THE HJMM EQUATION WITH REAL-WORLD DYNAMICS DRIVEN BY WIENER PROCESSES AND POISSON RANDOM MEASURES

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Abstract. We present a general framework of interest rate models driven by Wiener processes and Poisson random measures. Using as numéraire the growth optimal portfolio, we model the interest rate term structure under the real-world probability measure, and hence, we do not need the existence of an equivalent risk-neutral probability measure. Our investigations include the derivation of the growth optimal portfolio dynamics, the derivation of the drift condition, an existence proof of the corresponding term structure equation and a characterization of positivity preserving models.

1. Introduction

The time t value of a Euro at time $T \geq t$ is expressed by a *Zero Coupon Bond* with maturity T. This is a contract which guarantees the holder one Euro to be paid at the maturity date T. The corresponding bond prices can be written as the continuous discounting of one unit of cash

$$P_t(T) = \exp\bigg(- \int_t^T f_t(s) ds \bigg),$$

where $f_t(T)$ is the rate prevailing at time t for instantaneous borrowing at time T, also called the forward rate for date T.

In the spirit of [4, 6], we model the forward rate dynamics, which are driven by jump-diffusions, under the real-world probability measure. This is based on the Benchmark Approach presented in [27]. More precisely, we suppose that, under the real-world probability measure $\mathbb{P}$, for every date T the forward rates $f(T)$ follow an Itô process of the form

$$\begin{aligned} df_t(T) &= \alpha_t(T)dt + \sigma_t(T)dW_t + \int_B \gamma_t(T,x)(\mu(dt,dx) - F(dx)dt) \\ &\quad + \int_{B^c} \gamma_t(T,x)\mu(dt,dx), \quad t \in [0,T] \end{aligned} \tag{1.1}$$

with a (possibly infinite dimensional) Wiener process W and a homogeneous Poisson random measure μ. The integral $\int_B$ represents the small jumps of the forward rates, and $\int_{B^c}$ represents the large jumps. Note that (1.1) includes the original Heath-Jarrow-Morton (HJM) framework from [21] and its extensions such as [2, 3], [5] and [14, 13, 9, 10, 11, 12], which require the existence of a risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$.

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From a financial modeling point of view, the drift and the volatilities should be allowed to depend on the state of the prevailing forward curve. As argued in [18], after switching to the Musiela parametrization (see [24])

$$r_t(\xi) := f_t(t+\xi), \quad \xi \geq 0$$

this leads to a stochastic partial differential equation (SPDE)

$$\left\{ \begin{array}{rcl} dr_t & = & \big(\frac{d}{dx} r_t + \alpha(r_t)\big)dt + \sigma(r_t)dW_t \\ & & + \int_B \gamma(r_{t-},x)(\mu(dt,dx) - F(dx)dt) \\ & & + \int_{B^c} \gamma(r_{t-},x)\mu(dt,dx) \\ r_0 & = & h_0 \end{array} \right. \tag{1.2}$$

in the spirit of [7] (see also [29, 19]) and [25], the so-called Heath-Jarrow-Morton-Musiela (HJMM) equation.

Our goal of this paper is to present new results concerning the HJMM equation (1.2) with real-world dynamics which extend and generalize prior contributions to this field. The precise objectives are the following:

- The growth optimal portfolio S^{δ_*} plays a crucial role in the Benchmark Approach from [27]. We will show that it has the dynamics

$$\begin{aligned} dS_t^{\delta_*} = S_{t-}^{\delta_*} \bigg[\bigg(& r_t(0) + \|\theta_t\|^2_{L^0_2(\mathbb{R})} + \int_E \frac{\psi_t(x)^2}{1-\psi_t(x)} F(dx) \bigg) dt \\ & + \theta_t dW_t + \int_E \frac{\psi_t(x)}{1-\psi_t(x)} (\mu(dt,dx) - F(dx)dt) \bigg], \end{aligned} \tag{1.3}$$

  where θ and ψ denote the reference market prices of risk with respect to the Wiener process W and the Poisson random measure μ. This generalizes the dynamics that have been derived in [27, 6].
- We will show that in an arbitrage free bond market (in the spirit of [27]) the drift term in (1.1) is given by

$$\begin{aligned} \alpha_t(T) = & \Big\langle \sigma_t(T), \theta_t + \int_t^T \sigma_t(s)ds \Big\rangle_{L^0_2(\mathbb{R})} \\ & - \int_B \gamma_t(T,x) \bigg[\exp\bigg(-\phi_t(x) - \int_t^T \gamma_t(s,x)ds \bigg) - 1 \bigg] F(dx) \\ & - \int_{B^c} \gamma_t(T,x) \exp\bigg(-\phi_t(x) - \int_t^T \gamma_t(s,x)ds \bigg) F(dx), \end{aligned} \tag{1.4}$$

  where ϕ denotes the transformation

$$\phi_t(x) := -\ln(1-\psi_t(x)),$$

  of the reference market prices of risk with respect to the Poisson random measure μ. This generalizes the dynamics that have been derived in [27, 4]. Moreover, with $\theta \equiv \psi \equiv 0$ we obtain the HJM drift conditions for interest rate models under an assumed risk-neutral probability measure $\mathbb{Q}$, which have been derived in the aforementioned papers on HJM models.
- Under suitable regularity conditions, we will prove existence and uniqueness of mild solutions to the HJMM equation (1.2) with real-world dynamics. These evolutions are more general and provide more modeling freedom than term structure models under classical risk-neutral pricing, which require the existence of an equivalent risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$. Simultaneously, we relax the regularity conditions on the volatilities, which have been assumed to be Lipschitz and bounded in [18]. This is due to a recent result from [32], which shows that local Lipschitz and linear growth conditions are sufficient for existence and uniqueness of mild solutions.

- In practice, we are interested in term structure models producing positive forward curve, because negative forward rates are very rarely observed. Denoting by P the closed, convex cone of all nonnegative forward curves, we will show that the three conditions

$$\sigma^j(h)(\xi) = 0 \quad \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \quad \text{for all } j \in \mathbb{N},$$
$$h + \gamma(h,x) \in P \quad \text{for all } h \in P, \quad \text{for } F\text{–almost all } x \in E,$$
$$\gamma(h,x)(\xi) = 0 \quad \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \quad \text{for } F\text{–almost all } x \in E,$$

  are necessary and sufficient for the positivity preserving property of the HJMM equation (1.2). Surprisingly, these conditions do not involve the market prices of risk θ, ϕ and resemble those from [18], which have been derived under an assumed risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$.

The remainder of this paper is organized as follows: In Section 2 we introduce the general stochastic framework. In Section 3 we derive the dynamics (1.3) of the growth optimal portfolio and in Section 4 we derive the drift condition (1.4). In Section 5 we present our existence and uniqueness result for mild solutions to the HJMM equation (1.2) – see Theorem 5.1 – and in Section 6 we provide our result concerning positivity preserving models. Finally, in Section 7 we illustrate our results by focusing on Lévy process driven interest rate models with real-world forward rate dynamics. For the sake of lucidity, we have postponed the proof of Theorem 5.1 to Appendix A.

2. The stochastic framework

In this section, we shall present the general stochastic framework for our investigations in the forthcoming sections.

Throughout this text, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

Let U be a separable Hilbert space and let $Q \in L(U)$ be a nuclear, self-adjoint, positive definite linear operator. Then, there exist an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of U and a sequence $(\lambda_j)_{j\in\mathbb{N}} \subset (0,\infty)$ with $\sum_{j\in\mathbb{N}} \lambda_j < \infty$ such that

$$Qe_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N},$$

namely, the λ_j are the eigenvalues of Q, and each e_j is an eigenvector corresponding to λ_j. The space $U_0 := Q^{1/2}(U)$, equipped with the inner product

$$\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U,$$

is another separable Hilbert space and $(\sqrt{\lambda_j} e_j)_{j\in\mathbb{N}}$ is an orthonormal basis. Let W be an U-valued Q-Wiener process, see [7, p. 86, 87]. For another separable Hilbert space H, we denote by $L_2^0(H) := L_2(U_0, H)$ the space of Hilbert-Schmidt operators from U_0 into H, which, endowed with the Hilbert-Schmidt norm

$$\|\Phi\|_{L_2^0(H)} := \left(\sum_{j\in\mathbb{N}} \|\Phi(\sqrt{\lambda_j} e_j)\|^2 \right)^{1/2}, \quad \Phi \in L_2^0(H)$$

itself is a separable Hilbert space.

Let $(E, \mathcal{E})$ be a measurable space which we assume to be a Blackwell space (see [8, 20]). We remark that every Polish space with its Borel σ-field is a Blackwell space. Furthermore, let μ be a time-homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [22, Def. II.1.20]. Then its compensator is of the form $dt \otimes F(dx)$, where F is a σ-finite measure on $(E, \mathcal{E})$. In the sequel, $B \in \mathcal{E}$ denotes a set with $F(B^c) < \infty$.

3. The growth optimal portfolio

In this section, we will derive the dynamics (1.3) of the growth optimal portfolio in a market driven by a Wiener process and a Poisson random measure. The growth optimal portfolio is characterized as the portfolio maximizing the expected logarithmic utility from terminal wealth. For more detailed explanations about the growth optimal portfolio and related concepts, the reader is referred to [27], [6] and references therein.

We consider a financial market (S^0, S^1) consisting of the savings account S^0 and one risky asset S^1 with dynamics

$$dS_t^0 = S_t^0 r_t(0) dt, \tag{3.1}$$

$$\begin{aligned} dS_t^1 = S_{t-}^1 \bigg(& a_t dt + b_t dW_t + \int_B c_t(x)(\mu(dt,dx) - F(dx)dt) \\ & + \int_{B^c} c_t(x) \mu(dt,dx) \bigg), \end{aligned} \tag{3.2}$$

where $r_t(0)$ denotes the short rate and the process c satisfies $c > -1$. Important quantities are the *market price of risk* processes $\theta : \Omega \times \mathbb{R}_+ \to L_2^0(\mathbb{R})$ and $\psi : \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}$. In our financial market, the market prices of risks are specified as the solution (θ, ψ) with $\psi < 1$ of the equation

$$\langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) = a_t + \int_{B^c} c_t(x)F(dx) - r_t(0),$$

which we rewrite as

$$a_t = r_t(0) + \langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) - \int_{B^c} c_t(x)F(dx).$$

Inserting this equation into (3.2), we obtain the dynamics of the risky asset

$$\begin{aligned} dS_t^1 = S_{t-}^1 \bigg[& \bigg(r_t(0) + \langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) \bigg) dt \\ & + b_t dW_t + \int_E c_t(x)(\mu(dt,dx) - F(dx)dt) \bigg]. \end{aligned} \tag{3.3}$$

Let $\delta = (\delta^0, \delta^1)$ be a self-financing strategy and denote by S^δ the corresponding portfolio

$$S_t^\delta = \delta_t^0 S_t^0 + \delta_t^1 S_t^1.$$

Incorporating (3.1) and (3.3), by the self-financing property we obtain the portfolio dynamics

$$\begin{aligned} dS_t^\delta &= \delta_t^0 dS_t^0 + \delta_t^1 dS_t^1 \\ &= S_t^\delta r_t(0)dt + \delta_t^1 S_{t-}^1 \bigg[\bigg(\langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) \bigg) dt \\ &\quad + b_t dW_t + \int_E c_t(x)(\mu(dt,dx) - F(dx)dt) \bigg]. \end{aligned}$$

It will be convenient to introduce the fraction π_δ of S^δ that is invested in the risky asset S^1. This fraction is given by

$$\pi_{\delta,t} := \delta_t^1 \frac{S_t^1}{S_t^\delta}.$$

Then we obtain the portfolio dynamics

$$
\begin{aligned}
dS_t^\delta = S_{t-}^\delta \Bigg[r_t(0)dt + \pi_{\delta,t} \bigg(\Big(\langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) \Big) dt \\
+ b_t dW_t + \int_E c_t(x)(\mu(dt,dx) - F(dx)dt) \bigg) \Bigg].
\end{aligned} \tag{3.4}
$$

The growth optimal portfolio S^{δ_*} is characterized as the portfolio maximizing the expected log-utility $\mathbb{E}[\ln(S_t^\delta)]$ among all self-financing portfolios δ. In order to determine the dynamics of the growth optimal portfolio, we compute the dynamics of $\ln(S_t^\delta)$ for the portfolio dynamics (3.4). Applying Itô's formula we obtain

$$
\begin{aligned}
d\ln S_t^\delta = \Bigg[r_t(0) + \pi_{\delta,t} \bigg(\langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) \bigg) - \frac{1}{2}\pi_{\delta,t}^2 \|b_t\|_{L_2^0(\mathbb{R})}^2 \\
+ \int_E \big(\ln(1+\pi_{\delta,t}c_t(x)) - \pi_{\delta,t}c_t(x) \big) F(dx) \Bigg] dt \\
+ \pi_{\delta,t} b_t dW_t + \int_E \ln(1+\pi_{\delta,t}c_t(x))(\mu(dt,dx) - F(dx)dt).
\end{aligned}
$$

Since the growth optimal portfolio S^{δ_*} is characterized as the portfolio maximizing the expected log-utility, the fraction π_{δ^*} corresponding to the growth optimal portfolio should be chosen such that it maximizes the drift term. Differentiating the drift term with respect to π_{δ^*} and putting it equal to zero we obtain the equation

$$
\begin{aligned}
\langle b_t, \theta_t \rangle_{L_2^0(\mathbb{R})} + \int_E c_t(x)\psi_t(x)F(dx) - \pi_{\delta^*,t} \|b_t\|_{L_2^0(\mathbb{R})}^2 \\
+ \int_E \bigg(\frac{c_t(x)}{1+\pi_{\delta^*,t}c_t(x)} - c_t(x) \bigg) F(dx) = 0.
\end{aligned}
$$

The solution (θ, ψ) to this equation is given by

$$
\theta_t = \pi_{\delta^*,t} b_t \quad \text{and} \quad \psi_t(x) = \frac{\pi_{\delta^*,t}c_t(x)}{1+\pi_{\delta^*,t}c_t(x)}.
$$

Noting that

$$
\pi_{\delta^*,t}c_t(x) = \frac{\psi(x)}{1-\psi_t(x)},
$$

by (3.4) we obtain the dynamics of the growth optimal portfolio

$$
\begin{aligned}
dS_t^{\delta_*} = S_{t-}^{\delta_*} \Bigg[\bigg(r_t(0) + \|\theta_t\|_{L_2^0(\mathbb{R})}^2 + \int_E \frac{\psi_t(x)^2}{1-\psi_t(x)} F(dx) \bigg) dt \\
+ \theta_t dW_t + \int_E \frac{\psi_t(x)}{1-\psi_t(x)} (\mu(dt,dx) - F(dx)dt) \Bigg],
\end{aligned} \tag{3.5}
$$

which are just the dynamics (1.3) stated in the introduction. This generalizes the dynamics that have been derived in [27, 6]. Note that the dynamics (3.5) do not depend on the choice of the drift a of the risky asset S^1, which appears in (3.2).

4. The drift condition

In this section, we derive the drift condition (1.4) for arbitrage free real-world forward rate dynamics of the type (1.1). In the framework of the Benchmark Approach from [27], the absence of arbitrage is only subject to the existence of the growth optimal portfolio, which has the feature that benchmarked portfolios are nonnegative local martingales, and hence supermartingales. Using the dynamics

(3.5) of the growth optimal portfolio, we will first compute the dynamics of the bond prices

$$P_t(T) = \hat{P}_t(T) S_t^{\delta_*},$$

and then we will derive the dynamics of the forward rates $f_t(T) = -\frac{\partial}{\partial T} \ln P_t(T)$, which will yield the drift condition (1.4).

Since the benchmarked bond prices $\hat{P}(T) = \frac{P(T)}{S^{\delta_*}}$ are nonnegative local martingales, we assume that they have the dynamics

$$d\hat{P}_t(T) = -\hat{P}_{t-}(T)\bigg[\hat{\Sigma}_t(T)dW_t + \int_E \hat{\Lambda}_t(T,x)(\mu(dt,dx) - F(dx)dt)\bigg]$$

with $\hat{\Lambda} < 1$. Introducing the new process

$$\hat{\Gamma}_t(T,x) := \ln(1 - \hat{\Lambda}_t(T,x)),$$

we can express $\hat{\Lambda}$ as

$$\hat{\Lambda}_t(T,x) = 1 - e^{\hat{\Gamma}_t(T,x)},$$

and obtain the dynamics of the benchmarked bond prices

$$d\hat{P}_t(T) = -\hat{P}_{t-}(T)\bigg[\hat{\Sigma}_t(T)dW_t + \int_E \big(1 - e^{\hat{\Gamma}_t(T,x)}\big)(\mu(dt,dx) - F(dx)dt)\bigg]. \tag{4.1}$$

Furthermore, introducing the transform

$$\phi_t(x) := -\ln(1 - \psi_t(x))$$

of the reference market prices of risk with respect to the Poisson random measure μ, we get the identities

$$\begin{aligned} \psi_t(x) &= 1 - e^{-\phi_t(x)}, \\ \frac{\psi_t(x)}{1 - \psi_t(x)} &= \frac{1 - e^{-\phi_t(x)}}{e^{-\phi_t(x)}} = e^{\phi_t(x)} - 1, \\ \frac{\psi_t(x)^2}{1 - \psi_t(x)} &= \big(e^{\phi_t(x)} - 1\big)\big(1 - e^{-\phi_t(x)}\big). \end{aligned}$$

Incorporating these identities into (3.5), we obtain the dynamics of the growth optimal portfolio

$$\begin{aligned} dS_t^{\delta_*} = S_{t-}^{\delta_*}\bigg[&\bigg(r_t(0) + \|\theta_t\|_{L_2^0(\mathbb{R})}^2 + \int_E \big(e^{\phi_t(x)} - 1\big)\big(1 - e^{-\phi_t(x)}\big)F(dx)\bigg)dt \\ &+ \theta_t dW_t + \int_E \big(e^{\phi_t(x)} - 1\big)(\mu(dt,dx) - F(dx)dt)\bigg]. \end{aligned} \tag{4.2}$$

Because of the dynamics (4.1) and (4.2), the covariation of $\hat{P}(T)$ and S^{δ_*} is given by

$$\begin{aligned} d[\hat{P}(T), S^{\delta_*}]_t = -P_{t-}(T)\bigg[&\langle \hat{\Sigma}_t(T), \theta_t \rangle_{L_2^0(\mathbb{R})} dt \\ &+ \int_E \big(1 - e^{\hat{\Gamma}_t(T,x)}\big)\big(e^{\phi_t(x)} - 1\big)\mu(dt,dx)\bigg]. \end{aligned}$$

Therefore, by the definition of we the covariation (see [22, Def. I.4.45]) obtain

$$\begin{aligned}
dP_t(T) = d(\hat{P}_t(T)S_t^{\delta_*}) &= \hat{P}_{t-}(T)dS_t^{\delta_*} + S_{t-}^{\delta_*}d\hat{P}_t(T) + d[\hat{P}(T), S^{\delta_*}]_t \\
&= P_{t-}(T)\Bigg[\bigg(r_t(0) + \|\theta_t\|^2_{L_2^0(\mathbb{R})} + \int_E \big(e^{\phi_t(x)} - 1\big)\big(1 - e^{-\phi_t(x)}\big)F(dx)\bigg)dt \\
&\qquad + \theta_t dW_t + \int_E \big(e^{\phi_t(x)} - 1\big)(\mu(dt,dx) - F(dx)dt) \\
&\qquad - \hat{\Sigma}_t(T)dW_t - \int_E \big(1 - e^{\hat{\Gamma}_t(T,x)}\big)(\mu(dt,dx) - F(dx)dt) \\
&\qquad - \langle \hat{\Sigma}_t(T), \theta_t \rangle_{L_2^0(\mathbb{R})}dt - \int_E \big(1 - e^{\hat{\Gamma}_t(T,x)}\big)\big(e^{\phi_t(x)} - 1\big)F(dx)dt \\
&\qquad - \int_E \big(1 - e^{\hat{\Gamma}_t(T,x)}\big)\big(e^{\phi_t(x)} - 1\big)(\mu(dt,dx) - F(dx)dt)\Bigg].
\end{aligned}$$

These dynamics simplify to

$$\begin{aligned}
dP_t(T) = P_{t-}(T)\Bigg[\bigg(&r_t(0) + \langle \theta_t, \theta_t - \hat{\Sigma}_t(T)\rangle_{L_2^0(\mathbb{R})} \\
&+ \int_E \big(e^{\hat{\Gamma}_t(T,x)} - e^{-\phi_t(x)}\big)\big(e^{\phi_t(x)} - 1\big)F(dx)\bigg)dt \\
&+ (\theta_t - \hat{\Sigma}_t(T))dW_t \\
&+ \int_E \big(e^{\hat{\Gamma}_t(T,x) + \phi_t(x)} - 1\big)(\mu(dt,dx) - F(dx)dt)\Bigg].
\end{aligned}$$

Hence, due to the boundary condition $P_T(T) = 1$ we obtain

$$\theta_t = \hat{\Sigma}_t(t) \quad \text{and} \quad \phi_t(x) = -\hat{\Gamma}_t(t,x).$$

Defining the new processes

$$\begin{aligned}
\Sigma_t(T) &:= \hat{\Sigma}_t(T) - \theta_t, \\
\Gamma_t(T,x) &:= \hat{\Gamma}_t(T,x) + \phi_t(x),
\end{aligned}$$

we thus have the boundary conditions

$$\Sigma_t(t) = 0 \quad \text{and} \quad \Gamma_t(t,x) = 0. \tag{4.3}$$

Furthermore, we can express the dynamics of the bond prices as

$$\begin{aligned}
dP_t(T) = P_{t-}(T)\Bigg[\bigg(&r_t(0) - \langle \theta_t, \Sigma_t(T)\rangle_{L_2^0(\mathbb{R})} \\
&+ \int_E \big(e^{\Gamma_t(T,x) - \phi_t(x)} - e^{-\phi_t(x)}\big)\big(e^{\phi_t(x)} - 1\big)F(dx)\bigg)dt \\
&- \Sigma_t(T)dW_t + \int_E \big(e^{\Gamma_t(T,x)} - 1\big)(\mu(dt,dx) - F(dx)dt)\Bigg].
\end{aligned}$$

Applying Itô's formula, we obtain

$$\begin{aligned}
d\ln P_t(T) = \bigg(&r_t(0) - \langle \theta_t, \Sigma_t(T)\rangle_{L_2^0(\mathbb{R})} \\
&+ \int_E \big(e^{\Gamma_t(T,x) - \phi_t(x)} - e^{-\phi_t(x)}\big)\big(e^{\phi_t(x)} - 1\big)F(dx) \\
&- \frac{1}{2}\|\Sigma_t(T)\|^2_{L_2^0(\mathbb{R})} - \int_E \big(e^{\Gamma_t(T,x)} - \Gamma_t(T,x) - 1\big)F(dx)\bigg)dt \\
&- \Sigma_t(T)dW_t + \int_E \Gamma_t(T,x)(\mu(dt,dx) - F(dx)dt).
\end{aligned}$$

We can express these dynamics as

$$
\begin{aligned}
d \ln P_t(T) = \bigg(& r_t(0) - \langle \theta_t, \Sigma_t(T) \rangle_{L_2^0(\mathbb{R})} - \frac{1}{2} \| \Sigma_t(T) \|_{L_2^0(\mathbb{R})}^2 \\
& - \int_E \big(e^{\Gamma_t(T,x) - \phi_t(x)} - \Gamma_t(T,x) - e^{-\phi_t(x)} \big) F(dx) \bigg) dt \\
& - \Sigma_t(T) dW_t + \int_E \Gamma_t(T,x)(\mu(dt,dx) - F(dx)dt)
\end{aligned}
$$

Defining the new processes

$$
\sigma_t(T) := \frac{\partial}{\partial T} \Sigma_t(T) \quad \text{and} \quad \gamma_t(T) := - \frac{\partial}{\partial T} \Gamma_t(T,x),
$$

by the boundary conditions (4.3) we have

$$
\Sigma_t(T) = \int_t^T \sigma_t(s) ds \quad \text{and} \quad \Gamma_t(T,x) = - \int_t^T \gamma_t(s,x) ds.
$$

Therefore, the forward rates $f_t(T) = -\frac{\partial}{\partial T} \ln P_t(T)$ have the dynamics

$$
\begin{aligned}
df_t(T) = \bigg[& \Big\langle \sigma_t(T), \theta_t + \int_t^T \sigma_t(s) ds \Big\rangle_{L_2^0(\mathbb{R})} \\
& - \int_E \gamma_t(T,x) \bigg[\exp \bigg(- \phi_t(x) - \int_t^T \gamma_t(s,x) ds \bigg) - 1 \bigg] F(dx) \bigg] dt \\
& + \sigma_t(T) dW_t + \int_E \gamma_t(T,x)(\mu(dt,dx) - F(dx)dt),
\end{aligned}
$$

which we can write as

$$
\begin{aligned}
df_t(T) = \bigg[& \Big\langle \sigma_t(T), \theta_t + \int_t^T \sigma_t(s) ds \Big\rangle_{L_2^0(\mathbb{R})} \\
& - \int_B \gamma_t(T,x) \bigg[\exp \bigg(- \phi_t(x) - \int_t^T \gamma_t(s,x) ds \bigg) - 1 \bigg] F(dx) \\
& - \int_{B^c} \gamma_t(T,x) \exp \bigg(- \phi_t(x) - \int_t^T \gamma_t(s,x) ds \bigg) F(dx) \bigg] dt \\
& + \sigma_t(T) dW_t + \int_B \gamma_t(T,x)(\mu(dt,dx) - F(dx)dt) + \int_{B^c} \gamma_t(T,x) \mu(dt,dx).
\end{aligned}
$$

Consequently, we arrive at real-world forward rate dynamics of the type (1.1) with drift term being of the form (1.4), as stated in the introduction. This generalizes the dynamics that have been derived in [27, 4]. In particular, with $\theta \equiv \psi \equiv 0$ it generalizes the various drift conditions that have been derived for risk-neutral HJM models under an assumed risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$

5. Existence and uniqueness of mild solutions to the HJMM equation

In this section, we establish existence and uniqueness of mild solutions to the HJMM equation (1.2).

We fix an arbitrary constant $\beta > 0$ and denote by H_β the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$
\|h\|_\beta := \bigg(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \bigg)^{1/2} < \infty.
$$

Spaces of this kind have been introduced in [15]. We also refer to [30, Sec. 4], where some relevant properties have been summarized. The space H_β is a separable Hilbert space and the shift semigroup $(S_t)_{t\geq 0}$ given by $S_t h = h(t+\bullet)$ is a C_0-semigroup on H_β with infinitesimal generator $d/d\xi$ on the domain

$$\mathcal{D}(d/d\xi) = \{h \in H_\beta : (d/d\xi)h \in H_\beta\}.$$

Let H_β^0 be the subspace

$$H_\beta^0 := \Big\{h \in H_\beta : \lim_{x\to\infty} h(x) = 0\Big\}.$$

We fix arbitrary constants $0 < \beta < \beta'$ and set

$$C_\beta := 1 + \frac{1}{\sqrt{\beta}}. \tag{5.1}$$

Let $\sigma : \Omega \times H_\beta \to L_2^0(H_{\beta'}^0)$, $\theta : \Omega \times H_\beta \to L_2^0(\mathbb{R})$ and $\gamma : \Omega \times H_\beta \times E \to H_{\beta'}^0$, $\phi : \Omega \times H_\beta \times E \to \mathbb{R}$ be measurable mappings. Note that the volatilities σ, γ and the market prices of risk θ, ϕ may depend on $\omega \in \Omega$ and on the current state of the forward curve. In particular, they could also depend on some underlying factor process. We define the sequences $(\sigma^j)_{j\in\mathbb{N}}$ and $(\theta^j)_{j\in\mathbb{N}}$ as

$$\sigma^j : \Omega \times H_\beta \to H_{\beta'}^0, \quad \sigma^j(h) := \sigma(h)\sqrt{\lambda_j} e_j,$$
$$\theta^j : \Omega \times H_\beta \to \mathbb{R}, \quad \theta^j(h) := \theta(h)\sqrt{\lambda_j} e_j.$$

The following standing assumptions prevail throughout this section:

- There exist sequences $(L_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $(\kappa^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$ with
$$\sum_{j\in\mathbb{N}} (\kappa^j)^2 < \infty$$
such that $\mathbb{P}$–almost surely for all $j, n \in \mathbb{N}$ and all $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have
$$\|\sigma^j(h_1) - \sigma^j(h_2)\|_{\beta'} \leq L_n \kappa^j \|h_1 - h_2\|_\beta, \tag{5.2}$$
$$|\theta^j(h_1) - \theta^j(h_2)| \leq L_n \kappa^j \|h_1 - h_2\|_\beta, \tag{5.3}$$
and for all $j \in \mathbb{N}$ and $h \in H_\beta$ we have
$$\|\sigma^j(h)\|_{\beta'} \leq \kappa^j \sqrt{1 + \|h\|_\beta}, \tag{5.4}$$
$$|\theta^j(h)| \leq \kappa^j \sqrt{1 + \|h\|_\beta}. \tag{5.5}$$
- There exist a measurable mapping $\rho : B \to \mathbb{R}_+$ with
$$\int_B \rho(x)^2 F(dx) < \infty \tag{5.6}$$
and constants $M_\gamma, M_\phi \geq 0$ with $M_\gamma \leq 1$ and
$$\sqrt{\frac{1}{\beta'(\beta - \beta')}} M_\gamma + M_\phi \leq 1$$
such that $\mathbb{P}$–almost surely for all $x \in B$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have
$$\|\gamma(h_1, x) - \gamma(h_2, x)\|_{\beta'} \leq L_n \rho(x) \|h_1 - h_2\|_\beta, \tag{5.7}$$
$$|\phi(h_1, x) - \phi(h_2, x)| \leq L_n \rho(x) \|h_1 - h_2\|_\beta, \tag{5.8}$$
and for all $x \in B$ and $h \in H_\beta$ we have
$$\|\gamma(h, x)\|_{\beta'} \leq M_\gamma \ln^*\big(\rho(x)\sqrt{1 + \|h\|_\beta}\big), \tag{5.9}$$
$$|\phi(h, x)| \leq M_\phi \ln^*\big(\rho(x)\sqrt{1 + \|h\|_\beta}\big), \tag{5.10}$$

where $\ln^* : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the inverse of the strictly increasing function

$$\exp^* : \mathbb{R}_+ \to \mathbb{R}_+, \quad x \mapsto x \cdot \exp(C_\beta x).$$

Recall that the constant $C_\beta > 0$ was defined in (5.1).

- There exist measurable mappings $\tau : B^c \to \mathbb{R}_+$, $\zeta : B^c \to [1,\infty)$ with

$$\int_{B^c} (1 \vee \tau(x))\zeta(x)(1 \vee \ln^{**}(\zeta(x)))F(dx) < \infty \tag{5.11}$$

such that $\mathbb{P}$–almost surely for all $x \in B^c$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have

$$\|\gamma(h_1,x) - \gamma(h_2,x)\|_{\beta'} \leq L_n \tau(x) \|h_1 - h_2\|_\beta, \tag{5.12}$$

$$|\phi(h_1,x) - \phi(h_2,x)| \leq L_n \tau(x) \|h_1 - h_2\|_\beta, \tag{5.13}$$

and for all $x \in B^c$ and $h \in H_\beta$ we have

$$\|\gamma(h,x)\|_{\beta'} \leq M_\gamma \ln^{**}(\zeta(x)), \tag{5.14}$$

$$|\phi(h,x)| \leq M_\phi \ln^{**}(\zeta(x)), \tag{5.15}$$

where $\ln^{**} : [1,\infty) \to \mathbb{R}_+$ denotes the inverse of the strictly increasing function

$$\exp^{**} : \mathbb{R}_+ \to [1,\infty), \quad x \mapsto (1+x) \cdot \exp(C_\beta x).$$

Recall that the constant $C_\beta > 0$ was defined in (5.1).

- In view of (1.4), we assume that the drift $\alpha : \Omega \times H_\beta \to H_\beta$ is given by

$$\begin{aligned} \alpha(h) = &\sum_{j \in \mathbb{N}} \sigma^j(h) \bigg(\theta^j(h) + \int_0^\bullet \sigma^j(h)(\eta) d\eta \bigg) \\ &- \int_B \gamma(h,x) \bigg[\exp \bigg(-\phi(h,x) - \int_0^\bullet \gamma(h,x)(\eta) d\eta \bigg) - 1 \bigg] F(dx) \\ &- \int_{B^c} \gamma(h,x) \exp \bigg(-\phi(h,x) - \int_0^\bullet \gamma(h,x)(\eta) d\eta \bigg) F(dx). \end{aligned} \tag{5.16}$$

Then the HJMM equation (1.2) is a SPDE on the state space H_β. In order to state our main result of this section, we recall that *existence of mild solutions* to (1.2) holds, if for each $\mathcal{F}_0$-measurable random variable $h_0 : \Omega \to H_\beta$ there exists a H_β-valued, càdlàg, adapted process r such that $\mathbb{P}$–almost surely

$$\begin{aligned} r_t = S_t h_0 &+ \int_0^t S_{t-s} \alpha(r_s) ds + \int_0^t S_{t-s} \sigma(r_s) dW_s \\ &+ \int_0^t \int_B S_{t-s} \gamma(r_{s-}, x)(\mu(ds,dx) - F(dx)ds) \\ &+ \int_0^t \int_{B^c} S_{t-s} \gamma(r_{s-}, x) \mu(ds,dx), \quad t \geq 0. \end{aligned}$$

Furthermore, we say that *existence of mild solutions* to (1.2) holds, if for two mild solutions r and r' with the same initial condition h_0 we have $r = r'$ up to indistinguishability.

5.1. **Theorem.** *Under the previous conditions, existence and uniqueness of mild solutions to the HJMM equation (1.2) holds.*

We will provide the proof of Theorem 5.1 in Appendix A.

5.2. **Remark.** *Note that we have relaxed the regularity conditions on the volatilities σ and γ, which have been assumed to be Lipschitz and bounded in* [18]. *Indeed, conditions (5.4), (5.5) show that the growth of the mappings σ and θ is of order*

$\sqrt{1+\|h\|_\beta}$, *and conditions (5.9), (5.10) show that for fixed* $x \in B$ *the growth of the mappings* $\gamma(\bullet, x)$ *and* $\phi(\bullet, x)$ *is of order* $\ln^*(\rho(x)\sqrt{1+\|h\|_\beta})$.

6. Positivity preserving models

In applications, we are often interested in interest rate models producing positive forward curves. In this section, we characterize those real-world forward rate dynamics of the form (1.2) which are positivity preserving.

We start with a general result. Let $\alpha : \Omega \times H_\beta \to H_\beta$, $\sigma : \Omega \times H_\beta \to L_2^0(H_\beta^0)$ and $\gamma : \Omega \times H_\beta \times E \to H_\beta^0$ be measurable mappings. We denote by

$$P := \{h \in H_\beta : h \geq 0\}$$

the closed, convex cone of nonnegative forward curves.

6.1. **Definition.** *The HJMM equation (1.2) is called* positivity preserving, *if for each* $\mathcal{F}_0$*-measurable random variable* $h_0 : \Omega \to H_\beta$ *with* $\mathbb{P}(h_0 \in P) = 1$ *there exists a mild solution* r *to (1.2) with* $\mathbb{P}(r_t \in P) = 1$ *for all* $t \geq 0$.

Extending the arguments from [18], we obtain the following general result concerning positivity preserving models.

6.2. **Theorem.** *Suppose that* $(\alpha, \sigma, \gamma|_B)$ *are locally Lipschitz and satisfy the linear growth condition. Furthermore, suppose that* $\mathbb{P}$*–almost surely we have* $\sigma \in C^2(H_\beta; L_2^0(H_\beta))$ *and that the mapping*

$$\Omega \times H_\beta \to H_\beta, \quad h \mapsto \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

is locally Lipschitz. Then, the HJMM equation (1.2) is positivity preserving if and only if we have $\mathbb{P}$*–almost surely*

$$\int_B \gamma(h,x)(\xi) F(dx) < \infty \quad \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0 \tag{6.1}$$

$$\alpha(h)(\xi) - \int_B \gamma(h,x)(\xi) F(dx) \geq 0 \quad \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0 \tag{6.2}$$

$$\sigma^j(h)(\xi) = 0 \quad \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \quad \text{for all } j \in \mathbb{N} \tag{6.3}$$

$$h + \gamma(h,x) \in P \quad \text{for all } h \in P, \quad \text{for } F\text{–almost all } x \in E. \tag{6.4}$$

Now, let $\theta : \Omega \times H_\beta \to L_2^0(\mathbb{R})$ and $\phi : \Omega \times H_\beta \times E \to \mathbb{R}$ be the market prices of risk. We suppose that the regularity conditions for σ, θ and γ, ϕ from Section 5 are fulfilled and that the drift α is given by (5.16). Furthermore, suppose that $\mathbb{P}$–almost surely we have $\sigma \in C^2(H_\beta; L_2^0(H_{\beta'}^0))$ and that the mapping

$$\Omega \times H_\beta \to H_{\beta'}^0, \quad h \mapsto \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)$$

is locally Lipschitz.

6.3. **Proposition.** *The following statements are equivalent:*

(1) *We have (6.1)–(6.4).*
(2) *We have (6.3), (6.4) and*

$$\begin{aligned} \gamma(h,x)(\xi) = 0 \quad & \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \\ & \text{for } F\text{–almost all } x \in E. \end{aligned} \tag{6.5}$$

Proof. Suppose that conditions (6.3), (6.4) are satisfied. We will prove the equivalence (6.1), (6.2) ⇔ (6.5). By (6.4) we have

$$\begin{aligned} \gamma(h,x)(\xi) \geq 0 \quad & \text{for all } (h,\xi) \in P \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \\ & \text{for } F\text{–almost all } x \in E. \end{aligned} \tag{6.6}$$

Because of (6.3) and the structure (5.16) of the drift term α, the conditions (6.1), (6.2) are satisfied if and only if

$$-\int_E \gamma(h,x)(\xi) \exp\bigg(-\phi(h,x) - \int_0^\xi \gamma(h,x)(\eta)d\eta \bigg) F(dx) \geq 0$$
$$\text{for all } (h,\xi) \in H_\beta \times \mathbb{R}_+ \text{ with } h(\xi) = 0, \quad \text{for } F\text{–almost all } x \in E.$$

By (6.6), we deduce that (6.1), (6.2) $\Leftrightarrow$ (6.5). $\square$

6.4. **Theorem.** *The HJMM equation (1.2) is positivity preserving if and only if we have (6.3), (6.4), (6.5).*

Proof. This is a direct consequence of Theorem 6.2 and Proposition 6.3. $\square$

Consequently, we have generalized the positivity result from [18] for interest rate models with real-world forward rate dynamics. We point out that the market prices of risk θ, ϕ do not affect the positivity preserving property.

7. Lévy process driven interest rate models with real-world dynamics

In this section, we illustrate our previous results by focusing on real-world interest rate models driven by Lévy processes. Let $X^1, \dots, X^d$ be independent Lévy processes with Lévy-Itô decompositions

$$X^j = W^j + Y^j + Z^j, \quad j = 1, \dots, d,$$

where the processes Y and Z are given by

$$Y_t^j = \int_{\{|x| \leq 1\}} x(\mu^{X^j}(ds,dx) - F_j(dx)ds),$$
$$Z_t^j = \int_{\{|x| > 1\}} x \mu^{X^j}(ds,dx).$$

Here, W^j denotes a standard Wiener process, μ^{X^j} denotes the random measure associated to the jumps of X^j, and F_j denotes its Lévy measure. We suppose that the dynamics of the forward rates are of the form

$$df_t(T) = \alpha_t(T)dt + \sum_{j=1}^d \big(\sigma_t^j(T)dW_t^j + \delta_t^j(T)dY_t^j + \eta_t^j(T)dZ_t^j \big), \quad t \in [0,T].$$

Note that these dynamics are of the type (1.1), where the state space of the Wiener process W is $U = \mathbb{R}^d$, the mark space of the Poisson random measure μ is $E = \mathbb{R}^d$, the set B is given by $B = \{\|x\| \leq 1\}$, and the volatility γ is

$$\gamma_t(T,x) = \sum_{j=1}^d \delta_t^j(T)x_j \quad \text{for } x \in B,$$
$$\gamma_t(T,x) = \sum_{j=1}^d \eta_t^j(T)x_j \quad \text{for } x \in B^c.$$

Then, the dynamics (1.3) of the growth optimal portfolio $S_t^{\delta_*}$ become

$$\begin{aligned} dS_t^{\delta_*} &= S_{t-}^{\delta_*}\bigg[r_t(0)dt + \sum_{j=1}^d \theta_t^j(\theta_t^j dt + dW_t^j) \\ &+ \sum_{j=1}^d \int_{\{|x|\leq 1\}} \frac{\psi_t^j(x)}{1-\psi_t^j(x)}\big(\psi_t^j(x)F_j(dx)dt + (\mu^{X^j}(dt,dx) - F_j(dx)dt)\big) \\ &+ \sum_{j=1}^d \int_{\{|x|>1\}} \frac{\chi_t^j(x)}{1-\chi_t^j(x)}\big(\chi_t^j(x)F_j(dx)dt + (\mu^{X^j}(dt,dx) - F_j(dx)dt)\big)\bigg], \end{aligned}$$

where $\theta^1,\dots,\theta^d$ denote the reference market prices of risk with respect to the Wiener processes $W^1,\dots,W^d$, where $\psi^1,\dots,\psi^d$ denote the reference market prices of risk with respect to the pure jump parts of $Y^1,\dots,Y^d$, and where $\chi^1,\dots,\chi^d$ denote the reference market prices of risk with respect to the pure jump parts of $Z^1,\dots,Z^d$. We suppose that the latter are of the form

$$\psi_t^j(x) = 1 - \exp(-\varphi_t^j x) \quad \text{and} \quad \chi_t^j = 1 - \exp(-\vartheta_t^j x)$$

for some mappings $\varphi^1,\dots,\varphi^d$ and $\vartheta^1,\dots,\vartheta^d$. In the case of finitely many square-integrable Lévy processes, such dynamics of the growth optimal portfolio have been used in [28]. In the present situation, the drift condition (1.4) becomes

$$\begin{aligned} \alpha_t(T) &= \sum_{j=1}^d \sigma_t^j(T)\bigg(\theta_t^j + \int_t^T \sigma_t^j(s)ds\bigg) \\ &- \sum_{j=1}^d \delta_t^j(T) \int_{\{|x|\leq 1\}} x\bigg[\exp\bigg(-\bigg(\varphi_t^j + \int_t^T \delta_t^j(s)ds\bigg)x\bigg) - 1\bigg]F_j(dx) \\ &- \sum_{j=1}^d \eta_t^j(T) \int_{\{|x|>1\}} x\exp\bigg(-\bigg(\vartheta_t^j + \int_t^T \eta_t^j(s)ds\bigg)x\bigg)F_j(dx). \end{aligned} \tag{7.1}$$

Now, let us consider the corresponding Lévy process driven HJMM equation

$$\left\{ \begin{array}{rcl} dr_t &=& \big(\frac{d}{dx}r_t + \alpha(r_t)\big)dt + \sum_{j=1}^d \big(\sigma^j(r_t)dW_t^j + \delta^j(r_{t-})dY_t^j + \eta^j(r_{t-})dZ_t^j\big) \\ r_0 &=& h_0. \end{array} \right. \tag{7.2}$$

As in Section 5, we fix constants $0 < \beta < \beta'$ and choose the state space H_β. We suppose that for each $j = 1,\dots,n$ there are constants $N_j, \epsilon_j > 0$ such that

$$\int_{\{|x|>1\}} e^{zx}F_j(dx) < \infty \quad \text{for all } z \in I_j. \tag{7.3}$$

Here I_j denotes the interval $I_j = [-(1+\epsilon_j)N_jC_\beta, (1+\epsilon_j)N_jC_\beta]$, where we recall that the constant $C_\beta > 0$ was defined in (5.1). We introduce the functions

$$\begin{aligned} \Phi_j &: \mathbb{R} \to \mathbb{R}, \quad \Phi_j(z) := \int_{\{|x|\leq 1\}} \big(e^{zx} - 1 - zx\big)F_j(dx), \\ \Psi_j &: I_j \to \mathbb{R}, \quad \Psi_j(z) := \int_{\{|x|>1\}} \big(e^{zx} - 1\big)F_j(dx). \end{aligned}$$

Let $\sigma^j : \Omega\times H_\beta \to H_{\beta'}^0$, $\delta^j : \Omega\times H_\beta \to H_{\beta'}^0$, $\eta^j : \Omega\times H_\beta \to H_{\beta'}^0$ and $\theta^j : \Omega\times H_\beta \to \mathbb{R}$, $\varphi^j : \Omega \times H_\beta \to \mathbb{R}$, $\vartheta^j : \Omega \times H_\beta \to \mathbb{R}$ for $j = 1,\dots,d$ be measurable mappings. We suppose that the following conditions are satisfied:

- The mappings σ^j, δ^j, η^j and θ^j, φ^j, ϑ^j for $j = 1,\dots,d$ are locally Lipschitz.

- There exists a constant $K \geq 0$ such that $\mathbb{P}$–almost surely for all $h \in H_\beta$ we have

$$\|\sigma^j(h)\|_{\beta'} \leq K\sqrt{1 + \|h\|_\beta}, \quad j = 1, \ldots, n, \tag{7.4}$$

$$|\theta^j(h)| \leq K\sqrt{1 + \|h\|_\beta}, \quad j = 1, \ldots, n. \tag{7.5}$$

- There exist constants $M_\delta^1, \ldots, M_\delta^d > 0$ and $M_\varphi^1, \ldots, M_\varphi^d > 0$ such that $M_\delta \leq 1$ and

$$\sqrt{\frac{1}{\beta'(\beta - \beta')}} M_\delta + M_\varphi \leq 1 \tag{7.6}$$

where $M_\delta := M_\delta^1 + \ldots + M_\delta^d$ and $M_\varphi := M_\varphi^1 + \ldots + M_\varphi^d$, and there exists a constant $C \geq 0$ such that $\mathbb{P}$–almost surely for all $n \in \mathbb{N}$ and $h \in H_\beta$ we have

$$\|\delta^j(h)\|_{\beta'} \leq M_\delta^j \ln^* \big(C\sqrt{1 + \|h\|_\beta}\big), \tag{7.7}$$

$$|\varphi^j(h)| \leq M_\varphi^j \ln^* \big(C\sqrt{1 + \|h\|_\beta}\big). \tag{7.8}$$

- We have $\mathbb{P}$–almost surely for all $h \in H_\beta$ the estimates

$$\|\eta^j(h)\|_{\beta'} \leq M_\delta N^j, \quad j = 1, \ldots, n, \tag{7.9}$$

$$|\vartheta^j(h)| \leq M_\varphi N^j, \quad j = 1, \ldots, n. \tag{7.10}$$

- In view of (7.1), we assume that the drift $\alpha : \Omega \times H_\beta \to H_\beta$ is given by

$$\begin{aligned}
\alpha(h) = &\sum_{j=1}^n \sigma^j(h)\bigg(\theta^j(h) + \int_0^\bullet \sigma^j(h)(\xi)d\xi\bigg) \\
&- \sum_{j=1}^n \delta^j(h)\Phi_j'\bigg(-\varphi^j - \int_0^\bullet \delta^j(h)(\xi)d\xi\bigg) \\
&- \sum_{j=1}^n \eta^j(h)\Psi_j'\bigg(-\vartheta^j - \int_0^\bullet \eta^j(h)(\xi)d\xi\bigg).
\end{aligned}$$

Before we state our main result of this section, we prepare an auxiliary result.

7.1. **Lemma.** *For all $m \in \mathbb{N}_0$ and $N, \epsilon > 0$ there exists a constant $C > 0$ such that*

$$x^m e^{Nx} \leq C e^{(1+\epsilon)Nx} \quad \textit{for each } x \in \mathbb{R}_+.$$

Proof. Setting $C := \frac{m!}{(\epsilon N)^m}$, for all $x \in \mathbb{R}_+$ we have

$$x^m e^{Nx} = \frac{m!}{(\epsilon N)^m}\frac{(\epsilon N x)^m}{m!}e^{Nx} = C\frac{(\epsilon N x)^m}{m!}e^{Nx} \leq C e^{\epsilon N x}e^{Nx} = C e^{(1+\epsilon)Nx},$$

finishing the proof. □

Here is our existence and uniqueness result regarding the Lévy process driven HJMM equation with real-world forward rate dynamics.

7.2. **Theorem.** *Under the previous conditions, existence and uniqueness of mild solutions to the Lévy process driven HJMM equation (7.2) holds.*

Proof. We will show that the conditions from Section 5 are fulfilled. Conditions (5.2)–(5.5) are satisfied by the local Lipschitz continuity of the σ^j, θ^j and by (7.4), (7.5). Note that the mappings γ and ϕ are given by

$$\gamma(h,x) = \sum_{j=1}^d \delta^j(h)x_j \quad \text{and} \quad \phi(h,x) = \sum_{j=1}^d \varphi^j(h)x_j \quad \text{for } x \in B,$$
$$\gamma(h,x) = \sum_{j=1}^d \eta^j(h)x_j \quad \text{and} \quad \phi(h,x) = \sum_{j=1}^d \vartheta^j(h)x_j \quad \text{for } x \in B^c.$$

We define the bounded, measurable mapping

$$\rho : B \to \mathbb{R}_+, \quad \rho(x) := \sum_{j=1}^d |x_j|.$$

Then, the integrability condition (5.6) is satisfied, because the measures F_j are Lévy measures. Moreover, conditions (5.7), (5.8) are fulfilled, because the mappings δ^j, φ^j are locally Lipschitz. By (7.7), for all $h \in H_\beta$, $x \in B$ and $j = 1, \dots, d$ we have $\mathbb{P}$–almost surely

$$\begin{aligned}
&\exp^*\left(\frac{\|\delta^j(h)\|_{\beta'}|x_j|}{M_\delta^j}\right) = \frac{\|\delta^j(h)\|_{\beta'}|x_j|}{M_\delta^j}\exp\left(C_\beta\frac{\|\delta^j(h)\|_{\beta'}|x_j|}{M_\delta^j}\right) \\
&\leq \frac{\|\delta^j(h)\|_{\beta'}|x_j|}{M_\delta^j}\exp\left(C_\beta\frac{\|\delta^j(h)\|_{\beta'}}{M_\delta^j}\right) = |x_j|\exp^*\left(\frac{\|\delta^j(h)\|_{\beta'}}{M_\delta^j}\right) \\
&\leq |x_j| C\sqrt{1+\|h\|_\beta} \leq C\rho(x)\sqrt{1+\|h\|_\beta},
\end{aligned}$$

and therefore

$$\|\delta^j(h)\|_{\beta'}|x_j| \leq M_\delta^j \ln^*\left(C\rho(x)\sqrt{1+\|h\|_\beta}\right),$$

which gives us

$$\|\gamma(h,x)\|_{\beta'} \leq \sum_{j=1}^d \|\delta^j(h)\|_{\beta'}|x_j| \leq M_\delta \ln^*\left(C\rho(x)\sqrt{1+\|h\|_\beta}\right).$$

Analogously, by using (7.8) we prove that for all $h \in H_\beta$ and $x \in B$ we have $\mathbb{P}$–almost surely

$$|\varphi(h,x)| \leq M_\varphi \ln^*\left(C\rho(x)\sqrt{1+\|h\|_\beta}\right),$$

and hence, conditions (5.9), (5.10) are fulfilled. Next, we define the measurable mappings $\tau : B^c \to \mathbb{R}_+$, $\zeta : B^c \to [1,\infty)$ as

$$\begin{aligned}
&\tau(x) := \sum_{j=1}^d |x_j| \quad \text{and} \\
&\zeta(x) := \exp^{**}\left(\sum_{j=1}^d N^j|x_j|\right) = \left(1+\sum_{j=1}^d N^j|x_j|\right)\exp\left(C_\beta\sum_{j=1}^d N^j|x_j|\right).
\end{aligned}$$

Then we have

$$\ln^{**}(\zeta(x)) = \sum_{j=1}^d N^j|x_j| \quad \text{for all } x \in B^c,$$

and hence, the integrability condition (5.11) is satisfied by virtue of (7.3) and Lemma 7.1. Moreover, conditions (5.12), (5.13) are fulfilled, because the mappings

η^j, ϑ^j are locally Lipschitz. Furthermore, by (7.9), for all $h \in H_\beta$ and $x \in B^c$ we have $\mathbb{P}$–almost surely

$$\|\gamma(h,x)\|_{\beta'} \leq \sum_{j=1}^d \|\eta_t^j(h)\|_{\beta'} |x_j| \leq M_\delta \sum_{j=1}^d N^j |x_j| = M_\delta \ln^{**}(\zeta(x)),$$

showing (5.14). Analogously, by using (7.8) we prove that (5.15) is fulfilled. Consequently, applying Theorem 5.1 provides the stated existence and uniqueness result. □

7.3. **Remark.** *Lévy process driven interest rate models as solutions of SPDEs have been studied in* [1, 16, 23, 26, 31] *under a risk-neutral measure* $\mathbb{Q} \sim \mathbb{P}$. *As conditions (7.4), (7.5) and (7.7), (7.8) show, the volatilities of the diffusion part and of the small jump part do not need to bounded, and hence, we have improved the conditions from* [16] *under real-world forward rate dynamics.*

Appendix A. Proof of Theorem 5.1

In this appendix, we will provide the proof of Theorem 5.1. For this purpose, we will first recall a general existence and uniqueness result (see Theorem A.3) and then apply it to the HJMM equation (1.2).

We fix constants $0 < \beta < \beta'$ and denote by H_β the space of forward curves from Section 5. Let $\alpha : \Omega \times H_\beta \to H_\beta$, $\sigma : \Omega \times H_\beta \to L_2^0(H_\beta)$ and $\gamma : \Omega \times H_\beta \times E \to H_\beta$ be measurable mappings.

A.1. **Definition.** *We say that the mappings* $(\alpha, \sigma, \gamma|_B)$ *are* locally Lipschitz*, if for each* $n \in \mathbb{N}$ *there is a constant* $L_n \in \mathbb{R}_+$ *such that* $\mathbb{P}$*–almost surely*

$$\|\alpha(h_1) - \alpha(h_2)\|_\beta \leq L_n \|h_1 - h_2\|_\beta,$$
$$\|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H_\beta)} \leq L_n \|h_1 - h_2\|_\beta,$$
$$\left(\int_B \|\gamma(h_1,x) - \gamma(h_2,x)\|^2 F(dx) \right)^{1/2} \leq L_n \|h_1 - h_2\|_\beta$$

for all $h_1, h_2 \in H_\beta$ *with* $\|h_1\|_\beta, \|h_2\|_\beta \leq n$.

A.2. **Definition.** *We say that the mappings* $(\alpha, \sigma, \gamma|_B)$ *satisfy the* linear growth condition*, if there exists a constant* $K \in \mathbb{R}_+$ *such that* $\mathbb{P}$*–almost surely*

$$\|\alpha(h)\|_\beta \leq K(1 + \|h\|_\beta),$$
$$\|\sigma(h)\|_{L_2^0(H_\beta)} \leq K(1 + \|h\|_\beta),$$
$$\left(\int_B \|\gamma(h,x)\|^2 F(dx) \right)^{1/2} \leq K(1 + \|h\|_\beta)$$

for all $h \in H_\beta$.

A.3. **Theorem.** *If* $(\alpha, \sigma, \gamma|_B)$ *are locally Lipschitz and satisfy the linear growth condition, then existence and uniqueness of mild solutions to (1.2) holds.*

Proof. According to [18, Thm. 2.1] there exists another separable Hilbert space $\mathcal{H}_\beta$, a C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathcal{H}_\beta$ and continuous linear operators $\ell \in L(H_\beta, \mathcal{H}_\beta)$, $\pi \in L(\mathcal{H}_\beta, H_\beta)$ such that $\pi U_t \ell = S_t$ for all $t \in \mathbb{R}_+$. Therefore, existence and uniqueness of mild solutions to (1.2) follows from [32, Thm. 4.5]. □

In order to prove Theorem 5.1, we will show that under the standing assumptions of Section 5 the mappings $(\alpha, \sigma, \gamma|_B)$ are locally Lipschitz and satisfy the linear growth condition, which allows an application of Theorem A.3. First, we provide some properties of the space H_β of forward curves. Proofs of the following auxiliary results (Lemmas A.4–A.8) can be found in [15], [18] and [30].

A.4. **Lemma.** *For all $h \in H_\beta$ we have*

$$\|h\|_\infty \leq C_\beta \|h\|_\beta,$$

where we recall that the constant $C_\beta > 0$ was defined in (5.1).

A.5. **Lemma.** *Each constant function belongs to H_β, and we have*

$$\|c\|_\beta = |c| \quad \text{for all } c \in \mathbb{R}.$$

A.6. **Lemma.** *We have $H_{\beta'} \subset H_\beta$ and*

$$\|h\|_\beta \leq \|h\|_{\beta'} \quad \text{for each } h \in H_{\beta'}.$$

A.7. **Lemma.** *For all $h_1, h_2 \in H_\beta$ we have $m(h_1, h_2) := h_1 h_2 \in H_\beta$, and the multiplication $m : H_\beta \times H_\beta \to H_\beta$ is a continuous bilinear operator.*

A.8. **Lemma.** *For each $h \in H_{\beta'}^0$ we have $\mathcal{I}h := \int_0^\bullet h(\eta) d\eta \in H_\beta$, and the integral operator $\mathcal{I} : H_{\beta'}^0 \to H_\beta$ is a continuous linear operator with operator norm*

$$\|\mathcal{I}\| \leq \sqrt{\frac{1}{\beta'(\beta - \beta')}}.$$

We will require the following auxiliary result concerning exponentials of functions from the space of forward curves.

A.9. **Lemma.** *The following statements are true:*

(1) *For each $h \in H_\beta$ we have $\exp(h) \in H_\beta$.*
(2) *For all $h \in H_\beta$ we have the estimate*

$$\|\exp(h)\|_\beta \leq (1 + \|h\|_\beta) \exp(C_\beta \|h\|_\beta). \tag{A.1}$$

(3) *There is a constant $K_\beta > 0$ such that for all $h_1, h_2 \in H_\beta$ we have*

$$\|\exp(h_1) - \exp(h_2)\|_\beta \leq K_\beta (1 + \|h_2\|_\beta) \exp(C_\beta \max\{\|h_1\|_\beta, \|h_2\|_\beta\}) \|h_1 - h_2\|_\beta. \tag{A.2}$$

In particular, the mapping $h \mapsto \exp(h)$ is locally Lipschitz.

(4) *For all $h \in H_\beta$ we have the estimate*

$$\|\exp(h) - 1\|_\beta \leq K_\beta \|h\|_\beta \exp(C_\beta \|h\|_\beta). \tag{A.3}$$

Proof. Let $h \in H_\beta$ be arbitrary. The function $\exp(h)$ is again absolutely continuous and, by Lemma A.4, for each $\xi \in \mathbb{R}_+$ we have

$$|\exp(h(\xi))| \leq \exp(\|h\|_\infty) \leq \exp(C_\beta \|h\|_\beta). \tag{A.4}$$

We deduce that

$$\begin{aligned} \|\exp(h)\|_\beta^2 &= |\exp(h(0))|^2 + \int_{\mathbb{R}_+} |h'(\xi) \exp(h(\xi))|^2 e^{\beta\xi} d\xi \\ &\leq (1 + \|h\|_\beta^2) \exp(2C_\beta \|h\|_\beta) \leq (1 + \|h\|_\beta)^2 \exp(2C_\beta \|h\|_\beta). \end{aligned}$$

Therefore, we obtain $\exp(h) \in H_\beta$ and estimate (A.1), proving the first two statements. Now, let $h_1, h_2 \in H_\beta$ be arbitrary. By Lemma A.4, for all $\xi \in \mathbb{R}_+$ we have

$$\begin{aligned} &|\exp(h_1(\xi)) - \exp(h_2(\xi))| \leq \max\{\exp(h_1(\xi)), \exp(h_2(\xi))\} |h_1(\xi) - h_2(\xi)| \\ &\leq \max\{\exp(\|h_1\|_\infty), \exp(\|h_2\|_\infty)\} \|h_1 - h_2\|_\infty \\ &\leq C_\beta \max\{\exp(C_\beta \|h_1\|_\beta), \exp(C_\beta \|h_2\|_\beta)\} \|h_1 - h_2\|_\beta. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\| \exp(h_1) - \exp(h_2)\|_\beta^2 \\
&= |\exp(h_1(0)) - \exp(h_2(0))|^2 + \int_{\mathbb{R}_+} |h_1'(\xi)\exp(h_1(\xi)) - h_2'(\xi)\exp(h_2(\xi))|^2 e^{\beta\xi} d\xi \\
&\leq |\exp(h_1(0)) - \exp(h_2(0))|^2 \\
&\quad + 2\int_{\mathbb{R}_+} |h_1'(\xi) - h_2'(\xi)|^2 |\exp(h_1(\xi))|^2 e^{\beta\xi} d\xi \\
&\quad + 2\int_{\mathbb{R}_+} |h_2'(\xi)|^2 |\exp(h_1(\xi)) - \exp(h_2(\xi))|^2 e^{\beta\xi} d\xi \\
&\leq (1 + 2\|h_2\|_\beta^2) C_\beta^2 \max\{\exp(2C_\beta\|h_1\|_\beta), \exp(2C_\beta\|h_2\|_\beta)\}\|h_1 - h_2\|_\beta^2 \\
&\quad + 2\exp(2C_\beta\|h_1\|_\beta)\|h_1 - h_2\|_\beta^2
\end{aligned}$$

This gives us

$$\begin{aligned}
&\| \exp(h_1) - \exp(h_2)\|_\beta \leq \|h_1 - h_2\|_\beta \\
&\times \big(C_\beta(1 + 2\|h_2\|_\beta)\max\{\exp(C_\beta\|h_1\|_\beta), \exp(C_\beta\|h_2\|_\beta)\} + 2\exp(C_\beta\|h_1\|_\beta)\big).
\end{aligned}$$

Therefore, estimate (A.2) is satisfied with $K_\beta := 2(C_\beta + 1)$. Moreover, setting $h_1 := h$ and $h_2 := 0$ in (A.2), we deduce estimate (A.3). □

The forthcoming results (Propositions A.10–A.12) are concerned with the structure of the drift term (5.16).

A.10. **Proposition.** *Let $(\sigma^j)_{j\in\mathbb{N}}$ and $(\Sigma^j)_{j\in\mathbb{N}}$ be sequences of mappings $\sigma^j, \Sigma^j : \Omega \times H_\beta \to H_\beta$. Suppose there exist sequences $(L_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $(\kappa^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$ with*

$$\sum_{j\in\mathbb{N}} (\kappa^j)^2 < \infty \tag{A.5}$$

such that $\mathbb{P}$–almost surely for all $j, n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have

$$\|\sigma^j(h_1) - \sigma^j(h_2)\|_\beta \leq L_n \kappa^j \|h_1 - h_2\|_\beta, \tag{A.6}$$

$$\|\Sigma^j(h_1) - \Sigma^j(h_2)\|_\beta \leq L_n \kappa^j \|h_1 - h_2\|_\beta, \tag{A.7}$$

and for all $j \in \mathbb{N}$ and $h \in H_\beta$ we have

$$\|\sigma^j(h)\|_\beta \leq \kappa^j \sqrt{1 + \|h\|_\beta}, \tag{A.8}$$

$$\|\Sigma^j(h)\|_\beta \leq \kappa^j \sqrt{1 + \|h\|_\beta}. \tag{A.9}$$

Then, for all $h \in H_\beta$ we have $\mathbb{P}$–almost surely

$$\sum_{j\in\mathbb{N}} \|\sigma^j(h)\Sigma^j(h)\|_\beta < \infty, \tag{A.10}$$

and the mapping

$$\alpha_1 : \Omega \times H_\beta \to H_\beta, \quad \alpha_1(h) := \sum_{j\in\mathbb{N}} \sigma^j(h)\Sigma^j(h)$$

is locally Lipschitz and satisfies the linear growth condition.

Proof. Let $h \in H_\beta$ be arbitrary. By Lemma A.7 and (A.8), (A.9), (A.5) we have $\mathbb{P}$–almost surely

$$\begin{aligned}\sum_{j \in \mathbb{N}} \| \sigma^j(h) \Sigma^j(h) \|_\beta &\leq \|m\| \sum_{j \in \mathbb{N}} \|\sigma^j(h)\|_\beta \|\Sigma^j(h)\|_\beta \\ &\leq \|m\| \bigg(\sum_{j \in \mathbb{N}} (\kappa^j)^2 \bigg) (1 + \|h\|_\beta) < \infty,\end{aligned}$$

showing that (A.10) is satisfied, and by the triangle inequality we obtain $\mathbb{P}$–almost surely

$$\|\alpha_1(h)\|_\beta = \bigg\| \sum_{j \in \mathbb{N}} \sigma^j(h) \Sigma^j(h) \bigg\|_\beta \leq \sum_{j \in \mathbb{N}} \|\sigma^j(h) \Sigma^j(h)\|_\beta,$$

proving that α_1 satisfies the linear growth condition. Now, let $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ be arbitrary. By Lemma A.7 and (A.6)–(A.9) we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}&\|\alpha_1(h_1) - \alpha_1(h_2)\|_\beta \leq \sum_{j \in \mathbb{N}} \|\sigma^j(h_1)\Sigma^j(h_1) - \sigma^j(h_2)\Sigma^j(h_2)\|_\beta \\ &\leq \sum_{j \in \mathbb{N}} \|\sigma^j(h_1)(\Sigma^j(h_1) - \Sigma^j(h_2))\|_\beta + \sum_{j \in \mathbb{N}} \|(\sigma^j(h_1) - \sigma^j(h_2))\Sigma^j(h_2)\|_\beta \\ &\leq \|m\| \sum_{j \in \mathbb{N}} \|\sigma^j(h_1)\|_\beta \|\Sigma^j(h_1) - \Sigma^j(h_2)\|_\beta + \|m\| \sum_{j \in \mathbb{N}} \|\sigma^j(h_1) - \sigma^j(h_2)\|_\beta \|\Sigma^j(h_2)\|_\beta \\ &\leq 2\|m\| L_n \bigg(\sum_{j \in \mathbb{N}} (\kappa^j)^2 \bigg) \sqrt{1+n} \|h_1 - h_2\|_\beta,\end{aligned}$$

showing that α_1 is locally Lipschitz. □

A.11. **Proposition.** *Let $\gamma, \Gamma : \Omega \times H_\beta \times B \to H$ be measurable mappings. Suppose there exist a sequence $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and a bounded, measurable mapping $\rho : B \to \mathbb{R}_+$ with*

$$\int_B \rho(x)^2 F(dx) < \infty \tag{A.11}$$

such that $\mathbb{P}$–almost surely for all $x \in B$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have

$$\|\gamma(h_1, x) - \gamma(h_2, x)\|_\beta \leq L_n \rho(x) \|h_1 - h_2\|_\beta, \tag{A.12}$$

$$\|\Gamma(h_1, x) - \Gamma(h_2, x)\|_\beta \leq L_n \rho(x) \|h_1 - h_2\|_\beta, \tag{A.13}$$

and for all $x \in B$ and $h \in H_\beta$ we have

$$\|\gamma(h, x)\|_\beta \leq \ln^* \big(\rho(x) \sqrt{1 + \|h\|_\beta} \big), \tag{A.14}$$

$$\|\Gamma(h, x)\|_\beta \leq \ln^* \big(\rho(x) \sqrt{1 + \|h\|_\beta} \big). \tag{A.15}$$

Then, for all $h \in H_\beta$ we have $\mathbb{P}$–almost surely

$$\int_B \|\gamma(h, x)(e^{\Gamma(h,x)} - 1)\|_\beta F(dx) < \infty, \tag{A.16}$$

and the mapping

$$\alpha_2 : \Omega \times H_\beta \to H_\beta, \quad \alpha_2(h) := \int_B \gamma(h, x)(e^{\Gamma(h,x)} - 1) F(dx)$$

is locally Lipschitz and satisfies the linear growth condition.

Proof. By Lemma A.9 and estimates (A.13), (A.15), we have $\mathbb{P}$–almost surely for all $x \in B$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ the estimate

$$\begin{aligned}
&\|e^{\Gamma(h_1,x)} - e^{\Gamma(h_2,x)}\|_\beta \\
&\leq K_\beta(1 + \|\Gamma(h_2,x)\|_\beta) \exp(C_\beta \max\{\|\Gamma(h_1,x)\|_\beta, \|\Gamma(h_2,x)\|_\beta\}) \\
&\quad \times \|\Gamma(h_1,x) - \Gamma(h_2,x)\|_\beta \\
&\leq K_\beta \Big(1 + \ln^* \big(\|\rho\|_\infty \sqrt{1 + \|h_2\|_\beta} \big) \Big) \\
&\quad \times \exp \Big(C_\beta \max \big\{ \ln^* \big(\|\rho\|_\infty \sqrt{1 + \|h_1\|_\beta} \big), \ln^* \big(\|\rho\|_\infty \sqrt{1 + \|h_2\|_\beta} \big) \big\} \Big) \\
&\quad \times L_n \rho(x) \|h_1 - h_2\|_\beta.
\end{aligned}$$

Hence, there exists a sequence $(K_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\mathbb{P}$–almost surely for all $x \in B$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ we have

$$\|e^{\Gamma(h_1,x)} - e^{\Gamma(h_2,x)}\|_\beta \leq K_n \rho(x) \|h_1 - h_2\|_\beta. \tag{A.17}$$

By (A.14) and the inequality

$$\ln^* x \leq x \quad \text{for all } x \in \mathbb{R}_+,$$

we have $\mathbb{P}$–almost surely

$$\|\gamma(h,x)\|_\beta \leq \rho(x) \sqrt{1 + \|h\|_\beta} \quad \text{for all } h \in H_\beta \text{ and } x \in B. \tag{A.18}$$

Moreover, by Lemma A.9 and assumption (A.15), we obtain $\mathbb{P}$–almost surely for all $x \in B$ and $h \in H_\beta$ the estimate

$$\begin{aligned}
\|e^{\Gamma(h,x)} - 1\|_\beta &\leq K_\beta \|\Gamma(h,x)\|_\beta \exp(C_\beta \|\Gamma(h,x)\|_\beta) \\
&= K_\beta \exp^*(\|\Gamma(h,x)\|_\beta) \leq K_\beta \rho(x) \sqrt{1 + \|h\|_\beta}.
\end{aligned} \tag{A.19}$$

Now, let $h \in H_\beta$ be arbitrary. By Lemma A.7, estimates (A.18), (A.19) and the integrability condition (A.11) we have $\mathbb{P}$–almost surely

$$\begin{aligned}
\int_B \|\gamma(h,x)(e^{\Gamma(h,x)} - 1)\|_\beta F(dx) &\leq \|m\| \int_B \|\gamma(h,x)\|_\beta \|e^{\Gamma(h,x)} - 1\|_\beta F(dx) \\
&\leq \|m\| K_\beta \bigg(\int_B \rho(x)^2 F(dx) \bigg) (1 + \|h\|_\beta) < \infty,
\end{aligned}$$

showing that (A.16) is satisfied, and by the triangle inequality we obtain $\mathbb{P}$–almost surely

$$\|\alpha_2(h)\|_\beta = \bigg\| \int_B \gamma(h,x)(e^{\Gamma(h,x)} - 1) F(dx) \bigg\|_\beta \leq \int_B \|\gamma(h,x)(e^{\Gamma(h,x)} - 1)\|_\beta F(dx),$$

proving that α_2 satisfies the linear growth condition. Now, let $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ be arbitrary. By Lemma A.7 and estimates (A.12), (A.17), (A.18) (A.19) we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}
&\|\alpha_2(h_1) - \alpha_2(h_2)\|_\beta \leq \int_B \|\gamma(h_1,x)(e^{\Gamma(h_1,x)} - 1) - \gamma(h_2,x)(e^{\Gamma(h_2,x)} - 1)\|_\beta F(dx) \\
&\leq \|m\| \int_B \|\gamma(h_1,x)\|_\beta \|e^{\Gamma(h_1,x)} - e^{\Gamma(h_2,x)}\|_\beta F(dx) \\
&\quad + \|m\| \int_B \|\gamma(h_1,x) - \gamma(h_2,x)\|_\beta \|e^{\Gamma(h_2,x)} - 1\|_\beta F(dx) \\
&\leq \|m\| \sqrt{1+n} (K_n + L_n K_\beta) \bigg(\int_B \rho(x)^2 F(dx) \bigg) \|h_1 - h_2\|_\beta,
\end{aligned}$$

showing that α_2 is locally Lipschitz. □

A.12. **Proposition.** *Suppose there exist a sequence* $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ *and measurable mappings* $\tau : B^c \to \mathbb{R}_+$, $\zeta : B^c \to [1, \infty)$ *with*

$$\int_{B^c} (1 \vee \tau(x)) \zeta(x) (1 \vee \ln^{**}(\zeta(x))) F(dx) < \infty \tag{A.20}$$

such that $\mathbb{P}$*–almost surely for all* $x \in B^c$, $n \in \mathbb{N}$ *and* $h_1, h_2 \in H_\beta$ *with* $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ *we have*

$$\|\gamma(h_1, x) - \gamma(h_2, x)\|_\beta \leq L_n \tau(x) \|h_1 - h_2\|_\beta, \tag{A.21}$$

$$\|\Gamma(h_1, x) - \Gamma(h_2, x)\|_\beta \leq L_n \tau(x) \|h_1 - h_2\|_\beta, \tag{A.22}$$

and for all $x \in B$ *and* $h \in H_\beta$ *we have*

$$\|\gamma(h, x)\|_\beta \leq \ln^{**}(\zeta(x)), \tag{A.23}$$

$$\|\Gamma(h, x)\|_\beta \leq \ln^{**}(\zeta(x)). \tag{A.24}$$

Then, for all $h \in H_\beta$ *we have* $\mathbb{P}$*–almost surely*

$$\int_{B^c} \|\gamma(h, x) e^{\Gamma(h,x)}\|_\beta F(dx) < \infty, \tag{A.25}$$

and the mapping

$$\alpha_3 : \Omega \times H_\beta \to H_\beta, \quad \alpha_3(h) := \int_{B^c} \gamma(h, x) e^{\Gamma(h,x)} F(dx)$$

is locally Lipschitz and satisfies the linear growth condition.

Proof. By Lemma A.9 and estimates (A.24), (A.22) we have $\mathbb{P}$–almost surely for all $x \in B^c$, $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ the estimate

$$\begin{aligned}
&\|e^{\Gamma(h_1,x)} - e^{\Gamma(h_2,x)}\|_\beta \\
&\leq K_\beta (1 + \|\Gamma(h_2, x)\|_\beta) \exp(C_\beta \max\{\|\Gamma(h_1, x)\|_\beta, \|\Gamma(h_2, x)\|_\beta\}) \\
&\quad \times \|\Gamma(h_1, x) - \Gamma(h_2, x)\|_\beta \\
&\leq K_\beta \exp^{**}(\max\{\|\Gamma(h_1, x)\|_\beta, \|\Gamma(h_2, x)\|_\beta\}) \|\Gamma(h_1, x) - \Gamma(h_2, x)\|_\beta \\
&\leq K_\beta \zeta(x) L_n \tau(x) \|h_1 - h_2\|_\beta.
\end{aligned} \tag{A.26}$$

Moreover, by Lemma A.9 and assumption (A.24), we obtain $\mathbb{P}$–almost surely for all $x \in B^c$ and $h \in H_\beta$ the estimate

$$\|e^{\Gamma(h,x)}\|_\beta \leq (1 + \|\Gamma(h, x)\|_\beta) \exp(C_\beta \|\Gamma(h, x)\|_\beta) = \exp^{**}(\|\Gamma(h, x)\|_\beta) \leq \zeta(x). \tag{A.27}$$

Now, let $h \in H_\beta$ be arbitrary. By Lemma A.7, estimates (A.23), (A.27) and the integrability condition (A.20), we have $\mathbb{P}$–almost surely

$$\begin{aligned}
\int_{B^c} \|\gamma(h, x) e^{\Gamma(h,x)}\|_\beta F(dx) &\leq \|m\| \int_B \|\gamma(h, x)\|_\beta \|e^{\Gamma(h,x)}\|_\beta F(dx) \\
&\leq \|m\| \int_B \zeta(x) \ln^{**}(\zeta(x)) F(dx) < \infty,
\end{aligned}$$

showing that (A.25) is satisfied, and by the triangle inequality we obtain $\mathbb{P}$–almost surely

$$\|\alpha_3(h)\|_\beta = \left\| \int_{B^c} \gamma(h, x) e^{\Gamma(h,x)} F(dx) \right\|_\beta \leq \int_{B^c} \|\gamma(h, x) e^{\Gamma(h,x)}\|_\beta F(dx),$$

proving that α_3 satisfies the linear growth condition. Now, let $n \in \mathbb{N}$ and $h_1, h_2 \in H_\beta$ with $\|h_1\|_\beta, \|h_2\|_\beta \leq n$ be arbitrary. By Lemma A.7 and estimates (A.21), (A.23), (A.26), (A.27) we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}
\|\alpha_3(h_1) - \alpha_3(h_2)\|_\beta &\leq \int_{B^c} \|\gamma(h_1,x)e^{\Gamma(h_1,x)} - \gamma(h_2,x)e^{\Gamma(h_2,x)}\|_\beta F(dx) \\
&\leq \|m\| \int_{B^c} \|\gamma(h_1,x)\|_\beta \|e^{\Gamma(h_1,x)} - e^{\Gamma(h_2,x)}\|_\beta F(dx) \\
&\quad + \|m\| \int_{B^c} \|\gamma(h_1,x) - \gamma(h_2,x)\|_\beta \|e^{\Gamma(h_2,x)}\|_\beta F(dx) \\
&\leq \|m\| L_n \bigg(\int_{B^c} \tau(x)\zeta(x)\big(1 + K_\beta \ln^{**}(\zeta(x))\big) F(dx) \bigg) \|h_1 - h_2\|_\beta,
\end{aligned}$$

which by (A.20) shows that α_3 is locally Lipschitz. □

Now, we suppose that the regularity conditions for σ, θ and γ, ϕ from Section 5 are fulfilled and that the drift α is given by (5.16). Combining the previous results shows that $(\alpha, \sigma, \gamma|_B)$ are locally Lipschitz and satisfy the linear growth condition. Consequently, applying Theorem A.3 constitutes the proof of Theorem 5.1.

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