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Sixth sheet of exercises

1. Two independent random variables

Let X and Y be two independent random variables with distribution functions F and G. Prove the following statements :

(a) The distribution function of $X+Y$ is given by $F * G$.

(b) If X is absolutely continuous with density f, then $X+Y$ is absolutely continuous with density

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = \int_{\mathbb{R}} f(x-t) G(dt).$$

(c) If X and Y are absolutely continuous with densities f and g, then we have

$$h(x) = \int_{\mathbb{R}} f(x-t) g(t) dt = \int_{\mathbb{R}} f(t) g(x-t) dt, \;\; x \in \mathbb{R}.$$

2. Interesting results

The following two questions are independent.

(1) We know from the course that the negative Binomial distribution $\mathrm{NB}(\beta, p)$ with parameters $\beta > 0$ and $p \in (0,1)$ is specified by the stochastic vector

$$\pi(k) = \frac{\Gamma(\beta + k)}{k! \Gamma(\beta)} p^{\beta} (1-p)^k, \quad k \in \mathbb{N}_0.$$

Prove that for $k \in \mathbb{N}$, we have

$$\frac{\Gamma(\beta + k)}{k! \Gamma(\beta)} = \frac{(\beta + k - 1)(\beta + k - 2) \cdot \ldots \cdot \beta}{k!}.$$

(2) Let N be a random variable with values in $\mathbb{N}_0$. Prove that

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}(N \geq k).$$

3. Probability generating functions

Let N be a $\mathbb{N}_0$-valued random variable. Let $\mathbb{P} \circ N$ denotes the law of N (it could be denoted $\mathbb{P}_N$).

(a) If $\mathbb{P} \circ N = \mathrm{Bi}(n, p)$, then we have $\mathscr{M}_N^p = (0, \infty)$ and

$$\phi_N(t) = (1 - p + pt)^n.$$

(b) If $\mathbb{P} \circ N = \text{Pois}(\lambda)$, then we have $\mathscr{M}_N^p = (0, \infty)$ and

$$\phi_N(t) = e^{-\lambda(1-t)}.$$

(c) If $\mathbb{P} \circ N = \text{NB}(\beta, p)$, then we have $\mathscr{M}_N^p = (0, \frac{1}{1-p})$ and

$$\phi_N(t) = \left(\frac{1 - (1-p)t}{p} \right)^{-\beta}.$$

4. Interesting Lemma

We define the process $M = (M_n)_{n \in \mathbb{N}_0}$ as

$$M_n := \sum_{k=1}^{n} Y_k Z_k,$$

where Y_k and Z_k are defined in Lemma 3.2.21. in the course. Prove that M is a square-integrable $\mathbb{F}$-martingale with $M_0 = 0$ and

$$\mathbb{E}[M_n^2] = \sum_{k=1}^{n} \mathbb{E}[(Y_k Z_k)^2] \quad \text{for all } n \in \mathbb{N}.$$