Stefan Tappe, Raghid Zeineddine

Third sheet of exercises : Foundations of life insurance mathematics

1. Thiele differential equation

(a) Prove that the Thiele differential equation has the unique solution :

$$V(t) = \int_0^t (\pi(s) - \lambda(s)A(s)) \exp\Big(\int_s^t (\phi(u) + \lambda(u))du \Big) ds, \;\; t \in [0, \tau).$$

(b) We assume that there exists non-negative, continuous functions $k, f, \pi : \mathbb{R}_+ \to \mathbb{R}_+$ such that on $[0, \tau)$, the capital function K is given by $K(t) = 1 + \int_0^t k(s)ds$, the distribution function F of the future lifetime is given by $F(t) = \int_0^t f(s)ds$ and the premium function Π is given by $\Pi(t) = \int_0^t \pi(s)ds$.

It is shown in Theorem 1.2.34. in the course that the net premium reserve V satisfies the Thiele integral equation

$$\frac{V(t)}{K(t)} = \int_{[0,t)} \frac{1}{K(s)} d\Pi(s) - \int_{(0,t]} \frac{A(u) - V(u)}{K(u)} d\Lambda(u), \;\; t \in [0, \tau)$$

Prove, using the previous equation, that V satisfies the Thiele differential equation.

2. Natural premium

A person aged 60 years old signed in the first of January a life insurance police with a terminal time after 9 years ($\tau = 9$). According to the study, done by the insurance company, of the health situation of the person, the company assumed that the future lifetime of this person is exponentially distributed and that its expected future lifetime is 10 years. We suppose also that we have the following other quantities determining the life insurance police :

(a) The payment spectrum A is given by $A(s) = s$, $s \in \mathbb{R}_+$,

(b) The capital function is given by $K(t) = (1 + i)^t$, with interest rate $i = 3\%$.

Determine the **natural premium** payable each year in the first of January, starting from the date of signing the contract (that is payable at $t_0 = 0, t_1 = 1, \dots, t_9 = 9 = \tau$).

3. Net single premium

It has been proved in Proposition 2.1.2. in the course that if A/K is decreasing, then for any net premium function Π, we have

$$\text{Var}[\bar{A}(Y)] \leq \text{Var}[B],$$

and that $\text{Var}[B]$ is minimal ($=\text{Var}[\bar{A}(Y)]$) under all net premium functions if and only if Π is a net single premium.

The purpose of the exercise is to show that this proposition is not true if we don't suppose that A/K is decreasing. We consider a Pure endowment (a life insurance police (LIP) such that $\tau < \infty$, $A(s) = 0$ for $s < \tau$ and $A(\tau) > 0$) with the following quantities determining the LIP :

(a) The future lifetime $T \sim \exp(1/3)$,

(b) The terminal time of the police is $\tau = 10$,

(c) $A(10) = K(10)$, where K is the capital function,

(d) The premium function Π is given by $\Pi(t) = C \cdot \int_0^t K(s)ds$, where $C := \frac{e^{-10/3}}{3(1-e^{-10/3})}$.

Then,

1. Prove that Π is a net premium function.

2. Calculate $\text{Var}(\bar{\Pi}(Y))$ and $\text{Cov}(\bar{A}(Y), \bar{\Pi}(Y))$.

3. Verify that $\text{Var}(\bar{\Pi}(Y)) - 2\text{Cov}(\bar{A}(Y), \bar{\Pi}(Y)) < 0$.

4. Let B_Π denotes the present value of the LIP (with premium function Π defined above) in view of the policyholder. Deduce that $\text{Var}(B_\Pi) < \text{Var}(\bar{A}(Y))$.

5. Let $B_{\tilde{\Pi}}$ denotes the present value of the LIP (with a **net single premium** $\tilde{\Pi}$) in view of the policyholder. Verify that $\text{Var}(B_{\tilde{\Pi}}) = \text{Var}(\bar{A}(Y))$.

We deduce from the last two questions that $\text{Var}(B_\Pi) < \text{Var}(B_{\tilde{\Pi}})$. This, means that although $\tilde{\Pi}$ is a net single premium , $\text{Var}(B_{\tilde{\Pi}})$ is not minimal under all net premium functions. Thus, Proposition 2.1.2. does not apply (because A/K is not decreasing).

4. Martingales

Let M be a square-integrable martingale. Prove that :

(a) $\text{Cov}(M_t - M_s, M_v - M_u) = 0$ for all $0 \le s \le t \le u \le v < \infty$.

(b) $E[(M_t - M_s)^2] = E[M_t^2 - M_s^2]$ for all $0 \le s \le t < \infty$.