First sheet of exercises : Foundations of life insurance mathematics

1. An important lemma

Let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of directed payment flows such that

$$Z(t) := \sum_{j=0}^{\infty} Z_j(t) < \infty \text{ for every } t \in \mathbb{R}_+.$$

(a) Prove that $Z \in \mathscr{Z}_g$ (the set of all directed payment flows).

(b) Prove that for every m_Z-integrable function $f : \mathbb{R}_+ \to \mathbb{R}$ we have

$$\int_{\mathbb{R}_+} f(s)dZ(s) = \sum_{j=0}^{\infty} \int_{\mathbb{R}_+} f(s)dZ_j(s).$$

2. Discrete annuity

For any $t \in \mathbb{R}_+$, we define $K(t) := t+1$. Let $Z(t) := 0$ for $t \in [0,1)$ and $Z(t) := \sum_{k=1}^{n} \frac{1}{k}$ if $n \leq t < n+1$ where $n \in \mathbb{N}^*$.

(a) Prove that K is a capital function and that Z is a directed payment flow.

(b) Write Z in the form $Z(t) = \sum_{j=0}^{\infty} z_j \mathbf{1}_{[t_j,\infty)}$, $t \in \mathbb{R}_+$ where $(z_j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$ and $(t_j)_{j\in\mathbb{N}}$ is a sequence with $t_0 = 0$ and $\lim_{j\to\infty} t_j = \infty$ (this means, by definition, that Z is a discrete annuity).

(c) Calculate the present value $a(Z)$ of the payment flow Z.

3. Payment flow

Let $(Z_j)_{j\in\mathbb{N}^*}$ be a sequence of directed payment flows where $Z_j(t) := \frac{e^t}{j^2}$, $t \in \mathbb{R}_+$. Let Z be the sum of all the payment flows. Let K be the capital function given by $K(t) = e^{\delta t}$, $t \in \mathbb{R}_+$, with nominal interest rate $0 < \delta < 1$.

(a) Calculate the terminal value of Z until time $t > 0$.

(b) Calculate the present value of Z until time $t > 0$.

(c) Calculate the present value of the total payment flow Z.

4. Nominal interest rates

An interest rate is always stated in conjunction with a *basic time unit*; for example, one might speak of an annual interest rate of 6%. In addition, the *conversion* period has to be stated; this is the time interval at the end of which interest is credited or "compounded". An interest rate is called *effective* if the conversion period and the basic time unit are identical; in that case interest is credited at the end of the basic time unit. For example, if an initial capital C is invested with an effective annual interest rate i, then the accumulated value of C after one year is $C + Ci = C(1+i)$, the accumulated value of C after two years is $C(1+i) + C(1+i)i = C(1+i)^2$. By induction, the accumulated value of C after n years is $C(1+i)^n$.

When the conversion period does not coincide with the basic time unit, the interest rate is called *nominal*.

An annual interest rate of 6% with a conversion period of 3 months means that interest of 6%/4 = 1.5% is credited at the end of each quarter. Thus an initial capital of 1 increases to $(1+0.015)^4 =$ 1.06136 at the end of one year. Therefore, an annual nominal interest rate of 6%, convertible quarterly, is equivalent to an annual effective interest rate of 6.136% (because $1.06136 = 1 + i$ with $i = 6.136\%$).

(a) Let i be a given annual effective interest rate. We define $i^{(m)}$ as the nominal interest rate, convertible m times per year, which is equivalent to i. Equality of the accumulation value after one year leads to the equation

$$(1 + \frac{i^{(m)}}{m})^m = 1 + i.$$

Prove that $i^{(m)} \to \delta := \log(1+i)$ as $m \to \infty$.

<u>Remark</u> : δ is called the *force of interest* equivalent to i. We have obviously $1 + i = e^{\delta}$, thus $(1+i)^n = e^{n\delta}$ for any $n \in \mathbb{N}^*$.