eta, \quad k = 1, \ldots, e.$$

3.5. **Assumption.** *We assume there exist constants $N, \epsilon > 0$ such that for all $k = 1, \ldots, e$ we have*

$$(3.23) \qquad \int_{\{|x|>1\}} e^{zx} F_k(dx) < \infty, \quad z \in [-(1+\epsilon)N, (1+\epsilon)N]$$

$$(3.24) \qquad |\Delta^k(h)(\xi)| \leq N, \quad h \in H_\beta, \xi \in \mathbb{R}_+$$

a constant $L > 0$ such that (3.4) and

$$(3.25) \qquad \|\delta^k(h_1) - \delta^k(h_2)\|_{\beta'} \leq L\|h_1 - h_2\|_\beta, \quad k = 1, \ldots, e$$

are satisfied for all $h_1, h_2 \in H_\beta$, and a constant $M > 0$ such that (3.6) and

$$(3.26) \qquad \|\delta^k(h)\|_{\beta'} \leq M, \quad k = 1, \ldots, e$$

are satisfied for all $h \in H_\beta$.

Now, we obtain the statement of [19, Thm. 4.6] as a corollary.

3.6. **Corollary.** *Suppose Assumption 3.5 is fulfilled. Then, for each initial curve $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ there exists a unique adapted, càdlàg, mean-square continuous $\mathcal{H}_\beta$-valued solution $(f_t)_{t \geq 0}$ for the HJM equation (1.11) with $f_0 = \ell h_0$ satisfying (3.15), and there exists a unique adapted, càdlàg, mean-square continuous mild and weak H_β-valued solution $(r_t)_{t \geq 0}$ for the HJMM equation (1.10) with $r_0 = h_0$ satisfying (3.16), which is given by $r_t := \pi U_t f_t$, $t \geq 0$. Moreover, the implied bond market (1.4) is free of arbitrage.*

Proof. Using (3.24), the measurable function $\Phi : \mathbb{R}^e \to \mathbb{R}_+$ defined as

$$\Phi(x) := N \sum_{k=1}^e |x_k|, \quad x \in \mathbb{R}^e$$

satisfies (3.3). For each $k = 1, \ldots, e$ and every $m \in \mathbb{N}$ with $m \geq 2$ we have

$$(3.27) \qquad \int_{\mathbb{R}} |x|^m e^{|zx|} F_k(dx) < \infty, \quad z \in (-(1+\epsilon)N, (1+\epsilon)N).$$

Indeed, let $z \in (-(1+\epsilon)N, (1+\epsilon)N)$ be arbitrary. There exists $\eta \in (0, \epsilon)$ such that $|z| \leq (1+\eta)N$. By (3.20), (3.23) and the basic inequality $x^m \leq m!e^x$ for $x \geq 0$ we obtain

$$\int_{\mathbb{R}} |x|^m e^{|zx|} F_k(dx) \leq \int_{\mathbb{R}} |x|^m e^{(1+\eta)N|x|} F_k(dx)$$

$$\leq 2 \int_{\{|x| \leq \frac{\ln 2}{(1+\eta)N}\}} |x|^m F_k(dx) + \frac{m!}{((\epsilon-\eta)N)^m} \int_{\{|x| > \frac{\ln 2}{(1+\eta)N}\}} e^{(1+\epsilon)N|x|} F_k(dx) < \infty,$$

proving (3.27). Taking into account (3.21), (3.27), relations (3.25), (3.26) imply (3.5), (3.7). Furthermore, (3.27), the elementary inequalities

$$|e^x - 1 - x| \leq \frac{1}{2}x^2 e^{|x|}, \quad x \in \mathbb{R}$$

$$|e^x - 1| \leq |x|e^{|x|}, \quad x \in \mathbb{R}$$

and Lebesgue's theorem show that the cumulant generating functions

$$\Psi_k(z) = \int_{\mathbb{R}} (e^{zx} - 1 - zx) F_k(dx), \quad k = 1, \ldots, e$$

belong to class C^∞ on the open interval $(-(1+\epsilon)N, (1+\epsilon)N)$ with derivatives

$$\Psi'_k(z) = \int_{\mathbb{R}} x(e^{zx} - 1)F_k(dx),$$

$$\Psi_k^{(m)}(z) = \int_{\mathbb{R}} x^m e^{zx} F_k(dx), \quad m \geq 2.$$

Therefore, and because of (3.21), we can, for an arbitrary $h \in H_\beta$, write $\alpha_2(h)$, which is defined in (3.8), as

$$\alpha_2(h) = -\sum_{k=1}^{e} \delta^k(h)\Psi'_k\bigg(-\int_0^\bullet \delta^k(\eta)d\eta\bigg).$$

Hence, $\alpha_2(h)$ is absolutely continuous with weak derivative (3.9). Consequently, Assumption 3.1 is fulfilled and Theorem 3.4 applies. $\qquad\square$

Note that the boundedness assumptions (3.6), (3.7) of Theorem 3.4 resp. (3.6), (3.26) of Corollary 3.6 cannot be weakened substantially. For example, for arbitrage free term structure models driven by a single Brownian motion, it was shown in [31, Sec. 4.7] that for the simple case of proportional volatility, that is $\sigma(h) = \sigma_0 h$ for some constant $\sigma_0 > 0$, solutions necessarily explode. We mention, however, that [36, Sec. 6] contains some existence results for Lévy term structure models with linear volatility.

4. Positivity preserving term structure models driven by Wiener processes and Poisson measures

In applications, we are often interested in term structure models producing positive forward curves. In this section, we characterize HJMM forward curve evolutions of the type (1.10), which preserve positivity, by means of the characteristics of the SPDE. In the case of short rate models this can be characterized by the positivity of the short rate, a one-dimensional Markov process. In case of an infinite-factor evolution, as described by a generic HJMM equation (see for instance [2]), this problem is much more delicate. Indeed, one has to find conditions such that a Markov process defined by the HJMM equation (on a Hilbert space of forward rate curves) stays in a "small" set of curves, namely the convex cone of positive curves bounded by a non-smooth set. Our strategy to solve this problem is the following: First we show by general semimartingale methods necessary conditions for positivity. These necessary conditions are basically described by the facts that the Itô drift is inward pointing and that the volatilities are parallel at the boundary of the set of non-negative functions. Taking those conditions we can also prove that the Stratonovich drift is inward pointing, since parallel volatilities produce parallel Stratonovich corrections (a fact which is not true for general closed convex sets, but holds true for the set of non-negative functions P). Then we reduce the sufficiency proof to two steps: First we essentially apply results from [34] in order to solve the pure diffusion case and then we "slowly" switch on the jumps to see the general result.

Let H_β be the space of forward curves introduced in Section 2 for some fixed $\beta > 0$. We introduce the half spaces

$$H_\xi^+ := \{h \in H_\beta \,|\, h(\xi) \geq 0\}, \quad \xi \in \mathbb{R}_+$$

and define the closed, convex cone

$$P := \bigcap_{\xi \in \mathbb{R}_+} H_\xi^+$$

consisting of all nonnegative forward curves from H_β. In what follows, we shall use that, by the continuity of the functions from H_β, we can write P as

$$P = \bigcap_{\xi \in (0,\infty)} H_\xi^+.$$

Furthermore, we define the edges

$$\partial P_\xi := \{h \in P \,|\, h(\xi) = 0\}, \quad \xi \in (0,\infty).$$

First, we consider the positivity problem for general forward curve evolutions, where the HJM drift condition (1.6) is not necessarily satisfied, and afterwards we apply our results to the arbitrage free situation.

We emphasize that, in the sequel, we *assume* the existence of solutions. Sufficient conditions for existence and uniqueness are provided in [20] (we also mention the related articles [1] and [29]) for general stochastic partial differential equations and in the previous Section 3 for the HJMM term structure equation (1.10).

As in the previous section, we work on the space H_β of forward curves from Section 2 for some $\beta > 0$. At first glance, it looks reasonable to treat the positivity problem by working with *weak* solutions on H_β. However, this is unfeasible, because the point evaluations at $\xi \in (0,\infty)$, i.e., a linear functional $\zeta \in H_\beta$ such that $h(\xi) = \langle \zeta, h \rangle$ for all $h \in H_\beta$, do never belong to the domain $\mathcal{D}((\frac{d}{d\xi})^*)$ of the adjoint operator. Indeed, a well-known mollifying technique shows that for each $\xi \in (0,\infty)$ the linear functional $h \mapsto h'(\xi) : \mathcal{D}(\frac{d}{d\xi}) \to \mathbb{R}$ is unbounded.

Therefore treating the positivity problem with weak solutions does not bring an immediate advantage, hence we shall work with *mild* solutions on H_β.

Let measurable vector fields $\alpha : H_\beta \to H_\beta$, $\sigma : H_\beta \to L_2^0(H_\beta)$ and $\gamma : H_\beta \times E \to H_\beta$ be given. Currently, we do not assume that the drift term α is given by the HJM drift condition (1.6). For each j we define $\sigma^j : H_\beta \to H_\beta$ as $\sigma^j(h) := \sqrt{\lambda_j} \sigma(h) e_j$. We assume that for each $h_0 \in P$ the HJM equation

$$(4.1) \quad \begin{cases} df_t &= U_{-t}\ell\alpha(\pi U_t f_t)dt + U_{-t}\ell\sigma(\pi U_t f_t)dW_t \\ &\quad + \int_E U_{-t}\ell\gamma(\pi U_t f_{t-}, x)(\mu(dt,dx) - F(dx)dt) \\ f_0 &= \ell h_0, \end{cases}$$

has at least one $\mathcal{H}_\beta$-valued solution $(f_t)_{t \geq 0}$. Then, because of (2.7), the transformation $r_t := \pi U_t f_t$, $t \geq 0$ is a mild H_β-valued solution of the HJMM equation

$$(4.2)$$
$$\begin{cases} dr_t &= (\frac{d}{d\xi}r_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt,dx) - F(dx)dt) \\ r_0 &= h_0. \end{cases}$$

4.1. Definition. *The HJMM equation (4.2) is said to be* positivity preserving *if for all $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ with $\mathbb{P}(h_0 \in P) = 1$ there exists a solution $(f_t)_{t \geq 0}$ of (4.1) with $f_0 = \ell h_0$ such that $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+}\{r_t \in P\}) = 1$, where $r_t := \pi U_t f_t$, $t \geq 0$.*

4.2. Remark. *Note that the seemingly weaker condition $\mathbb{P}(r_t \in P) = 1$ for all $t \in \mathbb{R}_+$ is equivalent to the condition of the previous definition due to the càdlàg property of the trajectories.*

4.3. Definition. *The HJMM equation (4.2) is said to be* locally positivity preserving *if for all $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ with $\mathbb{P}(h_0 \in P) = 1$ there exists a solution $(f_t)_{t \geq 0}$ of (4.1) with $f_0 = \ell h_0$ and a strictly positive stopping time τ such that $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+}\{r_{t \wedge \tau} \in P\}) = 1$, where $r_t := \pi U_t f_t$, $t \geq 0$.*

4.4. Lemma. *Let $h_0 \in P$ be arbitrary and let $(f_t)_{t \geq 0}$ be a solution for (4.1) with $f_0 = \ell h_0$. Set $r_t := \pi U_t f_t$, $t \geq 0$. The following two statements are equivalent:*

(1) *We have* $\mathbb{P}(\bigcap_{t\in\mathbb{R}_+}\{r_t \in P\}) = 1$.
(2) *We have* $\mathbb{P}(\bigcap_{t\in[0,T]}\{f_t(T) \geq 0\}) = 1$ *for all* $T \in (0,\infty)$.

Proof. The claim follows, because the processes $(r_t)_{t\geq 0}$ and $(f_t(T))_{t\in[0,T]}$ for an arbitrary $T \in (0,\infty)$ are càdlàg, and because the functions from H_β are continuous. $\square$

4.5. Assumption. *We assume that the vector fields* $\alpha : H_\beta \to H_\beta$ *and* $\sigma : H_\beta \to L_2^0(H_\beta)$ *are continuous and that* $h \mapsto \int_B \gamma(h,x)F(dx)$ *is continuous on* H_β *for all* $B \in \mathcal{E}$ *with* $F(B) < \infty$.

4.6. Remark. *Notice that, by Hölder's inequality, Assumption 4.5 is implied by Assumptions 4.10, 4.11 below, and therefore in particular by Assumption 3.1.*

4.7. Proposition. *Suppose Assumption 4.5 is fulfilled. If equation (4.2) is positivity preserving, then we have*

$$(4.3) \qquad \int_E \gamma(h,x)(\xi)F(dx) < \infty, \text{ for all } \xi \in (0,\infty),\ h \in \partial P_\xi$$

$$(4.4) \qquad \alpha(h)(\xi) - \int_E \gamma(h,x)(\xi)F(dx) \geq 0, \text{ for all } \xi \in (0,\infty),\ h \in \partial P_\xi$$

$$(4.5) \qquad \sigma^j(h)(\xi) = 0, \text{ for all } \xi \in (0,\infty),\ h \in \partial P_\xi \text{ and all } j$$

$$(4.6) \qquad h + \gamma(h,x) \in P, \text{ for all } h \in P \text{ and } F\text{-almost all } x \in E.$$

4.8. Remark. *Observe that condition (4.6) implies*

$$(4.7) \qquad \gamma(h,x)(\xi) \geq 0, \text{ for all } \xi \in (0,\infty),\ h \in \partial P_\xi \text{ and } F\text{-almost all } x \in E.$$

Therefore, condition (4.3) is equivalent to

$$\int_E |\gamma(h,x)(\xi)|F(dx) < \infty, \text{ for all } \xi \in (0,\infty),\ h \in \partial P_\xi.$$

Consequently, conditions (4.3) and (4.4) can be unified to

$$\int_E |\gamma(h,x)(\xi)|F(dx) \leq \alpha(h)(\xi)$$

for all $\xi \in (0,\infty)$ *and* $h \in \partial P_\xi$.

Proof. Let $h_0 \in P$ be arbitrary and let $(f_t)_{t\geq 0}$ be a solution for (4.1) with $f_0 = \ell h_0$ such that $\mathbb{P}(\bigcap_{t\in\mathbb{R}_+}\{r_t \in P\}) = 1$, where $r_t := \pi U_t f_t$, $t \geq 0$. By Lemma 4.4, for each $T \in (0,\infty)$ and every stopping time $\tau \leq T$ we have

$$(4.8) \qquad\qquad\qquad \mathbb{P}(f_\tau(T) \geq 0) = 1.$$

Let $\phi \in U_0'$ be a linear functional such that $\phi^j := \phi e_j \neq 0$ for only finitely many j, and let $\psi : E \to \mathbb{R}$ be a measurable function of the form $\psi = c\mathbb{1}_B$ with $c > -1$ and $B \in \mathcal{E}$ satisfying $F(B) < \infty$. Let Z be the Doléans-Dade Exponential

$$Z_t = \mathcal{E}\bigg(\sum_j \phi^j \beta^j + \int_0^\bullet \int_E \psi(x)(\mu(ds,dx) - F(dx)ds)\bigg)_t, \quad t \geq 0.$$

By [25, Thm. I.4.61] the process Z is a solution of

$$Z_t = 1 + \sum_j \phi^j \int_0^t Z_s d\beta_s^j + \int_0^t \int_E Z_{s-}\psi(x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0$$

and, since $\psi > -1$, the process Z is a strictly positive local martingale. There exists a strictly positive stopping time τ_1 such that Z^{τ_1} is a martingale. Due to the method of the moving frame, see [20], we can use standard stochastic analysis, to

proceed further. For an arbitrary $T \in (0, \infty)$, integration by parts yields (see [25, Thm. I.4.52])

$$(4.9) \quad \begin{aligned} f_t(T)Z_t &= \int_0^t f_{s-}(T)dZ_s + \int_0^t Z_{s-}df_s(T) + \langle f(T)^c, Z^c \rangle_t \\ &\quad + \sum_{s \le t} \Delta f_s(T) \Delta Z_s, \quad t \ge 0. \end{aligned}$$

Taking into account the dynamics

$$(4.10) \quad \begin{aligned} f_t(T) &= \ell h_0(T) + \int_0^t U_{-s}\ell\alpha(\pi U_s f_s)(T)ds + \sum_j \int_0^t U_{-s}\ell\sigma^j(\pi U_s f_s)(T)d\beta_s^j \\ &\quad + \int_0^t \int_E U_{-s}\ell\gamma(\pi U_s f_{s-}, x)(T)(\mu(ds,dx) - F(dx)ds), \quad t \ge 0 \end{aligned}$$

we have

$$(4.11) \quad \langle f(T)^c, Z^c \rangle_t = \sum_j \phi^j \int_0^t Z_s U_{-s}\ell\sigma^j(\pi U_s f_s)(T)ds, \quad t \ge 0$$

$$(4.12) \quad \sum_{s \le t} \Delta f_s(T)\Delta Z_s = \int_0^t \int_E Z_{s-}\psi(x)U_{-s}\ell\gamma(\pi U_s f_{s-}, x)(T)\mu(ds,dx), \quad t \ge 0.$$

Incorporating (4.10), (4.11) and (4.12) into (4.9), we obtain

$$(4.13) \quad \begin{aligned} f_t(T)Z_t &= M_t + \int_0^t Z_{s-}\bigg(U_{-s}\ell\alpha(\pi U_s f_{s-})(T) + \sum_j \phi^j U_{-s}\ell\sigma^j(\pi U_s f_{s-})(T) \\ &\quad + \int_E \psi(x)U_{-s}\ell\gamma(\pi U_s f_{s-}, x)(T)F(dx) \bigg)ds, \quad t \ge 0 \end{aligned}$$

where M is a local martingale with $M_0 = 0$. There exists a strictly positive stopping time τ_2 such that M^{τ_2} is a martingale.

By Assumption 4.5 there exist a strictly positive stopping time τ_3 and a constant $\tilde{\alpha} > 0$ such that

$$|U_{-(t \wedge \tau_3)}\ell\alpha(\pi U_{t \wedge \tau_3} f_{(t \wedge \tau_3)-})(T)| \le \tilde{\alpha}, \quad t \ge 0.$$

Let $B := \{x \in E : h_0 + \gamma(h_0, x) \notin P\}$. In order to prove (4.6), it suffices, since F is σ-finite, to show that $F(B \cap C) = 0$ for all $C \in \mathcal{E}$ with $F(C) < \infty$. Suppose, on the contrary, there exists $C \in \mathcal{E}$ with $F(C) < \infty$ such that $F(B \cap C) > 0$. By the continuity of the functions from H_β, there exists $T \in (0, \infty)$ such that $F(B_T \cap C) > 0$, where $B_T := \{x \in E : h_0(T) + \gamma(h_0, x)(T) < 0\}$. We obtain

$$\int_{B_T \cap C} \gamma(h_0, x)(T)F(dx) \le \int_{B_T \cap C} (h_0(T) + \gamma(h_0, x)(T))F(dx) < 0.$$

By Assumption 4.5 and the left-continuity of the process $f_{.-}$, there exist $\eta > 0$ and a strictly positive stopping time $\tau_4 \le T$ such that

$$\int_{B_T \cap C} U_{-(t \wedge \tau_4)}\ell\gamma(\pi U_{(t \wedge \tau_4)} f_{(t \wedge \tau_4)-}, x)(T)F(dx) \le -\eta, \quad t \ge 0.$$

Let $\phi := 0$, $\psi := \frac{\tilde{\alpha}+1}{\eta}\mathbb{1}_{B_T \cap C}$ and $\tau := \bigwedge_{i=1}^4 \tau_i$. Taking expectation in (4.13) we obtain $\mathbb{E}[f_\tau(T)Z_\tau] < 0$, implying $\mathbb{P}(f_\tau(T) < 0) > 0$, which contradicts (4.8). This yields (4.6).

From now on, we assume that $h_0 \in \partial P_T$ for an arbitrary $T \in (0, \infty)$.

Suppose that $\sigma^j(h_0)(T) \ne 0$ for some j. By the continuity of σ (see Assumption 4.5) there exist $\eta > 0$ and a strictly positive stopping time $\tau_4 \le T$ such that

$$|U_{-(t \wedge \tau_4)}\ell\sigma^j(\pi U_{t \wedge \tau_4} f_{(t \wedge \tau_4)-})(T)| \ge \eta, \quad t \ge 0.$$

Let $\phi \in U_0'$ be the linear functional with $\phi^j = -\text{sign}(\sigma^j(h_0)(T))\frac{\tilde{\alpha}+1}{\eta}$ and $\phi^k = 0$ for $k \neq j$. Furthermore, let $\psi := 0$ and $\tau := \bigwedge_{i=1}^4 \tau_i$. Taking expectation in (4.13) yields $\mathbb{E}[f_\tau(T)Z_\tau] < 0$, implying $\mathbb{P}(f_\tau(T) < 0) > 0$, which contradicts (4.8). This proves (4.5).

Now suppose $\int_E \gamma(h_0, x)(T)F(dx) = \infty$. Using Assumption 4.5, relation (4.7) and the σ-finiteness of F, there exist $B \in \mathcal{E}$ with $F(B) < \infty$ and a strictly positive stopping time $\tau_4 \leq T$ such that

$$-\frac{1}{2}\int_B U_{-(t \wedge \tau_4)}\ell\gamma(\pi U_{t \wedge \tau_4}f_{(t \wedge \tau_4)-}, x)(T)F(dx) \leq -(\tilde{\alpha}+1), \quad t \geq 0.$$

Let $\phi := 0$, $\psi := -\frac{1}{2}\mathbb{1}_B$ and $\tau := \bigwedge_{i=1}^4 \tau_i$. Taking expectation in (4.13) we obtain $\mathbb{E}[f_\tau(T)Z_\tau] < 0$, implying $\mathbb{P}(f_\tau(T) < 0) > 0$, which contradicts (4.8). This yields (4.3).

Since F is σ-finite, there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ with $B_n \uparrow E$ and $F(B_n) < \infty$, $n \in \mathbb{N}$. Next, we show for all $n \in \mathbb{N}$ the relation

$$(4.14) \qquad \alpha(h_0)(T) + \int_E \psi_n(x)\gamma(h_0, x)(T)F(dx) \geq 0,$$

where $\psi_n := -(1 - \frac{1}{n})\mathbb{1}_{B_n}$. Suppose, on the contrary, that (4.14) is not satisfied for some $n \in \mathbb{N}$. Using Assumption 4.5, there exist $\eta > 0$ and a strictly positive stopping time $\tau_4 \leq T$ such that

$$U_{-(t \wedge \tau_4)}\ell\alpha(\pi U_t f_{(t \wedge \tau_4)-})(T)$$
$$+ \int_E \psi_n(x)U_{-(t \wedge \tau_4)}\ell\gamma(\pi U_{t \wedge \tau_4}f_{(t \wedge \tau_4)-}, x)(T)F(dx) \leq -\eta, \quad t \geq 0.$$

Let $\phi := 0$ and $\tau := \bigwedge_{i=1}^4 \tau_i$. Taking expectation in (4.13) we obtain $\mathbb{E}[f_\tau(T)Z_\tau] < 0$, implying $\mathbb{P}(f_\tau(T) < 0) > 0$, which contradicts (4.8). This yields (4.14). By (4.14), (4.3) and Lebesgue's theorem, we conclude (4.4). □

4.9. **Remark.** *The Cox-Ingersoll-Ross (CIR) model [7] is celebrated for its feature to produce nonnegative interest rates. At this point, it is worth pointing out that the Hull-White extension of the CIR model (HWCIR) is not positivity preserving. Indeed, in Musiela parametrization its term structure dynamics are given by*

$$\begin{cases} dr_t &= (\frac{d}{d\xi}r_t + \alpha_{\text{HJM}}(r_t))dt + \sigma(r_t)dW_t \\ r_0 &= h_0, \end{cases}$$

where W is a one-dimensional Wiener process and the vector fields $\alpha_{\text{HJM}}, \sigma : H_\beta \to H_\beta^0$ are defined as

$$\alpha_{\text{HJM}}(h) := \rho^2|h(0)|\lambda\Lambda, \quad h \in H_\beta$$
$$\sigma(h) := \rho\sqrt{|h(0)|}\lambda, \quad h \in H_\beta$$

where $\rho > 0$ is a constant, $h \mapsto h(0) : H_\beta \to \mathbb{R}$ denotes the evaluation of the short rate, and where $\lambda \in H_\beta^0$ is a function with $\lambda(x) > 0$ for all $x \geq 0$ such that $\Lambda = \int_0^\bullet \lambda(\eta)d\eta$ satisfies a certain Riccati equation, see [21, Sec. 6.2] for more details. Note that Assumption 4.5 is satisfied, because the vector fields $\alpha_{\text{HJM}}, \sigma : H_\beta \to H_\beta^0$ are continuous, but condition (4.5) from Proposition 4.7 does not hold, because σ only depends on the current state of the short rate. Hence, the HWCIR model cannot be positivity preserving.

Indeed, we can also verify this directly as follows. According to [21, Sec. 6.2], for any initial curve $h_0 \in H_\beta$ the short rate $R_t = r_t(0)$, $t \geq 0$ has the dynamics

$$(4.15) \qquad \begin{cases} dR_t & = & (b(t) - c|R_t|)dt + \rho\sqrt{|R_t|}dW_t \\ R_0 & = & h_0(0) \end{cases}$$

for some constant $c \in \mathbb{R}$ and a time-dependent function $b = b(h_0) : \mathbb{R}_+ \to \mathbb{R}$. Due to [40], for each starting point $h_0(0) \in \mathbb{R}$ the stochastic differential equation (4.15) has a unique strong solution, and, according to [17], this solution is nonnegative if and only if $h_0(0) \geq 0$ and $b(t) \geq 0$ for all $t \in \mathbb{R}_+$. By [18, Prop. 5.2], the forward rates are given by

$$f(t,T) = \int_t^T b(s)\lambda(T-s)ds + \lambda(T-t)R_t,$$

which implies for the initial forward curve

$$h_0(T) = \int_0^T b(s)\lambda(T-s)ds + \lambda(T)h_0(0), \quad T \geq 0.$$

Having in mind that $\lambda(t) > 0$ for all $t \geq 0$, we see that for certain nonnegative initial curves $h_0 \in P$ the function b can also reach negative values, which yields negative short rates. For example, take an initial curve $h_0 \in P$ with $h_0(0) > 0$ and $h_0(T) = 0$ for some $T > 0$.

We shall now present sufficient conditions for positivity preserving term structure models. In the sequel, we will require the following linear growth and Lipschitz conditions.

4.10. **Assumption.** *We assume $\int_E \|\gamma(0,x)\|_\beta^2 F(dx) < \infty$ and that there is a constant $K > 0$ such that*

$$\left(\int_E \|\gamma(h_1,x) - \gamma(h_2,x)\|_\beta^2 F(dx) \right)^{1/2} \leq K\|h_1 - h_2\|_\beta$$

for all $h_1, h_2 \in H_\beta$.

4.11. **Assumption.** *We assume there is a constant $L > 0$ such that*

$$\|\alpha(h_1) - \alpha(h_2)\|_\beta \leq L\|h_1 - h_2\|_\beta,$$
$$\|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H_\beta)} \leq L\|h_1 - h_2\|_\beta$$

for all $h_1, h_2 \in H_\beta$.

This ensures existence and uniqueness of solutions by [20, Cor. 10.9].

4.12. **Lemma.** *Suppose Assumptions 4.10, 4.11 are fulfilled, and for each $h_0 \in P$ we have $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{r_t \in P\}) = 1$, where $(r_t)_{t \geq 0}$ denotes the mild solution for (4.2) with $r_0 = h_0$. Then, the HJMM equation (4.2) is positivity preserving.*

Proof. Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ with $\mathbb{P}(h_0 \in P) = 1$ be arbitrary. There exists a sequence $(h_n)_{n \in \mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ such that $h_n \to h_0$ in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ and for each $n \in \mathbb{N}$ we have $\mathbb{P}(h_n \in P) = 1$ and h_n has only a finite number of values. By assumption we have $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{r_t^n \in P\}) = 1$ for all $n \in \mathbb{N}$. Applying [20, Prop. 9.1] yields

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|r_t - r_t^n\|_\beta^2 \right] \to 0 \quad \text{for all } T \in \mathbb{R}_+,$$

showing that $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{r_t \in P\}) = 1$. $\qquad \square$

For the following Lemma 4.14 we prepare an auxiliary result, which is proven in Appendix A.

4.13. Lemma. *Let τ be a bounded stopping time. We define the new filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ by $\tilde{\mathcal{F}}_t := \mathcal{F}_{\tau+t}$, the new U-valued process $\tilde{W}$ by $\tilde{W}_t := W_{\tau+t} - W_\tau$ and the new random measure $\tilde{\mu}$ on $\mathbb{R}_+ \times E$ by $\tilde{\mu}(\omega; B) := \mu(\omega; B_{\tau(\omega)})$, $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$, where*

$$B_\tau := \{(t+\tau, x) \in \mathbb{R}_+ \times E : (t,x) \in B\}.$$

Then $\tilde{W}$ is a Q-Wiener process with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and $\tilde{\mu}$ is a homogeneous Poisson random measure on $\mathbb{R}_+ \times E$ with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ having the compensator $dt \otimes F(dx)$. Moreover, we have the expansion

$$(4.16) \qquad \tilde{W} = \sum_j \sqrt{\lambda_j} \tilde{\beta}^j e_j,$$

where $\tilde{\beta}^j$ defined as $\tilde{\beta}_t^j := \beta_{\tau+t}^j - \beta_\tau^j$ is a sequence of real-valued independent $(\tilde{\mathcal{F}}_t)$-Brownian motions. Furthermore, if $(r_t)_{t\geq 0}$ is a weak solution for (4.2), then the $(\tilde{\mathcal{F}}_t)$-adapted process $(\tilde{r}_t)_{t\geq 0}$ defined by $\tilde{r}_t := r_{\tau+t}$ is a weak solution for

(4.17)
$$\begin{cases} d\tilde{r}_t &= (\frac{d}{d\xi}\tilde{r}_t + \alpha(\tilde{r}_t))dt + \sigma(\tilde{r}_t)d\tilde{W}_t + \int_E \gamma(\tilde{r}_{t-}, x)(\tilde{\mu}(dt, dx) - F(dx)dt) \\ \tilde{r}_0 &= r_\tau. \end{cases}$$

Proof. See Appendix A. $\qquad\qquad\square$

4.14. Lemma. *Suppose Assumptions 4.10, 4.11 are fulfilled. If equation (4.2) is locally positivity preserving and we have (4.6), then equation (4.2) is positivity preserving.*

Proof. Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ with $\mathbb{P}(h_0 \in P) = 1$ be arbitrary. Moreover, let $(r_t)_{t\geq 0}$ be the mild solution for (4.2) with $r_0 = h_0$. We define the stopping time

$$(4.18) \qquad \tau_0 := \inf\{t \geq 0 : r_t \notin P\}.$$

By the closedness of P and (4.6) we have $r_{\tau_0} \in P$ almost surely on $\{\tau_0 < \infty\}$. We claim that $\mathbb{P}(\tau_0 = \infty) = 1$. Assume, on the contrary, that

$$(4.19) \qquad \mathbb{P}(\tau_0 < N) > 0$$

for some $N \in \mathbb{N}$. Let τ be the bounded stopping time $\tau := \tau_0 \wedge N$. We define the new filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, the new Q-Wiener process $\tilde{W}$ and the new Poisson random measure $\tilde{\mu}$ as in Lemma 4.13. Note that $r_\tau \in L^2(\Omega, \tilde{\mathcal{F}}_0, \mathbb{P}; H_\beta)$, because, by (3.16), we have

$$\mathbb{E}[\|r_\tau\|_\beta^2] \leq \mathbb{E}\left[\sup_{t\in[0,N]} \|r_t\|_\beta^2\right] < \infty.$$

By Lemma 4.13, the $(\tilde{\mathcal{F}}_t)$-adapted process $\tilde{r}_t := r_{\tau+t}$ is the unique mild solution for (4.17). Since equation (4.2) is locally positivity preserving and $\mathbb{P}(r_\tau \in P) = 1$, there exists a strictly positive stopping time τ_1 such that $\mathbb{P}(\bigcap_{t\in\mathbb{R}_+} \tilde{r}_{t\wedge\tau_1} \in P) = 1$. Since $\{\tau_0 < N\} \subset \{\tau_0 = \tau\}$, we obtain

$$r_{\tau_0+t} \in P \quad \text{almost surely on } [0, \tau_1] \cap \{\tau_0 < N\},$$

which is a contradiction because of (4.19) and the Definition (4.18) of τ_0. Consequently, we have $\mathbb{P}(\tau_0 = \infty) = 1$, whence equation (4.2) is positivity preserving. $\quad\square$

4.15. Assumption. *We assume $\sigma \in C^2(H_\beta; L_2^0(H_\beta))$, and that the vector field*

$$(4.20) \qquad h \mapsto \sum_j D\sigma^j(h)\sigma^j(h)$$

is globally Lipschitz on H_β.

4.16. Remark. *Note that Assumption 4.15 is satisfied if $\sigma \in C_b^2(H_\beta; L_2^0(H_\beta))$ and the series (4.20) converges for every $h \in H_\beta$.*

4.17. Lemma. *Suppose Assumption 4.15 and relation (4.5) are fulfilled. Then we have*

$$\big(\sum_j D\sigma^j(h)\sigma^j(h)\big)(\xi) = 0, \text{ for all } \xi \in (0, \infty), \, h \in \partial P_\xi.$$

Proof. Let $\xi \in (0, \infty)$ be arbitrary. It suffices to show $\big(D\sigma^j(h)\sigma^j(h)\big)(\xi) = 0$ for all $h \in \partial P_\xi$ and all j. Therefore let j be fixed and denote $\sigma = \sigma^j$. By assumption for all $h \geq 0$ with $h(\xi) = 0$ we have that $\sigma(h)(\xi) = 0$. In other words the volatility vector field σ is parallel to the boundary at boundary elements of P. We denote the local flow of the Lipschitz vector field σ by Fl being defined on a small time interval $]-\epsilon, \epsilon[$ around time 0 and a small neighborhood of each element $h \in P$. We state first that the flow Fl leaves the set P invariant, i.e., $\mathrm{Fl}_t(h) \geq 0$ if $h \geq 0$, by convexity and closedness of the cone of positive functions due to [39]. Indeed, P is a closed and convex cone, whose supporting hyperplanes l (a linear functional l is called supporting hyperplane of P at h if $l(P) \geq 0$ and $l(h) = 0$) are given by appropriate positive measures μ on $\mathbb{R}_+$ via

$$l(h) = \int_{\mathbb{R}_+} h(\xi)\mu(d\xi),$$

whence condition (4) from [39] is fulfilled due to (4.5). Next we show that even more holds: the solution $\mathrm{Fl}_t(h)$ evaluated at ξ vanishes if $h(\xi) = 0$, which we show directly. Indeed, let us additionally fix $h \in \partial P_\xi$, i.e., $h \geq 0$ and $h(\xi) = 0$. Looking now at the Picard-Lindelöf approximation scheme

$$c^{(n+1)}(t) = h + \int_0^t \sigma(c^{(n)}(s))ds$$

with $c^{(n)}(0) = h$ and $c^{(0)}(s) = h$ for $s, t \in]-\epsilon, \epsilon[$ and $n \geq 0$, we see by induction that under our assumptions

$$c^{(n)}(t)(\xi) = 0$$

for all $n \geq 0$ and $t \in]-\epsilon, \epsilon[$ for the given fixed element h. Consequently – as $n \to \infty$ – we obtain that $\mathrm{Fl}_t(h)(\xi) = 0$, which is the limit of $c^{(n)}(t)$. Therefore

$$(D\sigma(h)\sigma(h))(\xi) = \frac{d}{ds}\big|_{s=0}\sigma(\mathrm{Fl}_s(h))(\xi) = 0,$$

since $\mathrm{Fl}_t(h) \geq 0$ by invariance and $\mathrm{Fl}_t(h)(\xi) = 0$ by the previous consideration lead to $\sigma(\mathrm{Fl}_t(h))(\xi) = 0$ for $t \in]-\epsilon, \epsilon[$. Notice that we did not need the global Lipschitz property of the Stratonovich correction for the proof of this lemma.   □

Before we show sufficiency for the HJMM equation (4.2) with jumps, we consider the pure diffusion case. Notice that, due to Lemma 4.17, the condition (4.4) is in fact equivalent to the very same condition formulated with the Stratonovich drift σ^0, defined in (4.21) below, instead of α, since the Stratonovich correction vanishes at the boundary of P.

In order to treat the pure diffusion case, we apply [34], which, by using the support theorem provided in [33], offers a general characterization of stochastic invariance of closed sets for SPDEs.

Other results for positivity preserving SPDEs, where, in contrast to our framework, the state space is an L^2-space, can be found in [27] and [30]. The results from [30] have been used in [36] in order to derive some positivity results for Lévy term structure models on L^2-spaces.

4.18. **Proposition.** *Suppose Assumptions 4.11, 4.15 are fulfilled and $\gamma \equiv 0$. If conditions (4.4), (4.5) are satisfied, then equation (4.2) is positivity preserving.*

Proof. In view of Lemma 4.12, it suffices to show that for all $h_0 \in P$ we have $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} \{r_t \in P\}) = 1$, where $(r_t)_{t \geq 0}$ denotes the mild solution for (4.2) with $r_0 = h_0$. Moreover, we may assume that the vector fields α, σ and

$$(4.21) \qquad h \mapsto \sigma^0(h) := \alpha(h) - \frac{1}{2} \sum_j D\sigma^j(h)\sigma^j(h)$$

are bounded in order to apply Nakayama's beautiful support theorem from [34]. Indeed, for $n \in \mathbb{N}$ we choose a bump function $\psi_n \in C^\infty(H_\beta; [0,1])$ such that $\psi_n \equiv 1$ on $\overline{B_n(0)}$ and $\mathrm{supp}(\psi_n) \subset B_{n+1}(0)$ and define the vector fields

$$\alpha_n(h) := \psi_n(h)\alpha(h), \quad h \in H_\beta$$
$$\sigma_n(h) := \psi_n(h)\sigma(h), \quad h \in H_\beta.$$

These vector fields and

$$h \mapsto \sigma_n^0(h) := \alpha_n(h) - \frac{1}{2} \sum_j D\sigma_n^j(h)\sigma_n^j(h)$$

are bounded by the Lipschitz continuity of α, σ and σ^0, and Assumptions 4.11, 4.15 as well as conditions (4.4), (4.5) are again satisfied.

Now, we show that the semigroup Nagumo's condition (3) from [33, Prop. 1.1] is fulfilled due to conditions (4.4) and (4.5). Introducing the distance $d_P(h)$ from P as minimal distance of $h \in H_\beta$ from P, we can formulate Nagumo's condition as

$$(4.22) \qquad \liminf_{t \downarrow 0} \frac{1}{t} d_P(S_t h + t\sigma^0(h) + t\sigma(h)u) = 0$$

for all $u \in U_0$ and $h \in P$. Fix now $h \in P$ and $u \in U_0$ and introduce the abbreviation $\tilde{\sigma} = \sigma^0 + \sigma(\cdot)u$, then we obviously have

$$\|S_t h + t\sigma^0(h) + t\sigma(h)u - S_t \mathrm{Fl}_t^{\tilde{\sigma}}(h)\|_\beta = t \left\| \tilde{\sigma}(h) - S_t \frac{\mathrm{Fl}_t^{\tilde{\sigma}}(h) - h}{t} \right\|_\beta, \quad t > 0$$

which means that

$$\lim_{t \downarrow 0} \frac{1}{t} \|S_t h + t\sigma^0(h) + t\sigma(h)u - S_t \mathrm{Fl}_t^{\tilde{\sigma}}(h)\|_\beta = 0.$$

Hence, Nagumo's condition (4.22) can equivalently be formulated as

$$(4.23) \qquad \liminf_{t \downarrow 0} \frac{1}{t} d_P(S_t \mathrm{Fl}_t^{\tilde{\sigma}}(h)) = 0,$$

for the particular choice of $u \in U_0$ and $h \in P$, since the shortest distance projector onto P is a Lipschitz continuous map. Due to conditions (4.4), (4.5) and Lemma 4.17, the semiflow $\mathrm{Fl}^{\tilde{\sigma}}$ leaves P invariant by [39], the semigroup $(S_t)_{t \geq 0}$ certainly, too, and therefore we have $d_P(S_t \mathrm{Fl}_t^{\tilde{\sigma}}(h)) = 0$, $t \geq 0$ whence Nagumo's condition (4.23) is more than satisfied. $\square$

Finally, the next result states the sufficient conditions under which we can conclude that equation (4.2) is positivity preserving.

4.19. **Proposition.** *Suppose Assumptions 4.10, 4.11, 4.15 and conditions (4.3)– (4.6) are fulfilled. Then, equation (4.2) is positivity preserving.*

Proof. Since the measure F is σ-finite, there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ with $B_n \uparrow E$ and $F(B_n) < \infty$ for all $n \in \mathbb{N}$. Let $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ be arbitrary. Relations (4.4), (4.7), (4.5), Proposition 4.18 and (4.6) together with the closedness

of P yield that, for each $n \in \mathbb{N}$, the mild solution $(r_t^n)_{t \geq 0}$ of the stochastic partial differential equation

$$(4.24) \quad \begin{cases} dr_t^n &= (\frac{d}{d\xi} r_t^n + \alpha(r_t^n) - \int_{B_n} \gamma(r_t^n, x) F(dx)) dt + \sigma(r_t^n) dW_t \\ & \quad + \int_{B_n} \gamma(r_{t-}^n, x) \mu(dt, dx) \\ r_0^n &= h_0 \end{cases}$$

satisfies $\mathbb{P}(\bigcap_{t \in \mathbb{R}_+} r_{t \wedge \tau}^n \in P) = 1$, where τ denotes the strictly positive stopping time

$$\tau := \inf\{t \geq 0 : \mu([0, t] \times B_n) = 1\}.$$

By virtue of Lemma 4.14, for each $n \in \mathbb{N}$ equation (4.24) is positivity preserving. According to [20, Prop. 9.1] we have

$$\mathbb{E}\left[\sup_{t \in [0, T]} \|r_t - r_t^n\|_\beta^2 \right] \to 0 \quad \text{for all } T \in \mathbb{R}_+,$$

proving that equation (4.2) is positivity preserving. $\qquad \square$

4.20. **Theorem.** *Suppose Assumptions 4.10, 4.11, 4.15 are fulfilled. Then, for each initial curve $h_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\beta)$ there exists a unique adapted, càdlàg, mean-square continuous $\mathcal{H}_\beta$-valued solution $(f_t)_{t \geq 0}$ for the HJM equation (4.1) with $f_0 = \ell h_0$ satisfying (3.15), and there exists a unique adapted, càdlàg, mean-square continuous mild and weak H_β-valued solution $(r_t)_{t \geq 0}$ for the HJMM equation (4.2) with $r_0 = h_0$ satisfying (3.16), which is given by $r_t := \pi U_t f_t$, $t \geq 0$. Moreover, equation (4.2) is positivity preserving if and only if we have (4.3)–(4.6).*

Proof. The statement follows from [20, Cor. 10.9], Proposition 4.7 (see also Remark 4.6) and Proposition 4.19. $\qquad \square$

4.21. **Remark.** *Note that Theorem 4.20 is also valid on other state spaces. The only requirements are that the Hilbert space H consists of real-valued, continuous functions, on which the point evaluations are continuous linear functionals, and that the shift semigroup extends to a strongly continuous group on a larger Hilbert space $\mathcal{H}$.*

4.22. **Remark.** *For the particular situation where equation (4.2) has no jumps, Theorem 4.20 corresponds to the statement of [30, Thm. 3], where positivity on weighted L^2-spaces is investigated. Since point evaluations are discontinuous functionals on L^2-spaces, the conditions in [30] are formulated by taking other appropriate linear functionals.*

We shall now consider the arbitrage free situation. Let $\alpha = \alpha_{\mathrm{HJM}} : H_\beta \to H_\beta$ in (4.2) be defined according to the HJM drift condition (1.6).

4.23. **Proposition.** *Conditions (4.3)–(4.6) are satisfied if and only if we have (4.5), (4.6) and*

$$(4.25) \quad \gamma(h, x)(\xi) = 0, \quad \xi \in (0, \infty), h \in \partial P_\xi \text{ and } F\text{-almost all } x \in E.$$

Proof. Provided (4.5), (4.6) are fulfilled, conditions (4.3), (4.4) are satisfied if and only if we have (4.3) and

$$(4.26) \quad -\int_E \gamma(h, x)(\xi) e^{\Gamma(h, x)(\xi)} F(dx) \geq 0, \quad \xi \in (0, \infty), h \in \partial P_\xi$$

because the drift α is given by (1.6). By (4.7), relations (4.3), (4.26) are fulfilled if and only if we have (4.25). $\qquad \square$

Now let, as in Section 3, measurable vector fields $\sigma : H_\beta \to L_2^0(H_\beta^0)$ and $\gamma : H_\beta \times E \to H_{\beta'}^0$ be given, where $\beta' > \beta$ is a real number.

4.24. Theorem. *Suppose Assumptions 3.1, 4.15 are fulfilled. Then, the statement of Theorem 3.4 is valid, and, in addition, the HJMM equation (1.10) is positivity preserving if and only if we have (4.5), (4.6), (4.25).*

Proof. The statement follows from Theorem 3.4, Theorem 4.20 and Proposition 4.23. □

Finally, let us consider the Lévy case, treated at the end of Section 3. In this framework, the following statement is valid.

4.25. Proposition. *Conditions (4.5), (4.6) and (4.25) are satisfied if and only if we have (4.5) and*

$$(4.27) \quad h + \delta^k(h)x \in P, \quad h \in P, \, k = 1, \ldots, e \text{ and } F_k\text{-almost all } x \in \mathbb{R}$$

$$(4.28) \quad \delta^k(h)(\xi) = 0, \quad \xi \in (0, \infty), \, h \in \partial P_\xi \text{ and all } k = 1, \ldots, e \text{ with } F_k(\mathbb{R}) > 0.$$

Proof. The claim follows from the Definition (3.18) of F and the Definition (3.22) of γ. □

4.26. Corollary. *Suppose Assumption 3.5, 4.15 are fulfilled. Then, the statement of Corollary 3.6 is valid, and, in addition, the HJMM equation (1.10) is positivity preserving if and only if we have (4.5), (4.27), (4.28).*

Proof. The assertion follows from Theorem 4.24 and Proposition 4.25. □

Our above results on arbitrage free, positivity preserving term structure models apply in particular for local state dependent volatilities. The following two results are obvious.

4.27. Proposition. *Suppose for each j there exists $\tilde{\sigma}^j : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and there are $\tilde{\gamma} : \mathbb{R}_+ \times \mathbb{R} \times E \to \mathbb{R}$ and $\phi : \mathbb{R}_+ \to (0, \infty)$ such that*

$$\sigma^j(h)(\xi) = \phi(\|h\|_\beta)\tilde{\sigma}^j(\xi, h(\xi)), \quad (h, \xi) \in H_\beta \times \mathbb{R}_+, \quad \text{for all } j$$

$$\gamma(h, x)(\xi) = \phi(\|h\|_\beta)\tilde{\gamma}(\xi, h(\xi), x), \quad (h, x, \xi) \in H_\beta \times E \times \mathbb{R}_+.$$

Then, conditions (4.5), (4.6), (4.25) are fulfilled if and only if

$$(4.29) \qquad \tilde{\sigma}^j(\xi, 0) = 0, \quad \xi \in (0, \infty), \quad \text{for all } j$$

$$(4.30) \qquad y + z\tilde{\gamma}(\xi, y, x) \geq 0, \quad \xi \in (0, \infty), \, y \in \mathbb{R}_+, \, z \in \phi(\mathbb{R}_+)$$
$$\text{and } F\text{-almost all } x \in E$$

$$(4.31) \qquad \tilde{\gamma}(\xi, 0, x) = 0, \quad \xi \in (0, \infty) \text{ and } F\text{-almost all } x \in E.$$

Lévy term structure models with local state dependent volatilities have been studied in [36] and [28]. In the framework of Proposition 4.25 we obtain the following result.

4.28. Proposition. *Suppose for each j there is $\tilde{\sigma}^j : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, for all $k = 1, \ldots, e$ there is $\tilde{\delta}^k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and there exists $\phi : \mathbb{R}_+ \to (0, \infty)$ such that*

$$\sigma^j(h)(\xi) = \phi(\|h\|_\beta)\tilde{\sigma}^j(\xi, h(\xi)), \quad (h, \xi) \in H_\beta \times \mathbb{R}_+, \quad \text{for all } j$$

$$\delta^k(h)(\xi) = \phi(\|h\|_\beta)\tilde{\delta}^k(\xi, h(\xi)), \quad (h, \xi) \in H_\beta \times \mathbb{R}_+, \quad k = 1, \ldots, e.$$

Then, conditions (4.5), (4.27), (4.28) are fulfilled if and only if we have (4.29) and

$$(4.32) \qquad y + zx\tilde{\delta}^k(\xi, y) \geq 0, \quad \xi \in (0, \infty), \, y \in \mathbb{R}_+, \, z \in \phi(\mathbb{R}_+), \, k = 1, \ldots, e$$
$$\text{and } F_k\text{-almost all } x \in \mathbb{R}$$

$$(4.33) \qquad \tilde{\delta}^k(\xi, 0) = 0, \quad \xi \in (0, \infty) \text{ and all } k = 1, \ldots, e \text{ with } F_k(\mathbb{R}) > 0.$$

Section 5 in [36] contains some positivity results for Lévy driven term structure models on weighted L^2-spaces. Using Proposition 4.28, we can derive the analogous statements of [36, Thm. 4] on our H_β-spaces.

4.29. **Remark.** *For local state dependent volatilities we can establish sufficient conditions on the mappings $\tilde{\sigma}^j$, $\tilde{\gamma}$, ϕ resp. $\tilde{\sigma}^j$, $\tilde{\delta}^k$, ϕ such that Assumptions 3.1, 4.15 resp. Assumptions 3.5, 4.15 are fulfilled, which allows us to combine Theorem 4.24 and Proposition 4.27 resp. Corollary 4.26 and Proposition 4.28. We obtain such sufficient conditions by modifying the conditions from [16, Prop. 5.4.1] in an appropriate manner.*

Appendix A. Attached proofs

In this appendix we gather the proofs of results which we have postponed for the sake of lucidity.

Proof. (of Theorem 2.1) Note that H_β is the space H_w from [16, Sec. 5.1] with weight function $w(\xi) = e^{\beta\xi}$, $\xi \in \mathbb{R}_+$. Hence, the first six statements follow from [16, Thm. 5.1.1, Cor. 5.1.1].

For each $\beta' > \beta$, the observation

$$\int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta\xi} d\xi \leq \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta'\xi} d\xi, \quad h \in H_{\beta'}$$

shows $H_{\beta'} \subset H_\beta$ and (2.5). For an arbitrary $h \in H_{\beta'}$ we have, by Hölder's inequality,

$$\int_{\mathbb{R}_+} |h(\xi) - h(\infty)|^2 e^{\beta\xi} d\xi = \int_{\mathbb{R}_+} \left(\int_\xi^\infty h'(\eta) e^{\frac{1}{2}\beta'\eta} e^{-\frac{1}{2}\beta'\eta} d\eta \right)^2 e^{\beta\xi} d\xi$$

$$\leq \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} |h'(\eta)|^2 e^{\beta'\eta} d\eta \right) \left(\int_\xi^\infty e^{-\beta'\eta} d\eta \right) e^{\beta\xi} d\xi \leq \frac{1}{\beta'(\beta'-\beta)} \|h\|_{\beta'}^2.$$

Choosing $C_5 := \frac{1}{\beta'(\beta'-\beta)}$ proves (2.6).

It is clear that $\|\cdot\|_\beta$ is a norm on $\mathcal{H}_\beta$. First, we prove that there is a constant $K_1 > 0$ such that

(A.1) $$\|h'\|_{L^1(\mathbb{R})} \leq K_1 \|h\|_\beta, \quad h \in \mathcal{H}_\beta.$$

Setting $K_1 := \sqrt{\frac{2}{\beta}}$, this is established by Hölder's inequality

(A.2)
$$\int_{\mathbb{R}} |h'(\xi)| d\xi = \int_{\mathbb{R}} |h'(\xi)| e^{\frac{1}{2}\beta|\xi|} e^{-\frac{1}{2}\beta|\xi|} d\xi$$

$$\leq \left(\int_{\mathbb{R}} |h'(\xi)|^2 e^{\beta|\xi|} d\xi \right)^{1/2} \left(\int_{\mathbb{R}} e^{-\beta|\xi|} d\xi \right)^{1/2} = \sqrt{\frac{2}{\beta}} \left(\int_{\mathbb{R}} |h'(\xi)|^2 e^{\beta|\xi|} d\xi \right)^{1/2}.$$

As a consequence of (A.1), for each $h \in \mathcal{H}_\beta$ the limits $h(\infty) := \lim_{\xi\to\infty} h(\xi)$ and $h(-\infty) := \lim_{\xi\to-\infty} h(\xi)$ exist. This allows us to the define the new norm

$$|h|_\beta := \left(|h(-\infty)|^2 + \int_{\mathbb{R}} |h'(\xi)|^2 e^{\beta|\xi|} d\xi \right)^{1/2}, \quad h \in \mathcal{H}_\beta.$$

From (A.2) we also deduce that

(A.3) $$\|h'\|_{L^1(\mathbb{R})} \leq K_1 |h|_\beta, \quad h \in \mathcal{H}_\beta.$$

Setting $K_2 := 1 + K_1$, from (A.1) and (A.3) is follows that

(A.4) $$\|h\|_{L^\infty(\mathbb{R})} \leq K_2 \|h\|_\beta,$$

(A.5) $$\|h\|_{L^\infty(\mathbb{R})} \leq K_2 |h|_\beta$$

for all $h \in \mathcal{H}_\beta$. Estimate (A.4) shows that, for each $\xi \in \mathbb{R}$, the point evaluation $h \mapsto h(\xi)$, $\mathcal{H}_\beta \to \mathbb{R}$ is a continuous linear functional.

Using (A.4) and (A.5) we conclude that

$$\frac{1}{(1 + K_2^2)^{\frac{1}{2}}} \|h\|_\beta \leq |h|_\beta \leq (1 + K_2^2)^{\frac{1}{2}} \|h\|_\beta, \quad h \in \mathcal{H}_\beta$$

which shows that $\| \cdot \|_\beta$ and $| \cdot |_\beta$ are equivalent norms on $\mathcal{H}_\beta$.

Consider the separable Hilbert space $\mathbb{R} \times L^2(\mathbb{R})$ equipped with the norm $(| \cdot |^2 + \| \cdot \|_{L^2(\mathbb{R})}^2)^{\frac{1}{2}}$. Then the linear operator $T : (\mathcal{H}_\beta, | \cdot |_\beta) \to \mathbb{R} \times L^2(\mathbb{R})$ given by

$$Th := (h(-\infty), h' e^{\frac{1}{2}\beta|\cdot|}), \quad h \in \mathcal{H}_\beta$$

is an isometric isomorphism with inverse

$$(T^{-1}(u, g))(x) = u + \int_{-\infty}^{x} g(\eta) e^{-\frac{1}{2}\beta|\eta|} d\eta, \quad (u, g) \in \mathbb{R} \times L^2(\mathbb{R}).$$

Since $\| \cdot \|_\beta$ and $| \cdot |_\beta$ are equivalent, $(\mathcal{H}_\beta, \| \cdot \|_\beta)$ is a separable Hilbert space.

Next, we claim that

$$\mathcal{D}_0 := \{g \in \mathcal{H}_\beta \,|\, g' \in \mathcal{H}_\beta\}$$

is dense in $\mathcal{H}_\beta$. Indeed, $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, see [5, Cor. IV.23]. Fix $h \in \mathcal{H}_\beta$ and let $(g_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be an approximating sequence of $h' e^{\frac{1}{2}\beta|\cdot|}$ in $L^2(\mathbb{R})$. Then we have $h_n := T^{-1}(h(-\infty), g_n) \in \mathcal{D}_0$ for all $n \in \mathbb{N}$ and $h_n \to h$ in $\mathcal{H}_\beta$.

For each $t \in \mathbb{R}$ and $h \in \mathcal{H}_\beta$, the function $U_t h$ is again absolutely continuous. We claim that there exists a constant $K_3 > 0$ such that

(A.6) $$\|U_t h\|_\beta^2 \leq (K_3 + e^{\beta|t|})\|h\|_\beta^2, \quad (t, h) \in \mathbb{R} \times \mathcal{H}_\beta.$$

Indeed, using (A.4) we obtain

$$\|U_t h\|_\beta^2 = |h(t)|^2 + \int_0^\infty |h'(\xi + t)|^2 e^{\beta\xi} d\xi + \int_{-\infty}^0 |h'(\xi + t)|^2 e^{-\beta\xi} d\xi$$

$$= |h(t)|^2 + e^{-\beta t} \int_t^\infty |h'(\xi)|^2 e^{\beta\xi} d\xi + e^{\beta t} \int_{-\infty}^t |h'(\xi)|^2 e^{-\beta\xi} d\xi$$

$$\leq (K_2^2 + 1 + e^{\beta|t|})\|h\|_\beta^2, \quad h \in \mathcal{H}_\beta.$$

Setting $K_3 := 1 + K_2^2$, this establishes (A.6). Hence, we have $U_t h \in \mathcal{H}_\beta$ for all $t \in \mathbb{R}$ and $h \in \mathcal{H}_\beta$ and $U_t \in L(\mathcal{H}_\beta)$, $t \in \mathbb{R}$.

It remains to show strong continuity of the group $(U_t)_{t \in \mathbb{R}}$. Using the observation

$$h(\xi + t) - h(\xi) = t \int_0^1 h'(\xi + st) ds, \quad (\xi, t, h) \in \mathbb{R} \times \mathbb{R} \times \mathcal{H}_\beta$$

and (A.6), we obtain for each $g \in \mathcal{D}_0$ the convergence

$$\|U_t g - g\|_\beta^2 = |g(t) - g(0)|^2 + \int_{\mathbb{R}} |g'(\xi + t) - g'(\xi)|^2 e^{\beta|\xi|} d\xi$$

$$\leq |g(t) - g(0)|^2 + t^2 \int_0^1 \int_{\mathbb{R}} |g''(\xi + st)|^2 e^{\beta|\xi|} d\xi ds$$

$$\leq |g(t) - g(0)|^2 + t^2 \int_0^1 \|U_{st} g'\|_\beta^2 ds$$

$$\leq |g(t) - g(0)|^2 + t^2 \|g'\|_\beta^2 \int_0^1 (K_3 + e^{\beta s|t|}) ds$$

$$= |g(t) - g(0)|^2 + \left(K_3 t^2 + \frac{|t|}{\beta}(e^{\beta|t|} - 1)\right)\|g'\|_\beta^2 \to 0 \quad \text{as } t \to 0.$$

Hence, $(U_t)_{t \in \mathbb{R}}$ is strongly continuous on $\mathcal{D}_0$. But for any $h \in \mathcal{H}_\beta$ and $\epsilon > 0$ there exists $g \in \mathcal{D}_0$ with $\|h - g\|_\beta < \frac{\epsilon}{4\sqrt{K_3 + e^\beta}}$. Combining this with (A.6) yields

$$\|U_t h - h\|_\beta \leq \|U_t(h - g)\|_\beta + \|U_t g - g\|_\beta + \|g - h\|_\beta$$
$$< \sqrt{K_3 + e^{\beta|t|}} \frac{\epsilon}{4\sqrt{K_3 + e^\beta}} + \|U_t g - g\|_\beta + \frac{\epsilon}{4\sqrt{K_3 + e^\beta}} < \epsilon$$

for $t \in \mathbb{R}$ small enough. We conclude that $(U_t)_{t \in \mathbb{R}}$ is a C_0-group on $\mathcal{H}_\beta$.

Finally, relation (2.7) follows from the definitions of ℓ and π. $\qquad \square$

Proof. (of Lemma 4.13) Note that $\tilde{W}$ is a continuous $(\tilde{\mathcal{F}}_t)$-adapted process with $\tilde{W}_0 = 0$, and $\tilde{\mu}$ is an integer-valued random measure on $\mathbb{R}_+ \times E$.

We fix an arbitrary $u \in U$. The process

$$M_t := \frac{\exp(i\langle u, W_t \rangle)}{\mathbb{E}[\exp(i\langle u, W_t \rangle)]}, \quad t \geq 0$$

is a complex-valued martingale, because for all $s, t \in \mathbb{R}_+$ with $s < t$ the random variable $W_t - W_s$ and the σ-algebra $\mathcal{F}_s$ are independent. The martingale $(M_t)_{t \geq 0}$ admits the representation

$$M_t = \exp\left(i\langle u, W_t \rangle + \frac{t}{2} \langle Qu, u \rangle \right), \quad t \geq 0.$$

According to the Optional Stopping Theorem, the process $(M_{t+\tau})_{t \geq 0}$ is a nowhere vanishing complex $(\tilde{\mathcal{F}}_t)$-martingale. Thus, for $s, t \in \mathbb{R}_+$ with $s < t$ we obtain

$$\mathbb{E}\left[\frac{M_{t+\tau}}{M_{s+\tau}} \,|\, \tilde{\mathcal{F}}_s \right] = 1.$$

For each $C \in \tilde{\mathcal{F}}_s$ we get

$$\mathbb{E}[\mathbb{1}_C \exp(i\langle u, \tilde{W}_t - \tilde{W}_s \rangle)] = \mathbb{P}(C) \exp\left(-\frac{t-s}{2} \langle Qu, u \rangle \right).$$

Hence, the random variable $\tilde{W}_t - \tilde{W}_s$ and the σ-algebra $\tilde{\mathcal{F}}_s$ are independent, and $\tilde{W}_t - \tilde{W}_s$ has a Gaussian distribution with covariance operator $(t-s)Q$. The expansion (4.16) follows from (3.1).

Now we fix $v \in \mathbb{R}$ and $B \in \mathcal{E}$ with $F(B) < \infty$. The process

$$N_t := \frac{\exp(iv\mu([0,t] \times B))}{\mathbb{E}[\exp(iv\mu([0,t] \times B))]}, \quad t \geq 0$$

is a complex-valued martingale, because for all $s, t \in \mathbb{R}_+$ with $s < t$ the random variable $\mu((s,t] \times B)$ and the σ-algebra $\mathcal{F}_s$ are independent. By [25, Thm. II.4.8] the martingale $(N_t)_{t \geq 0}$ admits the representation

$$N_t = \exp\left(iv\mu([0,t] \times B) - (e^{iv} - 1)F(B)t \right), \quad t \geq 0.$$

According to the Optional Stopping Theorem, the process $(N_{t+\tau})_{t \geq 0}$ is a nowhere vanishing complex $(\tilde{\mathcal{F}}_t)$-martingale. Thus, for $s, t \in \mathbb{R}_+$ with $s < t$ we obtain

$$\mathbb{E}\left[\frac{N_{t+\tau}}{N_{s+\tau}} \,|\, \tilde{\mathcal{F}}_s \right] = 1.$$

For each $C \in \tilde{\mathcal{F}}_s$ we get

$$\mathbb{E}[\mathbb{1}_C \exp(iv\tilde{\mu}((s,t] \times B))] = \mathbb{P}(C) \exp\left((e^{iv} - 1)F(B)(t-s) \right).$$

Hence, the random variable $\tilde{\mu}((s,t] \times B)$ and the σ-algebra $\tilde{\mathcal{F}}_s$ are independent, and $\tilde{\mu}((s,t] \times B)$ has a Poisson distribution with mean $(t-s)F(B)$.

Next, we claim that

$$(A.7) \qquad \int_\tau^{\tau+t} \Phi_s dW_s = \int_0^t \Phi_{\tau+s} d\tilde{W}_s$$

for every predictable process $\Phi : \Omega \times \mathbb{R}_+ \to L_2^0(H_\beta)$ satisfying

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2^0(H_\beta)}^2 ds < \infty \bigg) = 1$$

for all $t \in \mathbb{R}_+$, and

$$(A.8) \qquad \int_\tau^{\tau+t} \Psi(s,x)(\mu(ds,dx) - F(dx)ds) = \int_0^t \Psi(\tau+s,x)(\tilde{\mu}(ds,dx) - F(dx)ds)$$

for every predictable process $\Psi : \Omega \times \mathbb{R}_+ \times E \to H_\beta$ satisfying

$$\mathbb{P}\bigg(\int_0^t \int_E \|\Psi(s,x)\|_\beta^2 F(dx)ds < \infty \bigg) = 1$$

for all $t \in \mathbb{R}_+$. If Φ, Ψ are elementary and τ a simple stopping time, then (A.7), (A.8) hold by inspection. The general case follows by localization.

If $(r_t)_{t \geq 0}$ is a weak solution to (4.2), for every $\zeta \in \mathcal{D}((\frac{d}{d\xi})^*)$ relations (A.7), (A.8) yield

$$\langle \zeta, r_{\tau+t} \rangle = \langle \zeta, r_\tau \rangle + \int_\tau^{\tau+t} \bigg(\Big\langle \Big(\frac{d}{d\xi}\Big)^* \zeta, r_s \Big\rangle + \langle \zeta, \alpha(r_s) \rangle \bigg) ds + \int_\tau^{\tau+t} \langle \zeta, \sigma(r_s) \rangle dW_s$$

$$+ \int_\tau^{\tau+t} \int_E \langle \zeta, \gamma(r_{s-},x) \rangle (\mu(ds,dx) - F(dx)ds)$$

$$= \langle \zeta, r_\tau \rangle + \int_0^t \bigg(\Big\langle \Big(\frac{d}{d\xi}\Big)^* \zeta, r_{\tau+s} \Big\rangle + \langle \zeta, \alpha(r_{\tau+s}) \rangle \bigg) ds + \int_0^t \langle \zeta, \sigma(r_{\tau+s}) \rangle d\tilde{W}_s$$

$$+ \int_0^t \int_E \langle \zeta, \gamma(r_{(\tau+s)-},x) \rangle (\tilde{\mu}(ds,dx) - F(dx)ds).$$

Hence, $(\tilde{r}_t)_{t \geq 0}$ is a weak solution for (4.17). $\qquad \square$

REFERENCES

[1] Albeverio, S., Mandrekar, V., Rüdiger, B. (2009): Existence of mild solutions for stochastic differential equations and semilinear equations with non Gaussian Lévy noise. *Stochastic Processes and Their Applications* **119**(3), 835–863.

[2] Baudoin, F., Teichmann, J. (2005): Hypoellipticity in infinite dimensions and an application in interest rate theory. *The Annals of Applied Probability*, **15**(3), 1765–1777.

[3] Björk, T., Di Masi, G., Kabanov, Y., Runggaldier, W. (1997): Towards a general theory of bond markets. *Finance and Stochastics* **1**(2), 141–174.

[4] Björk, T., Kabanov, Y., Runggaldier, W. (1997): Bond market structure in the presence of marked point processes. *Mathematical Finance* **7**(2), 211–239.

[5] Brézis, H. (1993): *Analyse fonctionnelle*, fourth edition, Masson, Paris, Théorie et applications.

[6] Carmona, R., Tehranchi, M. (2006): *Interest rate models: an infinite dimensional stochastic analysis perspective.* Berlin: Springer.

[7] Cox, J,. Ingersoll, J., Ross, S. (1985): A theory of the term structure of interest rates. *Econometrica* **53**, 385–408.

[8] Da Prato, G., Zabczyk, J. (1992): *Stochastic equations in infinite dimensions.* New York: Cambridge University Press.

[9] Dellacherie, C., Meyer, P. A. (1982): *Probabilités et potentiel.* Hermann: Paris.

[10] Eberlein, E., Raible, S. (1999): Term structure models driven by general Lévy processes. *Mathematical Finance* **9**(1), 31–53.

[11] Eberlein, E., Özkan, F. (2003): The defaultable Lévy term structure: ratings and restructuring. *Mathematical Finance* **13**, 277–300.

[12] Eberlein, E., Jacod, J., Raible, S. (2005): Lévy term structure models: no-arbitrage and completeness. *Finance and Stochastics* **9**, 67–88.

[13] Eberlein, E., Kluge, W. (2006): Exact pricing formulae for caps and swaptions in a Lévy term structure model. *Journal of Computational Finance* **9**(2), 99–125.

[14] Eberlein, E., Kluge, W. (2006): Valuation of floating range notes in Lévy term structure models. *Mathematical Finance* **16**, 237–254.

[15] Eberlein, E., Kluge, W. (2007): Calibration of Lévy term structure models. In *Advances in Mathematical Finance:* In Honor of Dilip Madan, M. Fu, R. A. Jarrow, J.-Y. Yen, and R. J. Elliott (Eds.), Birkhäuser, pp. 155–180.

[16] Filipović, D. (2001): *Consistency problems for Heath–Jarrow–Morton interest rate models.* Berlin: Springer.

[17] Filipović, D. (2005): Time-inhomogeneous affine processes. *Stochastic Processes and Their Applications* **115**(4) 639–659.

[18] Filipović, D. (2009): *Term-structure models: A graduate course.* Springer Finance.

[19] Filipović, D., Tappe, S. (2008): Existence of Lévy term structure models. *Finance and Stochastics* **12**(1), 83–115.

[20] Filipović, D., Tappe, S., Teichmann, J. (2010): Jump-diffusions in Hilbert spaces: Existence, stability and numerics. Forthcoming in *Stochastics*.

[21] Filipović, D., Teichmann, J. (2004): On the geometry of the term structure of interest rates. *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* **460**, 129–167.

[22] Getoor, R. K. (1979): On the construction of kernels. *Séminaire de Probabilités* IX, Lecture Notes in Mathematics 465, 443–463.

[23] Heath, D., Jarrow, R., Morton, A. (1992): Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**(1), 77–105.

[24] Hyll, M. (2000): Affine term structures and short-rate realizations of forward rate models driven by jump-diffusion processes. In *Essays on the term structure of interest rates* PhD thesis, Stockholm School of Economics.

[25] Jacod, J., Shiryaev, A. N. (2003): *Limit theorems for stochastic processes.* Springer, Berlin.

[26] Jarrow, A., Madan, D. B. (1995): Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Mathematical Finance* **5**(4), 311–336.

[27] Kotelenez, P. (1992): Comparison methods for a class of function valued stochastic partial differential equations. *Probability Theory and related fields* **93**(1), 1–19.

[28] Marinelli, C. (2008): Local well-posedness of Musiela's SPDE with Lévy noise. *Mathematical Finance*, to appear.

[29] Marinelli, C., Prévôt, C., Röckner, M. (2010): Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. *Journal of Functional Analysis* **258**(2), 616–649.

[30] Milian, A. (2002): Comparison theorems for stochastic evolution equations. *Stochastics* **72**, 79–108.

[31] Morton, A. (1989): Arbitrage and martingales. PhD thesis, Cornell University.

[32] Musiela, M. (1993): Stochastic PDEs and term structure models. *Journées Internationales de Finance*, IGR-AFFI, La Baule.

[33] Nakayama, T. (2004): Support theorem for mild solutions of SDE's in Hilbert spaces. *J. Math. Sci. Univ. Tokyo* **11**, 245–311.

[34] Nakayama, T. (2004): Viability Theorem for SPDE's including HJM framework. *J. Math. Sci. Univ. Tokyo* **11**, 313–324.

[35] Peszat, S., Zabczyk, J. (2007): *Stochastic partial differential equations with Lévy noise.* Cambridge University Press, Cambridge.

[36] Peszat, S., Zabczyk, J. (2007): Heath-Jarrow-Morton-Musiela equation of bond market. Preprint IMPAN 677, Warsaw. (www.impan.gov.pl/EN/Preprints/index.html)

[37] Raible, S. (2000): Lévy processes in finance: theory, numerics, and empirical facts. PhD thesis, University of Freiburg.

[38] Shirakawa, H. (1991): Interest rate option pricing with Poisson-Gaussian forward rate curve processes. *Mathematical Finance* **1**(4), 77–94.

[39] Volkmann, P. (1973): Über die Invarianz konvexer Mengen und Differentialgleichungen in einem normierten Raume. *Mathematische Annalen* **203**(1), 201–210.

[40] Watanabe, S., Yamada, T. (1971): On the uniqueness of solutions of stochastic differential equations. II. *J. Math. Kyoto Univ.* **11**, 553–563.

[41] Werner, D. (2002): *Funktionalanalysis.* Fourth Edition, Berlin: Springer.

Swiss Finance Institute, Ecole Polytechnique Fédérale de Lausanne, Quartier UNIL-Dorigny, Extranef 218, CH-1015 Lausanne, Switzerland; ETH Zürich, Department of Mathematics, Rämistrasse 101, CH-8092 Zürich, Switzerland

E-mail address: `damir.filipovic@epfl.ch, stefan.tappe@math.ethz.ch, josef.teichmann@math.ethz.ch`