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Insurance Mathematics
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Second sheet of exercises : Foundations of life insurance mathematics

1. Rate of return

Calculate (provided it exists) the rate of return of the payment flow $Z = Z_L - Z_P$ in the following cases :

(a) $Z_P = 3\mathbf{1}_{[2,\infty)}$ and $Z_L = 2\mathbf{1}_{[6,\infty)}$.

(b) $Z_P(s) = 2 + s$ and $Z_L(s) = 1 + s^2/2$.

2. Future life time and force of mortality

Let T_x be the future lifetime of a person of age x, and let F be its distribution function.

(a) We suppose that $t_{\max} < \infty$, where $t_{\max}$ is the maximal future lifetime of the person. Calculate $F(t_{\max})$.

(b) Let $\lambda(t)$ be the force of mortality at time t. Compute the distribution function F in the following cases :

(1) De Moivre (1724) : $\lambda(t) = \frac{1}{t_{\max}-t}$, $t \in (0, t_{\max})$ with $t_{\max} = 86$.

(2) Gompertz (1825) : $\lambda(t) = be^{ct}$ with $b, c > 0$.

(3) Makeham (1860) : $\lambda(t) = a + be^{ct}$ with $a, b, c > 0$.

(4) Weibull (1939) : $\lambda(t) = kt^{\gamma}$ with $k > 0$ and $\gamma > -1$.

3. Net single premium

Let $\bar{\Pi} \in \mathbb{R}_+$ be a net single premium. We suppose that the payment spectrum $A(\cdot)$ is constant ($A(t) = A$, $t \in \mathbb{R}_+$) and the capital function is $K(t) = e^{\delta t}$ with $\delta > 0$. We suppose that the future lifetime T is exponentially distributed with parameter $\lambda > 0$. Compute Π.

4. Regular payment flow

We consider $W : [0,\infty) \times \mathscr{Z} \to [-\infty, +\infty]$, defined by $W(t, Z) = K(t)a(Z)$, where K is a capital function and Z is a payment flow ($a(Z)$ being the present value of Z). $W(t, Z)$ is the value of the payment flow Z at time t. The purpose of the exercise is to prove that W is *regular*. That is to prove that W verifies the following properties :

(1) Finiteness : $\forall t \geq 0$, $W(t, Z) \in \mathbb{R}$, for $Z \in \mathscr{Z}_g$ such that $Z(\infty) < \infty$.

(2) Sensitivity : $\forall t, u \geq 0$, $W(t, \varepsilon_u) \neq 0$, where $\varepsilon_u := \mathbf{1}_{[u,\infty)}$.

(3) Additivity : $\forall Z_1, Z_2 \in \mathscr{Z}$, such that $Z_1 + Z_2 \in \mathscr{Z}$, we have

$$W(t, Z_1 + Z_2) = W(t, Z_1) + W(t, Z_2), \;\; t \geq 0,$$

provided the last quantity in the right side is well defined.

(4) Monotone continuity : Let $(Z_n)_n \subset \mathscr{Z}_g$ be a monotone increasing sequence. We suppose that $Z := \sup_n Z_n \in \mathscr{Z}_g$, then we have

$$W(\cdot, Z) = \sup_n W(\cdot, Z_n).$$

(5) Immediacy : $u \mapsto W(t, \varepsilon_u)$ is right-continuous $\forall t \geq 0$.

(6) Consistency : $\forall u \geq 0$, for $Z \in \mathscr{Z}_g$ such that $Z(\infty) < \infty$, we have

$$W(\cdot, Z) = W(\cdot, W(u, Z)\varepsilon_u).$$