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Lecture Notes

Insurance Mathematics

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Contents

Chapter 1

Foundations of life insurance mathematics

1.1 Elementary financial mathematics

1.1.1 Interest rates and capital functions

Definition 1.1.1. *A monotone increasing, right-continuous function $K : \mathbb{R}_+ \to [1, \infty)$ with $K(0) = 1$ is called a capital function (or accumulation function).*

Definition 1.1.2. *Let K be a capital function.*

(a) We call $r := K(1)$ the accumulation factor (for the first year).

(b) We call $i := r - 1$ the interest rate or the effective interest rate.

(c) We call $v := 1/r$ the discount factor.

Example 1.1.3 (Discrete interest rates (with compound interest))**.** *We set*

$$K(t) := (1 + i)^{\lfloor t \rfloor}, \quad t \in \mathbb{R}_+,$$

where $\lfloor t \rfloor := \max\{k \in \mathbb{N}_0 : k \leq t\}$. Here i is indeed the interest rate from Definition 1.1.2(b).

Example 1.1.4 (Continuous interest rates (with compound interest))**.** *We set*

$$K(t) := e^{\delta t}, \quad t \in \mathbb{R}_+,$$

where $\delta \in \mathbb{R}_+$ denotes the so-called nominal interest rate. Here we have $r = e^{\delta}$, $i = e^{\delta} - 1$ and $v = e^{-\delta}$. In general we have $\delta \neq i$. Furthermore, note that

$$\lim_{n \to \infty} \left(1 + \frac{i}{n}\right)^n = e^i.$$

Definition 1.1.5. *Let K be a capital function. If there is a non-negative, measurable function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$K(t) = 1 + \int_0^t k(s)ds, \quad t \in \mathbb{R}_+,$$

then $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\phi(t) := \frac{k(t)}{K(t)}, \quad t \in \mathbb{R}_+$$

is called the force of interest of K.

Example 1.1.6. *For $K(t) = e^{\delta t}$ we have*

$$K(t) = 1 + \int_0^t \delta e^{\delta s} ds, \quad t \in \mathbb{R}_+.$$

Thus we have $k(t) = \delta e^{\delta t}$, and obtain

$$\phi(t) = \frac{k(t)}{K(t)} = \frac{\delta e^{\delta t}}{e^{\delta t}} = \delta.$$

This is the reason for calling δ from Example 1.1.4 the interest rate.

Lemma 1.1.7. *Let K be a capital function as in Definition 1.1.5. Then we have*

$$K(t) = \exp\left(\int_0^t \phi(s)ds \right), \quad t \in \mathbb{R}_+.$$

Proof. Suppose that k is continuous. Then we have $K \in C^1(\mathbb{R}_+)$ with $K' = k$, and it follows

$$\frac{d}{dt}\big(\ln K(t)\big) = \frac{k(t)}{K(t)} = \phi(t).$$

Since $K(0) = 1$, we deduce

$$\ln K(t) = \int_0^t \phi(s)ds.$$

$\square$

Definition 1.1.8. *Let K be a capital function. We define die accumulated force of interest $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$\Phi(t) := \int_{(0,t]} \frac{1}{K(s-)} dK(s), \quad t \in \mathbb{R}_+,$$

where

$$K(s-) := \lim_{u \uparrow s} K(u).$$

Remark 1.1.9. *Let K be a capital function as in Definition 1.1.5. Then we have*

$$\Phi(t) = \int_0^t \phi(s)ds = \int_0^t \frac{k(s)}{K(s)}ds.$$

With Lemma 1.1.7 it follows

$$K(t) = e^{\Phi(t)} \quad \Leftrightarrow \quad \Phi(t) = \ln K(t).$$

But in general, we do not have $\Phi(t) = \ln K(t)$.

Example 1.1.10. *Let $K(t) = (1+i)^{\lfloor t \rfloor}$ as in Example 1.1.3. Then $K(s-) = 1$ for all $s \in [0,1]$. We obtain*

$$\Phi(1) = \int_{(0,1]} \frac{1}{K(s-)} dK(s) = \int_{(0,1]} dK(s) = K(1) - K(0) = i.$$

However $\ln K(1) = \ln(1+i)$.

1.1.2 Payment flows

Definition 1.1.11.

(a) A directed payment flow is a right-continuous, monotone increasing function $Z : \mathbb{R}_+ \to \mathbb{R}_+$.

(b) We denote by $\mathscr{Z}_g$ the set of all directed payment flows.

(c) A function $Z : \mathbb{R}_+ \to \mathbb{R}$ is called an undirected payment flow (or simply payment flow) if there are $Z_1, Z_2 \in \mathscr{Z}_g$ with $Z = Z_1 - Z_2$ such that $Z_1(\infty) := \lim_{t\to\infty} Z_1(t) < \infty$ or $Z_2(\infty) := \lim_{t\to\infty} Z_2(t) < \infty$.

(d) We denote by $\mathscr{Z}$ the set of all undirected payment flows.

Example 1.1.12. *If K is a capital function, then $Z := K - 1$ is a directed payment flow, which we call interest payment flow.*

Example 1.1.13. *Let $(z_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}_+$ be a sequence, and let $(t_j)_{j\in\mathbb{N}_0}$ be a strictly increasing sequence with $t_0 = 0$ and $\lim_{j\to\infty} t_j = \infty$. Then*

$$Z(t) := \sum_{j=0}^{\infty} z_j \mathbb{1}_{[t_j,\infty)}(t), \quad t \in \mathbb{R}_+$$

is a directed payment flow, which we call a discrete annuity.

Definition 1.1.14. *Let K be a capital function, and let $Z \in \mathscr{Z}_g$ be a discrete annuity. Then we call*

$$a(Z) := \sum_{j=0}^{\infty} \frac{z_j}{K(t_j)} \in [0, \infty]$$

the present value of the payment flow Z.

Remark 1.1.15. *We can express the present value as*

$$a(Z) = \sum_{j=0}^{\infty} \frac{\Delta Z(t_j)}{K(t_j)} = \sum_{t \geq 0} \frac{\Delta Z(t)}{K(t)},$$

where

$$\Delta Z(t) := Z(t) - Z(t-).$$

Remark 1.1.16. *For every $Z \in \mathscr{Z}_g$ there exists a unique measure m_Z on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $m_Z(\{0\}) = Z(0)$ and*

$$m_Z((s,t]) = Z(t) - Z(s), \quad 0 \leq s \leq t.$$

For every m_Z-integrable function $f : \mathbb{R}_+ \to \mathbb{R}$ we define

$$\int_{\mathbb{R}_+} f(s)\, dZ(s) := \int_{\mathbb{R}_+} f \, dm_Z.$$

Definition 1.1.17. *Let $Z \in \mathscr{Z}$ be a payment flow.*

(a) The terminal value of Z until time $t \in \mathbb{R}_+$ is given by

$$s(Z)(t) := K(t) \int_{[0,t]} \frac{1}{K(s)} dZ(s).$$

(b) The present value of Z until time t is given by

$$a(Z)(t) := \int_{[0,t]} \frac{1}{K(s)} dZ(s).$$

(c) The present value of the total payment flow Z is

$$a(Z) := \int_{\mathbb{R}_+} \frac{1}{K(s)} dZ(s).$$

Remark 1.1.18. *For $Z \in \mathscr{Z}_g$ we have $a(Z) \in [0, \infty]$, and for $Z \in \mathscr{Z}_g$ we have $a(Z) \in [-\infty, \infty]$.*

Proposition 1.1.19. *For a càdlàg-function $Z : \mathbb{R}_+ \to \mathbb{R}$ with $Z_0 = 0$ the following statements are equivalent:*

(i) Z is locally of bounded variation.

(ii) There are increasing, right-continuous functions $Z_1, Z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $Z_1(0) = Z_2(0) = 0$ such that $Z = Z_1 - Z_2$.

Proposition 1.1.20. *Let $Z : \mathbb{R}_+ \to \mathbb{R}$ be a càdlàg-function of locally bounded variation with $Z(0) = 0$. Then there are unique increasing, right-continuous functions $Z_1, Z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $Z_1(0) = Z_2(0) = 0$ such that $Z = Z_1 - Z_2$ and $\operatorname{Var}(Z) = Z_1 + Z_2$. They are given by*

$$Z_1 = \frac{Z + \operatorname{Var}(Z)}{2} \quad \textit{and} \quad Z_2 = Z_1 - Z.$$

1.1.3 Equivalence principle and premium reserve

Let K be a capital function.

Definition 1.1.21. *Two payment flows $Z_1, Z_2 \in \mathscr{Z}$ are called equivalent (relative to K) if $a(Z_1) = a(Z_2) \in \mathbb{R}$.*

Definition 1.1.22. *Let $Z_L, Z_P \in \mathscr{Z}_g$ be such that $\min\{a(Z_L), a(Z_P)\} < \infty$.*

(a) For every time $t \in \mathbb{R}_+$ the prospective premium reserve of (Z_L, Z_P) relative to the capital function K is defined as

$$V(t) := K(t)\left[\int_{[t,\infty)} \frac{dZ_L(s)}{K(s)} - \int_{[t,\infty)} \frac{dZ_P(s)}{K(s)}\right].$$

(b) If $V(t) \geq 0$ for all $t \in \mathbb{R}_+$, then we call (Z_L, Z_P) a savings plan.

(c) If $V(t) \leq 0$ for all $t \in \mathbb{R}_+$, then we call (Z_L, Z_P) a credit agreement, and $-V(t)$ is the residual debt at time t.

Remark 1.1.23. *Here P stands for premium (paid to a company) and L for benefit (German: Leistung) (paid to the policyholder).*

Example 1.1.24 (Savings account)**.** *Consider* $K(t) = e^{\delta t}$ *and* $Z_P = A$, $Z_L = B\mathbb{1}_{[5,\infty)}$, *say with* $\delta = 0.05$ *and* $A = 10000$. *How do we choose* B *such that* Z_P *and* Z_L *are equivalent? The equivalence* $a(Z_P) = a(Z_L)$ *means*

$$\frac{A}{K(0)} = \frac{B}{K(5)}.$$

Since $K(0) = 1$, *it follows*

$$B = K(5)A = e^{5\delta}A.$$

Furthermore, we have

$$V(t) = K(t)\Big(\frac{B}{K(5)}\mathbb{1}_{[0,5]}(t) - A\mathbb{1}_{\{0\}}(t)\Big) = K(t)\frac{B}{K(5)}\mathbb{1}_{(0,5]}(t).$$

Definition 1.1.25. *Let* $Z_L, Z_P \in \mathscr{Z}_g$ *be such that* $\min\{a(Z_L), a(Z_P)\} < \infty$. *For every time* $t \in \mathbb{R}_+$ *the* <u>*retrospective premium reserve*</u> *of* (Z_L, Z_P) *relative to the capital function* K *is defined as*

$${}^{(r)}V(t) := K(t)\bigg[\int_{[0,t)} \frac{dZ_P(s)}{K(s)} - \int_{[0,t)} \frac{dZ_L(s)}{K(s)}\bigg].$$

Lemma 1.1.26. *Let* $Z_L, Z_P \in \mathscr{Z}_g$ *be equivalent payment flows relative to the capital function* K. *Then we have*

$${}^{(r)}V(t) = V(t) \quad \textit{for all } t \in \mathbb{R}_+.$$

Proof. Since $a(Z_L) = a(Z_P)$, we obtain

$$\begin{aligned}
V(t) &= K(t)\bigg[\int_{[t,\infty)} \frac{dZ_L(s)}{K(s)} - \int_{[t,\infty)} \frac{dZ_P(s)}{K(s)}\bigg] \\
&= K(t)\bigg[a(Z_L) - \int_{[0,t)} \frac{dZ_L(s)}{K(s)} - a(Z_P) + \int_{[0,t)} \frac{dZ_P(s)}{K(s)}\bigg] \\
&= K(t)\bigg[\int_{[0,t)} \frac{dZ_P(s)}{K(s)} - \int_{[0,t)} \frac{dZ_L(s)}{K(s)}\bigg] =^{(r)} V(t).
\end{aligned}$$

□

Definition 1.1.27. *Let* $Z \in \mathscr{Z}$ *be a payment flow with decomposition* $Z = Z_P - Z_L$ *for some* $Z_P, Z_L \in \mathscr{Z}_g$. *We call (provided it exists) the minimal* $i \in \mathbb{R}_+$ *such that* $a(Z_P) = a(Z_L)$ *relative to the capital function* $K(t) = (1+i)^t$ *the* <u>*rate of return*</u> *of* Z.

Example 1.1.28. *Consider* $Z_P = \pi \mathbb{1}_{[t_P,\infty)}$ *and* $Z_L = A\mathbb{1}_{[t_L,\infty)}$ *with* $\pi, A \in (0,\infty)$ *and* $t_P, t_L \in \mathbb{R}_+$. *The equivalence* $a(Z_P) = a(Z_L)$ *means*

$$\frac{\pi}{(1+i)^{t_P}} = \frac{A}{(1+i)^{t_L}}.$$

For $t_L \neq t_P$ *it follows*

$$i = \left(\frac{A}{\pi}\right)^{\frac{1}{t_L - t_P}} - 1.$$

Since $i \in \mathbb{R}_+$, *one of the following two conditions has to be satisfied:*

- $t_L > t_P$ *and* $A \geq \pi$.
- $t_L < t_P$ *and* $A \leq \pi$.

1.2 Foundations of life insurance mathematics

1.2.1 Probabilities of death

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Furthermore, let $T_x : \Omega \to (0,\infty)$ die future lifetime of a person of age x. We denote by $F = F_{T_x} : \mathbb{R} \to [0,1]$ its distribution function, ans assume that $F(0) = 0$. We set

$${}_tq_x := \mathbb{P}(T_x \leq t) \in [0,1], \quad t \in \mathbb{R}_+$$

and $q_x := {}_1q_x$. Often, we agree to write $T = T_x$.

Definition 1.2.1. *Das maximal future lifetime is defined as*

$$t_{\max} := \sup\{t \in \mathbb{R}_+ : F(t) < 1\} = \sup\{t \in \mathbb{R}_+ : \mathbb{P}(T > t) > 0\} \in (0,\infty].$$

Definition 1.2.2. *We define the survival function* $\bar{F} : \mathbb{R} \to [0,1]$ *as*

$$\bar{F}(t) := 1 - F(t) = \mathbb{P}(T_x > t).$$

Furthermore, we set ${}_tp_x := \mathbb{P}(T_x > t)$ *and* $p_x := {}_1p_x$.

Definition 1.2.3. *If* T *is absolutely continuous with density* $f : \mathbb{R} \to \mathbb{R}_+$, *then we define*

$$\lambda : (0, t_{\max}) \to \mathbb{R}_+, \quad \lambda(t) := \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\bar{F}(t)}.$$

We call $\lambda(t)$ *the force of mortality at time* t.

Definition 1.2.4. *More generally, we define the* <u>*accumulated force of mortality*</u>

$$\Lambda(t) := \int_{[0,t]} \frac{1}{1-F(u-)} dF(u) \in [0,\infty], \quad t \in \mathbb{R}_+.$$

Remark 1.2.5. *If T is absolutely continuous with density $f : \mathbb{R} \to \mathbb{R}_+$, then we have*

$$\Lambda(t) = \int_0^t \lambda(s)ds, \quad t \in (0, t_{\max}).$$

The survival function satisfies the ODE

$$\begin{cases} \bar{F}'(t) &= -\lambda(t)\bar{F}(t), \quad t \in (0, t_{\max}) \\ \bar{F}(0) &= 1. \end{cases}$$

The unique solution is given by

$$\bar{F}(t) = \exp\left(- \int_0^t \lambda(s)ds \right) = \exp(-\Lambda(t)).$$

Important quantities in life tables:

- ${}_k p_x$ is the survival probability for k years of a person of age x.
- p_x is the survival probability for one year of a person of age x.
- ${}_k q_x$ is the probability of death within k years of a person of age x.
- q_x is the probability of death within one year of a person of age x.
- ℓ_x is the (expected) number of persons reaching age x; often on the basis $\ell_0 =$ 100.000.
- d_x is the (expected) number of persons dying at age x.
- e_x is the expected remaining lifetime of a person of age x.

Example 1.2.6 (Forces of mortality)**.**

- *De Moivre (1724):*

$$\lambda(t) = \frac{1}{t_{\max} - t}, \quad t \in (0, t_{\max}) \quad \text{with } t_{\max} = 86.$$

- *Gompertz (1825):*

$$\lambda(t) = be^{ct} \quad \text{with } b, c > 0.$$

- *Makeham (1860):*

$$\lambda(t) = a + be^{ct} \quad \text{with } a, b, c > 0.$$

- *Weibull (1939):*

$$\lambda(t) = kt^{\gamma} \quad \text{with } k > 0 \text{ and } \gamma > -1.$$

1.2.2 Elements of a life insurance police

Definition 1.2.7. *A stochastic process is a family $(X_t)_{t\in\mathbb{R}_+}$ of $\mathbb{R}$-valued random variables.*

Definition 1.2.8. *A random payment flow is a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ such that for every $\omega \in \Omega$ the path $t \mapsto X_t(\omega)$ belongs to $\mathscr{L}$.*

Definition 1.2.9. *A payment spectrum is a non-negative, measurable function $A : \mathbb{R}_+ \to \mathbb{R}_+$.*

Definition 1.2.10. *A premium function is a monotone increasing, right-continuous function $\Pi : \mathbb{R}_+ \to \mathbb{R}_+$.*

For $t \in \mathbb{R}_+$ the quantity $\Pi(t)$ denotes the sum of a premiums paid until time t.

Definition 1.2.11. *The quantities determining a life insurance police (LIP) are:*

- *F is the distribution function of the future lifetime $T : \Omega \to (0, \infty)$.*
- *$\tau \in (0, t_{\max}]$ is the terminal time of the police.*
- *$Y := \min\{T, \tau\}$ is the (random) time of benefit.*
- *The payment spectrum A. At time Y the amount $A(Y)$ is paid to the policyholder.*
- *The capital function K.*
- *The premium function Π.*

All quantities with exception of T (and hence Y) are assumed to be known and deterministic.

Definition 1.2.12.

(a) The (directed) benefit flow of a LIP is given by

$$Z_L := A(Y)\mathbb{1}_{[\![Y,\infty[\![}.$$

(b) The (undirected) premium flow is given by

$$Z_P := \Pi\mathbb{1}_{[\![0,Y[\![} + \Pi(Y-)\mathbb{1}_{[\![Y,\infty[\![}.$$

(c) The (undirected) payment flow of a LIP is given by

$$Z := Z_L - Z_P.$$

Examples 1.2.13. *Special cases:*

- $\tau = \infty$. *Whole life insurance.*
- $\tau < \infty$ *and* $A(\tau) = 0$. *Term insurance.*
- $\tau < \infty$ *and* $A(t) = 0$ *for* $t < \tau$ *as well as* $A(\tau) > 0$. *Pure endowment.*
- $\tau < \infty$ *and* $A(t) \geq 0$ *for* $t \leq \tau$. *Endowment.*

Definition 1.2.14. *The (random) present value of a LIP in view of the policyholder (PH) is*

$$B = a(Z_L) - a(Z_P).$$

Definition 1.2.15.

(a) The (expected) present value of benefit is given by $\mathbb{E}[a(Z_L)]$.

(b) The (expected) present value of premium is given by $\mathbb{E}[a(Z_P)]$.

(c) A premium function Π *is called net premium function if* $\mathbb{E}[B] = 0$*; that is* $\mathbb{E}[a(Z_L)] = \mathbb{E}[a(Z_P)]$.

Remark 1.2.16. *We have*

$$a(Z_L) = \int_{[0,\infty)} \frac{1}{K(s)} dZ_L(s) = \frac{A(Y)}{K(Y)}$$

and

$$a(Z_P) = \int_{[0,\infty)} \frac{1}{K(s)} dZ_P(s) = \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s).$$

Remark 1.2.17. *We have*

$$F_Y = F\mathbb{1}_{[0,\tau)} + \mathbb{1}_{[\tau,\infty)}$$

and

$$F_Y(ds) = \mathbb{1}_{[0,\tau)}(s)F(ds) + (1 - F(\tau-))\delta_\tau(ds).$$

Lemma 1.2.18.

(a) The expected present value of benefit is given by

$$\begin{aligned}\mathbb{E}[a(Z_L)] &= \int_{[0,\tau]} \frac{A(s)}{K(s)} dF_Y(s) \\ &= \int_{[0,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)}(1 - F(\tau-)).\end{aligned}$$

The first term is the expected present value of benefit in case of death, and the second term is the expected present value of benefit in case of survival.

(b) The expected present value of premium is given by

$$\mathbb{E}[a(Z_P)] = \int_{[0,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s).$$

Proof.

(a) We have

$$\begin{aligned}\mathbb{E}[a(Z_L)] = \mathbb{E}\left[\frac{A(Y)}{K(Y)}\right] &= \int_{\mathbb{R}_+} \frac{A(s)}{K(s)} dF_Y(s) \\ &= \int_{[0,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)}(1 - F(\tau-)).\end{aligned}$$

(b) We have

$$\begin{aligned}\mathbb{E}[a(Z_P)] &= \mathbb{E}\left[\int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s)\right] = \int_{\mathbb{R}_+} \frac{\mathbb{E}[\mathbb{1}_{\{s<Y\}}]}{K(s)} d\Pi(s) = \int_{\mathbb{R}_+} \frac{\mathbb{P}(Y > s)}{K(s)} d\Pi(s) \\ &= \int_{\mathbb{R}_+} \frac{1 - F_Y(s)}{K(s)} d\Pi(s) = \int_{[0,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s).\end{aligned}$$

□

Definition 1.2.19. *A real number $\tilde{\Pi} \in \mathbb{R}_+$ is called a* <u>*net single premium*</u> *(NSP) if $\Pi(t) = \tilde{\Pi}$, $t \in \mathbb{R}_+$ is a net premium function.*

Example 1.2.20. *For $\Pi(t) = \bar{\Pi}$, $t \in \mathbb{R}_+$ we have $a(Z_P) = \tilde{\Pi}$. Thus the NSP is given by*

$$\tilde{\Pi} = \mathbb{E}\left[\frac{A(Y)}{K(Y)}\right].$$

In the special case $A(t) \equiv A$ and $K(t) = e^{\delta t}$ we have

$$\tilde{\Pi} = A \cdot \mathbb{E}\left[e^{-\delta Y}\right].$$

Definition 1.2.21. *A* <u>*running constant premium in advance*</u> Π *at time points* $0 = t_0 < t_1 < \ldots < t_{N-1} < \tau$ *for some* $N \in \mathbb{N}$ *is given by*

$$\Pi = \sum_{k=0}^{N-1} \pi \mathbb{1}_{[t_k,\infty)},$$

where $\pi \in \mathbb{R}_+$ *sis chosen such that* Π *is a net premium function.*

Remark 1.2.22. *For* $N = 1$ *we have a NSP.*

Definition 1.2.23. *The* <u>*natural premium*</u> *(payable at time points* $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = \tau$ *with* $N \in \overline{\mathbb{N}}$*) is given by*

$$\Pi = \sum_{k=0}^{N-1} \pi_k \mathbb{1}_{[t_k,\infty)},$$

where

$$\pi_k = K(t_k)\mathbb{E}\left[\int_{(t_k,t_{k+1}]} \frac{1}{K(s)} dZ_L(s) \,\middle|\, T > t_k\right], \quad k = 0, \ldots, N-1.$$

Proposition 1.2.24. *The following statements are true:*

(a) We have

$$\pi_k = \frac{K(t_k)}{1 - F(t_k)} \int_{(t_k,t_{k+1}]} \frac{A(s)}{K(s)} dF_Y(s), \quad k = 0, \ldots, N-1.$$

(b) The natural premium is a net premium function.

Proof.

(a) We have $\mathbb{P}^{\{T>t_k\}} \ll \mathbb{P}$ with

$$\frac{d\mathbb{P}^{\{T>t_k\}}}{d\mathbb{P}} = \frac{\mathbb{1}_{\{T>t_k\}}}{\mathbb{P}(T > t_k)}.$$

Therefore

$$\begin{aligned}
\pi_k &= K(t_k)\mathbb{E}_{\mathbb{P}^{\{T>t_k\}}}\left[\int_{(t_k,t_{k+1}]} \frac{1}{K(s)} dZ_L(s)\right] \\
&= \frac{K(t_k)}{\mathbb{P}(T > t_k)}\mathbb{E}\left[\int_{(t_k,t_{k+1}]} \frac{1}{K(s)} dZ_L(s)\, \mathbb{1}_{\{T>t_k\}}\right] \\
&= \frac{K(t_k)}{1 - F(t_k)}\mathbb{E}\left[\frac{A(Y)}{K(Y)} \mathbb{1}_{\{Y\in(t_k,t_{k+1}]\}}\right] \\
&= \frac{K(t_k)}{1 - F(t_k)} \int_{(t_k,t_{k+1}]} \frac{A(s)}{K(s)} dF_Y(s).
\end{aligned}$$

(b) Since $F(0) = 0$, we have

$$\begin{aligned}
\mathbb{E}[a(Z_P)] &= \int_{[0,\tau)} \frac{1-F(s)}{K(s)} d\Pi(s) = \sum_{k=0}^{N-1} \frac{1-F(t_k)}{K(t_k)} \pi_k \\
&= \sum_{k=0}^{N-1} \frac{1-F(t_k)}{K(t_k)} \frac{K(t_k)}{1-F(t_k)} \int_{(t_k,t_{k+1}]} \frac{A(s)}{K(s)} dF_Y(s) \\
&= \int_{[0,\tau]} \frac{A(s)}{K(s)} dF_Y(s) = \mathbb{E}[a(Z_L)].
\end{aligned}$$

□

1.2.3 Net premium reserves

We consider a LIP with net premium function Π.

Definition 1.2.25. *The (expected) prospective net premium reserve (NPR) $V(t)$ of a LIP at time $t \in [0, t_{\max})$ is given by*

$$V(t) = K(t)\,\mathbb{E}\left[\frac{A(Y)}{K(Y)}\mathbb{1}_{\{t \le Y\}} - \int_{[\![t,Y[\![} \frac{1}{K(s)} d\Pi(s) \,\middle|\, T > t\right].$$

Lemma 1.2.26. *We have $V(0) = 0$.*

Proof. Indeed, we have

$$\begin{aligned}
V(0) &= K(0)\,\mathbb{E}\left[\frac{A(Y)}{K(Y)}\mathbb{1}_{\{0 \le Y\}} - \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s) \,\middle|\, T > 0\right] \\
&= \mathbb{E}\left[\frac{A(Y)}{K(Y)} - \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s)\right] \\
&= \mathbb{E}[a(Z_L)] - \mathbb{E}[a(Z_P)] = 0.
\end{aligned}$$

□

Lemma 1.2.27.

(a) For all $t \in [0, \tau)$ we have

$$V(t) = \frac{K(t)}{1-F(t)}\left(\int_{(t,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)}(1 - F(\tau-)) - \int_{[t,\tau)} \frac{1-F(s)}{K(s)} d\Pi(s)\right).$$

(b) If $\tau < t_{\max}$, then we have

$$V(\tau) = A(\tau) \quad and \quad V(t) = 0 \quad for\ all\ t \in (\tau, t_{\max}).$$

In particular, we have

$$\lim_{t \uparrow \tau} \frac{V(t)}{K(t)} = \frac{A(\tau)}{K(\tau)} = \frac{V(\tau)}{K(\tau)}.$$

Proof.

(a) For $t \in [0, \tau)$ we have $\{T > t\} = \{Y > t\}$, and hence

$$\begin{aligned} V(t) &= \frac{K(t)}{\mathbb{P}(T > t)} \mathbb{E}\Bigg[\bigg(\frac{A(Y)}{K(Y)} \mathbb{1}_{\{t \le Y\}} - \int_{[\![t,Y[\![} \frac{1}{K(s)} d\Pi(s)\bigg) \mathbb{1}_{\{T > t\}}\Bigg] \\ &= \frac{K(t)}{1 - F(t)} \bigg(\int_{(t,\tau]} \frac{A(s)}{K(s)} dF_Y(s) - \int_{[t,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s) \bigg) \\ &= \frac{K(t)}{1 - F(t)} \bigg(\int_{(t,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)} (1 - F(\tau-)) - \int_{[t,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s) \bigg). \end{aligned}$$

(b) We have $\tau < t_{\max}$. For $t \in (\tau, t_{\max})$ we have $V(t) = 0$, because $Y \le \tau < t$. For $t = \tau$ we have $\{T > t\} = \{T > \tau\} = \{Y = \tau\}$, and hence

$$V(\tau) = K(\tau) \, \mathbb{E}\bigg[\frac{A(\tau)}{K(\tau)} \mathbb{1}_{\{t \le \tau\}} \,\bigg|\, T > \tau\bigg] = A(\tau).$$

Moreover, we have

$$\lim_{t \uparrow \tau} \frac{V(t)}{K(t)} = \lim_{t \uparrow \tau} \frac{1}{1 - F(t)} \frac{A(\tau)}{K(\tau)} (1 - F(\tau-)) = \frac{A(\tau)}{K(\tau)} = \frac{V(\tau)}{K(\tau)}.$$

□

Lemma 1.2.28. *We have the retrospective representation*

$$V(t) = \frac{K(t)}{1 - F(t)} \bigg(- \int_{[0,t]} \frac{A(s)}{K(s)} dF(s) + \int_{[0,t)} \frac{1 - F(s)}{K(s)} d\Pi(s) \bigg).$$

for all $t \in [0, \tau)$.

Proof. By Lemma 1.2.18 we have

$$
\begin{aligned}
0 &= \mathbb{E}[a(Z_L)] - \mathbb{E}[a(Z_P)] \\
&= \int_{[0,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)}(1 - F(\tau-)) - \int_{[0,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s) \\
&= \int_{[0,t]} \frac{A(s)}{K(s)} dF(s) + \int_{(t,\tau)} \frac{A(s)}{K(s)} dF(s) + \frac{A(\tau)}{K(\tau)}(1 - F(\tau-)) \\
&\quad - \int_{[0,t)} \frac{1 - F(s)}{K(s)} d\Pi(s) - \int_{[t,\tau)} \frac{1 - F(s)}{K(s)} d\Pi(s)
\end{aligned}
$$

Therefore, by Lemma 1.2.27 we have

$$
(1 - F(t))\frac{V(t)}{K(t)} = -\int_{[0,t]} \frac{A(s)}{K(s)} dF(s) + \int_{[0,t)} \frac{1 - F(s)}{K(s)} d\Pi(s),
$$

and hence the claimed retrospective representation. □

1.2.4 The Thiele differential equation

We assume there exist non negative, continuous functions $k, f, \pi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$
\begin{aligned}
K(t) &= 1 + \int_0^t k(s)ds, \quad t \in [0, \tau), \\
F(t) &= \int_0^t f(s)ds, \quad t \in [0, \tau), \\
\Pi(t) &= \int_0^t \pi(s)ds \quad t \in [0, \tau).
\end{aligned}
$$

Furthermore, we assume that A is continuous on $[0, \tau)$. Recall the force of interest

$$
\phi(t) = \frac{k(t)}{K(t)}
$$

and the force of mortality

$$
\lambda(t) = \frac{f(t)}{1 - F(t)}.
$$

Theorem 1.2.29. *The net premium reserve V satisfies the* <u>*Thiele differential equation*</u>

$$
\begin{cases}
V'(t) = \phi(t)V(t) + \pi(t) + \lambda(t)(V(t) - A(t)), & t \in [0, \tau) \\
V(0) = 0.
\end{cases}
$$

Proof. By Lemma 1.2.26 we have $V(0) = 0$, and by Lemma 1.2.28 we have

$$V(t) = \frac{K(t)}{1-F(t)}\bigg(- \int_0^t \frac{A(s)}{K(s)} f(s)ds + \int_0^t \frac{1-F(s)}{K(s)}\pi(s)ds\bigg).$$

For the function

$$W(t) := \frac{K(t)}{1-F(t)}, \quad t \in [0,\tau)$$

we obtain

$$\begin{aligned} W'(t) &= \frac{(1-F(t))k(t) + K(t)f(t)}{(1-F(t))^2} = \frac{k(t)}{1-F(t)} + \frac{K(t)f(t)}{(1-F(t))^2} \\ &= \frac{k(t)}{K(t)}W(t) + \frac{f(t)}{1-F(t)}W(t) = \phi(t)W(t) + \lambda(t)W(t). \end{aligned}$$

Therefore

$$\begin{aligned} V'(t) &= \phi(t)V(t) + \lambda(t)V(t) + \frac{K(t)}{1-F(t)}\bigg(- \frac{A(t)}{K(t)}f(t) + \frac{1-F(t)}{K(t)}\pi(t)\bigg) \\ &= \phi(t)V(t) + \lambda(t)V(t) - \lambda(t)A(t) + \pi(t). \end{aligned}$$

□

Proposition 1.2.30. *The Thiele differential equation has the unique solution*

$$V(t) = \int_0^t (\pi(s) - \lambda(s)A(s)) \exp\bigg(\int_s^t (\phi(u) + \lambda(u))du\bigg)ds.$$

Proof. Exercise. □

Definition 1.2.31.

(a) We call

$$\pi^s(t) = V'(t) - \phi(t)V(t)$$

the <u>savings component</u>.

(b) We call

$$\pi^r(t) = (A(t) - V(t))\lambda(t)$$

the <u>risk component</u>.

Remark 1.2.32. *Then we have the decomposition*

$$\pi(t) = \pi^s(t) + \pi^r(t).$$

Remark 1.2.33. *Suppose that $A(\tau) = 0$. The continuous analogue of Definition 1.2.23 is*

$$\pi^{\text{nat}}(s) := \lambda(s)A(s) = \frac{f(s)}{1 - F(s)}A(s).$$

By Lemma 1.2.27 for all $t \in [0, \tau)$ we have

$$V(t) = \frac{K(t)}{1 - F(t)}\left(\int_t^\tau \frac{A(s)}{K(s)}f(s)ds - \int_t^\tau \frac{1 - F(s)}{K(s)}\pi^{\text{nat}}(s)ds\right) = 0.$$

Therefore $V = V' = 0$, and we obtain

$$\pi^{\text{nat}}(t) = A(t)\lambda(t) = \pi^r(t).$$

1.2.5 The Thiele integral equation

Recall the accumulated force of mortality

$$\Lambda(t) = \int_{[0,t]} \frac{1}{1 - F(s-)}dF(s).$$

Theorem 1.2.34. *The net premium reserve V satisfies the Thiele integral equation*

$$\frac{V(t)}{K(t)} = \int_{[0,t)} \frac{1}{K(s)}d\Pi(s) - \int_{(0,t]} \frac{A(u) - V(u)}{K(u)}d\Lambda(u), \quad t \in [0, \tau).$$

Proof. See [BOS17, Satz 2.83]. □

Remark 1.2.35. *Under the assumption of the previous section we obtain the Thiele differential equation from Theorem 1.2.29.*

Proof. Exercise. □

Chapter 2

Hattendorf's theorem

2.1 Net single premium and the variance of the present value

Lemma 2.1.1. *Let X be a random variable and let $f, g : \mathbb{R} \to \mathbb{R}$ be two measurable, increasing functions such that $f(X), g(X) \in \mathscr{L}^1$. Then $f(X)g(X)$ is quasi-integrable and*

$$\mathbb{E}[f(X)]\mathbb{E}[g(X)] \leq \mathbb{E}[f(X)g(X)] \in (-\infty, \infty].$$

Proof. By the monotonicity of f and g we have

$$\big(f(y) - f(x)\big)\big(g(y) - g(x)\big) \geq 0 \quad \text{for all } x, y \in \mathbb{R}.$$

Therefore

$$\mathbb{E}\big[\big(f(y) - f(X)\big)\big(g(y) - g(X)\big)\big] \geq 0 \quad \text{for all } y \in \mathbb{R}.$$

Hence, we have

$$f(y)g(y) - f(y)\mathbb{E}[g(X)] - g(y)\mathbb{E}[f(X)] + \mathbb{E}[f(X)g(X)] \geq 0 \quad \text{for all } y \in \mathbb{R}.$$

This shows the quasi-integrability of $f(X)g(X)$ with $\mathbb{E}[f(X)g(X)] \in (-\infty, \infty]$ and

$$f(X)g(X) - f(X)\mathbb{E}[g(X)] - g(X)\mathbb{E}[f(X)] + \mathbb{E}[f(X)g(X)] \geq 0,$$

and hence by taking expectation again

$$2\mathbb{E}[f(X)g(X)] - 2\mathbb{E}[f(X)]\mathbb{E}[g(X)] \geq 0.$$

$\square$

Proposition 2.1.2. *Suppose the function A/K is decreasing. Then for a net premium function Π the following statements are equivalent:*

(i) $\mathrm{Var}[B]$ *is minimal under all net premium functions.*

(ii) Π *is a net single premium.*

Proof. Let Π be a net premium function. We define

$$\bar{A}(t) := \frac{A(t)}{K(t)}, \quad \bar{\Pi}(t) := \int_{[0,t)} \frac{1}{K(s)} d\Pi(s).$$

Since $K(0) = 1$, we have $\bar{A}(t) \leq A(0)$ for all $t \in \mathbb{R}_+$. Hence $\bar{A}(Y)$ is a bounded random variable, and hence in particular $\mathrm{Var}[\bar{A}(Y)] < \infty$. Now we distinguish two cases:

- $\mathrm{Var}[\bar{\Pi}(Y)] = \infty$. Then we have

$$\mathrm{Var}[B] = \mathrm{Var}[\bar{A}(Y) - \bar{\Pi}(Y)] = \infty.$$

  Indeed, otherwise we would have $B \in \mathscr{L}^2$, which leads to the contradiction $\bar{\Pi}(Y) \in \mathscr{L}^2$.

- $\mathrm{Var}[\bar{\Pi}(Y)] < \infty$. Then we have

$$\begin{aligned}\mathrm{Var}[B] &= \mathrm{Var}[\bar{A}(Y) - \bar{\Pi}(Y)] \\ &= \mathrm{Var}[\bar{A}(Y)] - 2\mathrm{Cov}(\bar{A}(Y), \bar{\Pi}(Y)) + \mathrm{Var}[\bar{\Pi}(Y)].\end{aligned}$$

  If Π is a net single premium, then $\bar{\Pi}(Y) = \Pi(0)$ is deterministic, and we obtain

$$\mathrm{Var}[B] = \mathrm{Var}[\bar{A}(Y)].$$

  Now, we suppose that Π is not a net single premium. The function $\bar{\Pi}$ is increasing, and by hypothesis the function $-\bar{A}$ is increasing as well. By Lemma 2.1.1 we obtain

$$\begin{aligned}\mathrm{Cov}(\bar{A}(Y), \bar{\Pi}(Y)) &= \mathbb{E}[\bar{A}(Y)\bar{\Pi}(Y)] - \mathbb{E}[\bar{A}(Y)]\mathbb{E}[\bar{\Pi}(Y)] \\ &= \mathbb{E}[-\bar{A}(Y)]\mathbb{E}[\bar{\Pi}(Y)] - \mathbb{E}[-\bar{A}(Y)\bar{\Pi}(Y)] \leq 0.\end{aligned}$$

  Therefore, we have

$$\mathrm{Var}[\bar{A}(Y)] \leq \mathrm{Var}[B].$$

$\square$

Remark 2.1.3. *For a constant payment spectrum A the function A/K is decreasing.*

2.2 Martingales

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 2.2.1. *A family $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{R}_+}$ of sub-σ-algebras of $\mathscr{F}$ is called a filtration if $\mathscr{F}_s \subset \mathscr{F}_t$ for all $0 \leq s \leq t < \infty$.*

Let $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{R}_+}$ be a filtration.

Definition 2.2.2. *A stochastic process X is called adapted if for each $t \in \mathbb{R}_+$ the random variable X_t is $\mathscr{F}_t$-measurable.*

Definition 2.2.3. *Let X be an adapted process such that $X_t \in \mathscr{L}^1$ for all $t \in \mathbb{R}_+$.*

(a) X is called a martingale if

$$\mathbb{E}[X_t \,|\, \mathscr{F}_s] = X_s \quad \mathbb{P}\text{-almost surely} \quad \text{for all } 0 \leq s \leq t < \infty.$$

(b) X is called a submartingale if

$$\mathbb{E}[X_t \,|\, \mathscr{F}_s] \geq X_s \quad \mathbb{P}\text{-almost surely} \quad \text{for all } 0 \leq s \leq t < \infty.$$

(c) X is called a supermartingale if

$$\mathbb{E}[X_t \,|\, \mathscr{F}_s] \leq X_s \quad \mathbb{P}\text{-almost surely} \quad \text{for all } 0 \leq s \leq t < \infty.$$

Definition 2.2.4. *A martingale M is called square-integrable if $M_t \in \mathscr{L}^2$ for all $t \in \mathbb{R}_+$.*

Lemma 2.2.5. *Let M be a square-integrable martingale. Then we have*

$$\operatorname{Cov}(M_t - M_s, M_v - M_u) = 0 \quad \text{for all } 0 \leq s \leq t \leq u \leq v < \infty.$$

Proof. We have

$$\begin{aligned}
\operatorname{Cov}(M_t - M_s, M_v - M_u) &= \mathbb{E}[(M_t - M_s)(M_v - M_u)] \\
&= \mathbb{E}[\mathbb{E}[(M_t - M_s)(M_v - M_u) \,|\, \mathscr{F}_u]] \\
&= \mathbb{E}[(M_t - M_s) \underbrace{\mathbb{E}[M_v - M_u \,|\, \mathscr{F}_u]}_{=0}] = 0.
\end{aligned}$$

□

Lemma 2.2.6. *Let M be a square-integrable martingale. Then we have*

$$\mathbb{E}[(M_t - M_s)^2] = \mathbb{E}[M_t^2 - M_s^2] \quad \text{for all } 0 \leq s \leq t < \infty.$$

Proof. We have

$$\begin{aligned}\mathbb{E}[(M_t - M_s)^2 \,|\, \mathscr{F}_s] &= \mathbb{E}[M_t^2 - 2M_tM_s + M_s^2 \,|\, \mathscr{F}_s] \\ &= \mathbb{E}[M_t^2 \,|\, \mathscr{F}_s] - 2M_s\mathbb{E}[M_t \,|\, \mathscr{F}_s] + \mathbb{E}[M_s^2 \,|\, \mathscr{F}_s] \\ &= \mathbb{E}[M_t^2 \,|\, \mathscr{F}_s] - 2M_s^2 + M_s^2 = \mathbb{E}[M_t^2 - M_s^2 \,|\, \mathscr{F}_s],\end{aligned}$$

and hence

$$\mathbb{E}[(M_t - M_s)^2] = \mathbb{E}[\mathbb{E}[(M_t - M_s)^2 \,|\, \mathscr{F}_s]] = \mathbb{E}[\mathbb{E}[M_t^2 - M_s^2 \,|\, \mathscr{F}_s]] = \mathbb{E}[M_t^2 - M_s^2].$$

□

2.3 Hattendorf's theorem

Recall that $Y = \min\{T, \tau\}$.

Definition 2.3.1. *We define the process*

$$N := \mathbb{1}_{[\![Y,\infty[\![}.$$

Remark 2.3.2. *Then we have*

$$N_t = \mathbb{1}_{[\![Y,\infty[\![}(t) = \mathbb{1}_{\{Y\le t\}} = \mathbb{1}_{[0,t]}(Y)$$

for all $t \in \mathbb{R}_+$.

Definition 2.3.3. *We define the canonical filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ as*

$$\mathscr{F}_t := \sigma(N_s : s \in [0,t]), \quad t \in \mathbb{R}_+.$$

Remark 2.3.4. *Then we have*

$$\mathscr{F}_t = \sigma(\{Y \le s\} : s \in [0,t]) = \sigma(\min\{Y,t\}) \vee \{Y = t\}, \quad t \in \mathbb{R}_+.$$

Hence, the filtration contains at time t precisely the information, required in order to decide whether (and, if applicable, when) Y has occurred until time t or not.

Definition 2.3.5. *The <u>accumulated force of mortality</u> for Y is given by*

$$\Lambda_Y(t) := \int_{(0,t]} \frac{1}{1 - F_Y(u-)} dF_Y(u).$$

If Y is absolutely continuous with density f_Y, then we define the <u>force of mortality</u>

$$\lambda_Y(t) := \frac{f_Y(t)}{1 - F_Y(t)}.$$

Lemma 2.3.6. *The following statements are equivalent:*

(i) We have $\lim_{t\uparrow\tau}\Lambda_Y(t)=\infty$.

(ii) We have either $\tau=\infty$ *or both* $\tau<\infty$ *and* $F_Y(\tau-)=1$.

Proof. Exercise $\square$

Let us briefly consider the situation $\tau<\infty$ and $F_Y(\tau-)=1$. Then we have $\tau=t_{\max}$. Indeed, since $\tau\in(0,t_{\max}]$ we have $\tau\le t_{\max}$. Moreover, we have

$$\begin{aligned}\mathbb{P}(T>\tau)&\le\mathbb{P}(\min\{T,\tau\}=\tau)=\mathbb{P}(Y=\tau)\\&=1-\mathbb{P}(Y<\tau)=1-F_Y(\tau-)=0,\end{aligned}$$

and it follows

$$t_{\max}=\sup\{t\in\mathbb{R}_+:\mathbb{P}(T>t)>0\}\le\tau.$$

Therefore τ is the maximal future lifetime. Hence, it is intuitively clear that for $t\uparrow\tau$ the accumulated force of mortality at time t tends to ∞.

Remark 2.3.7. *Recall that*

$$F_Y=F\mathbb{1}_{[0,\tau)}+\mathbb{1}_{[\tau,\infty)}$$

and

$$\begin{aligned}F_Y(ds)&=\mathbb{1}_{[0,\tau)}(s)F(ds)+(1-F(\tau-))\delta_\tau(ds)\\&=\mathbb{1}_{[0,\tau)}(s)F(ds)+(1-F_Y(\tau-))\delta_\tau(ds).\end{aligned}$$

Lemma 2.3.8. *If* $\tau<\infty$*, then we have*

$$\Delta\Lambda_Y(\tau)=\begin{cases}1, & \textit{falls } F(\tau-)<1,\\0, & \textit{falls } F(\tau-)=1.\end{cases}$$

Proof. We have $\Delta F_Y(\tau)=1-F(\tau-)$ and

$$\begin{aligned}\Delta\Lambda_Y(\tau)&=\lim_{h\downarrow0}\big(\Lambda_Y(\tau)-\Lambda_Y(\tau-h)\big)=\lim_{h\downarrow0}\int_{(\tau-h,\tau]}\frac{1}{1-F_Y(u-)}dF_Y(u)\\&=\int_{\{\tau\}}\frac{1}{1-F_Y(u-)}dF_Y(u).\end{aligned}$$

$\square$

Remark 2.3.9. *From now on, we assume that* $F(\tau-)<1$*, provided that* $\tau<\infty$.

Lemma 2.3.10. *Let $A \subset (s, \infty)$ be an interval for some $s \in [0, \tau)$. Then we have*

$$\mathbb{P}(Y \in A \,|\, \mathscr{F}_s) = \frac{\mathbb{P}(Y \in A)}{1 - F_Y(s)} \mathbb{1}_{\{Y > s\}} \quad \mathbb{P}\text{-almost surely.}$$

Proof. The random variable on the right-hand side is $\mathscr{F}_s$-measurable. The system

$$\mathscr{G}_s = \{\{Y > r\} : r \in [0, s]\}$$

is a $\cap$-stable system such that $\mathscr{F}_s = \sigma(\mathscr{G}_s)$. Moreover, for each $r \in [0, s]$ we have

$$\begin{aligned} \mathbb{E}\left[\frac{\mathbb{P}(Y \in A)}{1 - F_Y(s)} \mathbb{1}_{\{Y > s\}} \mathbb{1}_{\{Y > r\}}\right] &= \frac{\mathbb{P}(Y \in A)}{\mathbb{P}(Y > s)} \mathbb{P}(\{Y > r\} \cap \{Y > s\}) \\ &= \mathbb{P}(Y \in A) = \mathbb{E}[\mathbb{1}_{\{Y \in A\}} \mathbb{1}_{\{Y > r\}}], \end{aligned}$$

since by hypothesis $A \subset (s, \infty) \subset (r, \infty)$, and hence

$$\{Y \in A\} \cap \{Y > r\} = \{Y \in A\}.$$

$\square$

Definition 2.3.11. *We define the process M as*

$$M_t := N_t - \int_{[\![0, t \wedge Y]\!]} d\Lambda_Y(u), \quad t \in \mathbb{R}_+.$$

Here we call the process

$$\left(\int_{[\![0, t \wedge Y]\!]} d\Lambda_Y(u)\right)_{t \in \mathbb{R}_+} = \left(\int_{[\![0, t \wedge Y]\!]} \frac{1}{1 - F_Y(u-)} dF_Y(u)\right)_{t \in \mathbb{R}_+}$$

the <u>compensator</u> of N.

Within the general semimartingale theory of stochastic processes, this process is the predictable compensator N^p.

Remark 2.3.12. *For all $t \in \mathbb{R}_+$ we have*

$$\begin{aligned} \int_{[\![0, t \wedge Y]\!]} d\Lambda_Y(u) &= \int_{[0, t]} \mathbb{1}_{[\![0, Y]\!]}(u) d\Lambda_Y(u) \\ &= \int_{[0, t]} \mathbb{1}_{[u, \infty)}(Y) d\Lambda_Y(u) = \int_{[0, t]} \mathbb{1}_{\{u \leq Y\}} d\Lambda_Y(u). \end{aligned}$$

If Y is absolutely continuous with density f_Y, then we have

$$\int_{[\![0, t \wedge Y]\!]} d\Lambda_Y(u) = \int_{[\![0, t \wedge Y]\!]} \lambda_Y(u) du.$$

Proposition 2.3.13.

(a) M is a right-continuous martingale with $M_0 = 0$ and $M_t = M_Y$ for all $t \geq Y$.

(b) If $\tau < \infty$, then M is continuous at τ.

Proof.

(a) Since $Y > 0$ and $F(0) = 0$, we have

$$M_0 = \mathbb{1}_{\{0\}}(Y) - \Lambda_Y(\{0\}) = 0.$$

Moreover, for $t \geq Y$ we have

$$N_t = 1 = N_Y \quad \text{and}$$
$$\int_{[\![0,t\wedge Y]\!]} d\Lambda_Y(u) = \int_{[\![0,Y]\!]} d\Lambda_Y(u),$$

and hence $M_t = M_Y$.

It is clear that M is an adapted process. Furthermore, we have $M_t \in \mathscr{L}^1$ for all $t \in \mathbb{R}_+$. Indeed, we have

$$\begin{aligned}
\mathbb{E}[|M_t|] &\leq \mathbb{E}[\mathbb{1}_{[0,t]}(Y)] + \mathbb{E}[\Lambda_Y(Y \wedge t)] \\
&= \mathbb{P}(Y \in [0,t]) + \mathbb{E}\left[\int_{[0,t]} \mathbb{1}_{[\![0,Y]\!]}(u) d\Lambda_Y(u) \right] \\
&= F_Y(t) + \int_{[0,t]} \mathbb{P}(Y \geq u) \frac{1}{1 - F_Y(u-)} dF_Y(u) \\
&= F_Y(t) + F_Y(t) = 2F_Y(t) \leq 2.
\end{aligned}$$

If $\tau < \infty$, then we have $M_t = M_\tau$ for all $t \geq \tau$. Hence, it suffices to show that $\mathbb{E}[M_t \,|\, \mathscr{F}_s] = M_s$ $\mathbb{P}$-almost surely for all $0 \leq s < t < \infty$ with $t \leq \tau$. Using Fubini's theorem for conditional expectations we obtain $\mathbb{P}$-almost surely

$$\begin{aligned}
\mathbb{E}[M_t \,|\, \mathscr{F}_s] &= \mathbb{E}[N_t \,|\, \mathscr{F}_s] - \mathbb{E}\left[\int_{[\![0,t\wedge Y]\!]} d\Lambda_Y(u) \,\middle|\, \mathscr{F}_s \right] \\
&= \mathbb{E}[\mathbb{1}_{[0,t]}(Y) \,|\, \mathscr{F}_s] - \int_{[0,t]} \mathbb{E}[\mathbb{1}_{[u,\infty)}(Y) \,|\, \mathscr{F}_s] d\Lambda_Y(u) \\
&= \mathbb{1}_{[0,s]}(Y) - \int_{[0,s]} \mathbb{1}_{[u,\infty)}(Y) d\Lambda_Y(u) \\
&\quad + \mathbb{E}[\mathbb{1}_{(s,t]}(Y) \,|\, \mathscr{F}_s] - \int_{(s,t]} \mathbb{E}[\mathbb{1}_{[u,\infty)}(Y) \,|\, \mathscr{F}_s] d\Lambda_Y(u) \\
&= M_s + \mathbb{P}(Y \in (s,t] \,|\, \mathscr{F}_s) - \int_{(s,t]} \mathbb{P}(Y \in [u,\infty) \,|\, \mathscr{F}_s) d\Lambda_Y(u).
\end{aligned}$$

Using Lemma 2.3.10 two times yields $\mathbb{P}$-almost surely

$$\begin{aligned}\int_{(s,t]} \mathbb{P}(Y \in [u,\infty) \,|\, \mathscr{F}_s) d\Lambda_Y(u) &= \int_{(s,t]} \frac{\mathbb{P}(Y \in [u,\infty))}{1 - F_Y(s)} \mathbb{1}_{\{Y>s\}} d\Lambda_Y(u) \\ &= \mathbb{1}_{\{Y>s\}} \int_{(s,t]} \frac{1 - F_Y(u-)}{1 - F_Y(s)} \frac{1}{1 - F_Y(u-)} dF_Y(u) \\ &= \frac{\mathbb{P}(Y \in (s,t])}{\mathbb{P}(Y > s)} \mathbb{1}_{\{Y>s\}} = \mathbb{P}(Y \in (s,t] \,|\, \mathscr{F}_s).\end{aligned}$$

The right-continuity of the martingale M immediately follows from Definition 2.3.11.

(b) Suppose $\tau < \infty$. We set

$$N_t^p := \int_{[\![0,t\wedge Y]\!]} d\Lambda_Y(u), \quad t \in \mathbb{R}_+.$$

Then we have $M = N - N^p$. By Definition 2.3.1 we also have $N = \mathbb{1}_{[\![Y,\infty[\![}$. If $Y < \tau$, then we have $\Delta N_\tau = \Delta N_\tau^p = 0$, and hence $\Delta M_\tau = 0$. If $Y = \tau$, then we have $\Delta N_\tau = 1$. Since $F_Y(\tau-) < 1$, by Lemma 2.3.8 we obtain $\Delta\Lambda_Y(\tau) = 1$. Therefore $\Delta M_\tau = 0$.

□

In the proof of Proposition 2.3.13 we have used:

Proposition 2.3.14 (Fubini's theorem for conditional expectations). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $(X, \mathscr{X}, \mu)$ be a finite measure space. Let $f : (\Omega \times X, \mathscr{F} \otimes \mathscr{X}) \to \mathbb{R}_+$ be a product-measurable, non-negative, bounded function. Furthermore, let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra, and let $g : (\Omega \times X, \mathscr{G} \otimes \mathscr{X}) \to \mathbb{R}_+$ be a product-measurable, non-negative, bounded function such that for each $x \in X$ the mapping $g(\cdot, x) : (\Omega, \mathscr{G}) \to \mathbb{R}_+$ is a version of the conditional expectation $\mathbb{E}[f(\cdot, x) \,|\, \mathscr{G}]$. Then we have*

$$\mathbb{E}\bigg[\int_X f(\cdot, x)\mu(dx) \bigg| \mathscr{G} \bigg] = \int_X g(\cdot, x)\mu(dx) \quad \mathbb{P}\text{-}almost\ surely.$$

Proof. By Fubini's theorem the mapping

$$\int_X f(\cdot, x)\mu(dx) : \Omega \to \mathbb{R}_+$$

is bounded and $\mathscr{F}$-measurable, and the mapping

$$\int_X g(\cdot, x)\mu(dx) : \Omega \to \mathbb{R}_+$$

is bounded and $\mathscr{G}$-measurable. Furthermore, for every non-negative $\mathscr{G}$-measurable random variable $Z : \Omega \to \mathbb{R}_+$ we have by Fubini's theorem

$$\begin{aligned}\mathbb{E}\Big[Z\int_X f(\cdot,x)\mu(dx)\Big] &= \int_X \mathbb{E}[Zf(\cdot,x)]\mu(dx) \\ &= \int_X \mathbb{E}[Zg(\cdot,x)]\mu(dx) = \mathbb{E}\Big[Z\int_X g(\cdot,x)\mu(dx)\Big].\end{aligned}$$

□

Remark 2.3.15. *In the proof of Proposition 2.3.13 we have used Fubini's theorem for conditional expectations (Proposition 2.3.14) for* $0 \le s < t < \infty$ *with*

$$\begin{aligned}(X,\mathscr{X},\mu) &= ((s,t],\mathcal{B}((s,t]),\Lambda_Y(du)), \\ f(\cdot,u) &= \mathbb{1}_{[u,\infty)}(Y), \\ \mathscr{G} &= \mathscr{F}_s \\ g(\cdot,u) &= \frac{\mathbb{P}(Y\in[u,\infty))}{1-F_Y(s)}\mathbb{1}_{\{Y>s\}},\end{aligned}$$

and so we have obtained

$$\mathbb{E}\Big[\int_{(s,t]}\mathbb{1}_{[u,\infty)}(Y)d\Lambda_Y(u)\,|\,\mathscr{F}_s\Big] = \int_{(s,t]}\frac{\mathbb{P}(Y\in[u,\infty))}{1-F_Y(s)}\mathbb{1}_{\{Y>s\}}d\Lambda_Y(u) \quad \mathbb{P}\text{-almost surely.}$$

Example 2.3.16. *We assume that* $\tau = \infty$ *and* $T \sim \mathrm{Exp}(1)$ *(whole life insurance). Then we also have* $Y \sim \mathrm{Exp}(1)$*, and for all* $t \in \mathbb{R}_+$ *we have*

$$\begin{aligned}F_Y(t) &= 1-\exp(-t), \\ f_Y(t) &= \exp(-t), \\ \lambda_Y(t) &= \frac{f_Y(t)}{1-F_Y(t)} = \frac{\exp(-t)}{1-(1-\exp(-t))} = 1.\end{aligned}$$

Therefore, by Remark 2.3.12 we have

$$\begin{aligned}M_t &= \mathbb{1}_{[\![Y,\infty[\![}(t) - \int_{[\![0,t\wedge Y]\!]} du \\ &= \mathbb{1}_{[\![Y,\infty[\![}(t) - \Big(t\mathbb{1}_{[\![0,Y[\![}(t) + Y\mathbb{1}_{[\![Y,\infty[\![}(t)\Big) \\ &= -t\mathbb{1}_{[\![0,Y[\![}(t) + (1-Y)\mathbb{1}_{[\![Y,\infty[\![}(t).\end{aligned}$$

In coincidence with Proposition 2.3.13 for each $t \in \mathbb{R}_+$ we have

$$\begin{aligned}\mathbb{E}[M_t] &= \mathbb{E}\big[-t\mathbb{1}_{[\![0,Y[\![}(t) + (1-Y)\mathbb{1}_{[\![Y,\infty[\![}(t)\big] \\ &= -t\mathbb{P}(Y>t) + \mathbb{E}[(1-Y)\mathbb{1}_{[0,t]}(Y)] \\ &= -t\exp(-t) + \int_0^t (1-u)\exp(-u)du \\ &= -t\exp(-t) + \Big(1-\exp(-t)\Big) + \Big(t\exp(-t) + \exp(-t) - 1\Big) = 0.\end{aligned}$$

Example 2.3.17. *We assume that $\tau = \infty$ and that T has a discrete distribution with*

$$\mathbb{P}(T=1) = \mathbb{P}(T=2) = \frac{1}{2}.$$

Then we have

$$F_Y = \frac{1}{2}\mathbb{1}_{[1,2)} + \mathbb{1}_{[2,\infty)},$$

and it follows

$$\begin{aligned}M_t &= \mathbb{1}_{[\![Y,\infty[\![}(t) - \int_{[\![0,t\wedge Y]\!]} \frac{1}{1-F_Y(u-)} dF_Y(u) \\ &= \mathbb{1}_{[\![Y,\infty[\![}(t) - \left(\frac{1}{2}\mathbb{1}_{[1,\infty)}(t\wedge Y) + \mathbb{1}_{[2,\infty)}(t\wedge Y)\right) \\ &= \mathbb{1}_{[\![Y,\infty[\![}(t) - \left(\frac{1}{2}\mathbb{1}_{[1,\infty)}(t) + \mathbb{1}_{[2,\infty)}(t)\mathbb{1}_{\{Y=2\}}\right) \\ &= \frac{1}{2}\big(\mathbb{1}_{\{Y=1\}} - \mathbb{1}_{\{Y=2\}}\big)\mathbb{1}_{[1,\infty)}(t).\end{aligned}$$

Here we also see that

$$\mathbb{E}[M_t] = 0 \quad \text{for all } t \in \mathbb{R}_+.$$

Proposition 2.3.18. *M is a square-integrable martingale, and we have*

$$\mathbb{E}[(M_t - M_s)^2] = \int_{(s,t]} (1-\Delta\Lambda_Y(u))dF_Y(u) \quad \text{for all } 0 \le s \le t < \infty.$$

Proof. We have

$$M_t = \mathbb{1}_{\{Y \le t\}} - \int_{(0,t]} \mathbb{1}_{\{u \le Y\}} d\Lambda_Y(u).$$

Hence, by Fubini's theorem we have

$$
\begin{aligned}
\mathbb{E}[(M_t - M_s)^2] &= \mathbb{E}\left[\left(\mathbb{1}_{\{s<Y\le t\}} - \int_{(s,t]} \mathbb{1}_{\{u\le Y\}} d\Lambda_Y(u)\right)^2\right] \\
&= \mathbb{E}\big[\mathbb{1}_{\{s<Y\le t\}}\big] - 2\mathbb{E}\left[\int_{(s,t]} \mathbb{1}_{\{s<Y\le t\}}\mathbb{1}_{\{u\le Y\}} d\Lambda_Y(u)\right] \\
&\quad + \mathbb{E}\left[\int_{(s,t]}\int_{(s,t]} \mathbb{1}_{\{u\le Y\}}\mathbb{1}_{\{v\le Y\}} d\Lambda_Y(u) d\Lambda_Y(v)\right] \\
&= F_Y(t) - F_Y(s) - 2\int_{(s,t]} \mathbb{P}(Y \in [u,t]) d\Lambda_Y(u) \\
&\quad + \int_{(s,t]}\int_{(s,t]} \mathbb{P}(Y \ge \max\{u,v\}) d\Lambda_Y(u) d\Lambda_Y(v).
\end{aligned}
$$

Using Fubini's theorem we obtain

$$
\begin{aligned}
&\int_{(s,t]}\int_{(s,t]} \mathbb{P}(Y \ge \max\{u,v\}) d\Lambda_Y(u) d\Lambda_Y(v) \\
&= \int_{(s,t]}\int_{(s,v]} \mathbb{P}(Y \ge v) d\Lambda_Y(u) d\Lambda_Y(v) + \int_{(s,t]}\int_{(v,t]} \mathbb{P}(Y \ge u) d\Lambda_Y(u) d\Lambda_Y(v) \\
&= \int_{(s,t]}\int_{[u,t]} \mathbb{P}(Y \ge v) d\Lambda_Y(v) d\Lambda_Y(u) + \int_{(s,t]}\int_{(v,t]} \mathbb{P}(Y \ge u) d\Lambda_Y(u) d\Lambda_Y(v) \\
&= \int_{(s,t]}\int_{[u,t]} \mathbb{P}(Y \ge v) d\Lambda_Y(v) d\Lambda_Y(u) + \int_{(s,t]}\int_{[u,t]} \mathbb{P}(Y \ge v) d\Lambda_Y(v) d\Lambda_Y(u) \\
&\quad - \int_{(s,t]}\int_{\{u\}} \mathbb{P}(Y \ge v) d\Lambda_Y(v) d\Lambda_Y(u) \\
&= 2\int_{(s,t]}\int_{[u,t]} (1 - F_Y(v-)) d\Lambda_Y(v) d\Lambda_Y(u) - \int_{(s,t]} (1 - F_Y(u-)) \Delta\Lambda_Y(u) d\Lambda_Y(u).
\end{aligned}
$$

Now, we show that both integrals are finite. Since

$$
\Lambda_Y(dv) = \frac{1}{1 - F_Y(v-)} dF_Y(v),
$$

we have

$$
\begin{aligned}
&\int_{(s,t]}\int_{[u,t]} (1 - F_Y(v-)) d\Lambda_Y(v) d\Lambda_Y(u) = \int_{(s,t]}\int_{[u,t]} dF_Y(v) d\Lambda_Y(u) \\
&= \int_{(s,t]} \mathbb{P}(Y \in [u,t]) d\Lambda_Y(u) = \int_{(s,t]} \frac{F_Y(t) - F_Y(u-)}{1 - F_Y(u-)} dF_Y(u) \le 1.
\end{aligned}
$$

Since

$$\Delta\Lambda_Y(u) = \frac{\Delta F_Y(u)}{1 - F_Y(u-)} = \frac{F_Y(u) - F_Y(u-)}{1 - F_Y(u-)} \leq 1,$$

we also have

$$\int_{(s,t]} (1 - F_Y(u-))\Delta\Lambda_Y(u) d\Lambda_Y(u) = \int_{(s,t]} \Delta\Lambda_Y(u) dF_Y(u) \leq 1.$$

Furthermore, it follows

$$\int_{(s,t]} \mathbb{P}(Y \in [u,t]) d\Lambda_Y(u) = \int_{(s,t]} \frac{F_Y(t) - F_Y(u-)}{1 - F_Y(u-)} dF_Y(u),$$

showing in particular that this integral is finite. We obtain $\mathbb{E}[(M_t - M_s)^2] < \infty$ with

$$\begin{aligned}
\mathbb{E}[(M_t - M_s)^2] &= F_Y(t) - F_Y(s) - 2\int_{(s,t]} \frac{F_Y(t) - F_Y(u-)}{1 - F_Y(u-)} dF_Y(u) \\
&\quad + 2\int_{(s,t]} \frac{F_Y(t) - F_Y(u-)}{1 - F_Y(u-)} dF_Y(u) - \int_{(s,t]} \Delta\Lambda_Y(u) dF_Y(u) \\
&= \int_{(s,t]} (1 - \Delta\Lambda_Y(u)) dF_Y(u).
\end{aligned}$$

□

A continuous linear operator $T \in L(X,Y)$ between two normed spaces X and Y is called an isometry if

$$\|Tx\| = \|x\| \quad \text{for all } x \in X.$$

If X and Y are Hilbert spaces, then $T \in L(X,Y)$ is an isometry if and only if

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for all } x, y \in X.$$

If $\mathcal{E} \subset X$ is a dense subspace and $T \in L(X,Y)$ is a continuous linear operator such that

$$\|Tx\| = \|x\| \quad \text{for all } x \in \mathcal{E},$$

then T is an isometry. Indeed, for all $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ such that $x_n \to x$, and we obtain

$$\|Tx\| = \left\| T\Big(\lim_{n\to\infty} x_n \Big) \right\| = \left\| \lim_{n\to\infty} Tx_n \right\| = \lim_{n\to\infty} \|Tx_n\| = \lim_{n\to\infty} \|x_n\| = \|x\|.$$

Proposition 2.3.19. *For every $f \in \mathscr{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), (1-\Delta\Lambda_Y)dF_Y)$ we have*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} f(u)dM_u\right)^2\right] = \int_{\mathbb{R}_+} f(u)^2(1-\Delta\Lambda_Y(u))dF_Y(u).$$

In other words, the linear mapping

$$I : L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), (1-\Delta\Lambda_Y)dF_Y) \to L^2(\Omega, \mathscr{F}, \mathbb{P}), \quad I(f) = \int_{\mathbb{R}_+} f(u)dM_u$$

is an isometry between Hilbert spaces.

Proof. Let $\mathcal{E}$ be the space of all simple functions

$$f = \sum_{j=1}^n c_j \mathbb{1}_{(t_j, t_{j+1}]}$$

with $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{R}$ and $0 \leq t_1 < \ldots < t_{n+1}$. Since $\mathcal{E}$ is dense in

$$L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), (1-\Delta\Lambda_Y)dF_Y),$$

it suffices to prove that I is an isometry on $\mathcal{E}$. Thus, let $f \in \mathcal{E}$ be arbitrary. Using Lemma 2.2.5 and Proposition 2.3.18 we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}_+} f(u)dM_u\right)^2\right] = \mathbb{E}\left[\left(\sum_{j=1}^n c_j(M_{t_{j+1}} - M_{t_j})\right)^2\right] \\
&= \sum_{j=1}^n \sum_{k=1}^n c_j c_k \mathbb{E}[(M_{t_{j+1}} - M_{t_j})(M_{t_{k+1}} - M_{k_l})] = \sum_{j=1}^n c_j^2 \mathbb{E}[(M_{t_{j+1}} - M_{t_j})^2] \\
&= \sum_{j=1}^n c_j^2 \int_{(t_j, t_{j+1}]} (1-\Delta\Lambda_Y(u))dF_Y(u) = \int_{\mathbb{R}_+} \sum_{j=1}^n c_j^2 \mathbb{1}_{(t_j, t_{j+1}]}(u)(1-\Lambda_Y(u))dF_Y(u) \\
&= \int_{\mathbb{R}_+} f(u)^2(1-\Delta\Lambda_Y(u))dF_Y(u).
\end{aligned}$$

□

Remark 2.3.20. *One can show that the predictable quadratic variation $\langle M, M\rangle$ is given by*

$$\langle M, M\rangle_t = \int_{(0,t]} (1-\Delta\Lambda_Y(u))dF_Y(u), \quad t \in \mathbb{R}_+.$$

Hence, Proposition 2.3.19 confirms the well-known Itô isometry

$$\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right], \quad t \in \mathbb{R}_+.$$

The present value of a LIP was defined as

$$B = \frac{A(Y)}{K(Y)} - \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s),$$

and by the equivalence principle we have $\mathbb{E}[B] = 0$. From now on, we always assume that

$$\mathbb{E}\left[\frac{A(Y)}{K(Y)}\right] < \infty.$$

Definition 2.3.21. *The* <u>*loss*</u> *of the insurance company until time $t \in \mathbb{R}_+$ is defined as*

$$L(t) := \mathbb{E}[B \,|\, \mathscr{F}_t].$$

Remark 2.3.22. *Hence, the loss is the conditional expected present value, given the information whether the time of benefit Y has occurred until time t or not.*

Proposition 2.3.23. *For every $t \in \mathbb{R}_+$ we have $\mathbb{P}$-almost surely*

$$L(t) = \left(\frac{A(Y)}{K(Y)} - \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s)\right) \mathbb{1}_{\{Y \le t\}} + \left(\frac{V(t)}{K(t)} - \int_{[0,t)} \frac{1}{K(s)} d\Pi(s)\right) \mathbb{1}_{\{Y > t\}}.$$

Proof. Exercise. □

Remark 2.3.24. *Noting the definition of the present value, Proposition 2.3.23 provides an intuitive characterization of the loss until time t. More precisely:*

- *If $t \ge Y$, which means that the time of benefit has already occurred, then we obtain the well-known definition of the present value.*
- *If $t < Y$, which means that the time of benefit is in the future, then we obtain an analogous representation where Y is replaced by t. Furthermore, the payment spectrum A is replaced by the net premium reserve V. Intuitively, this is clear, since the time of benefit has not yet occurred, and has to be covered by the insurance company.*

Remark 2.3.25. *L is a martingale with $L(0) = 0$ and $\lim_{t\to\infty} L(t) = B$ $\mathbb{P}$-almost surely.*

The martingale L according to Definition 2.3.21 is determined $\mathbb{P}$-almost surely for every time point t. From the theory of stochastic processes it is known that L admits a càdlàg (and hence in particular a right-continuous) version. In the current situation we can explicitly write down such a version.

Proposition 2.3.26. *For every $t \in \mathbb{R}_+$ we have the representation*

$$L(t) = \int_{(0,t]} \frac{A(u) - V(u)}{K(u)} dM_u \quad \mathbb{P}\text{-almost surely.}$$

In particular, L has a right-continuous version.

Proof. We have

$$\begin{aligned}
\int_{(0,t]} \frac{A(u) - V(u)}{K(u)} dM_u &= \int_{(0,t]} \frac{A(u) - V(u)}{K(u)} dN_u - \int_{]\!]0,t\wedge Y]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \\
&= \frac{A(Y) - V(Y)}{K(Y)} \mathbb{1}_{[\![Y,\infty[\![}(t) - \int_{]\!]0,t\wedge Y]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \\
&= \left(\frac{A(Y)}{K(Y)} - \frac{V(Y)}{K(Y)} - \int_{]\!]0,Y]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \right) \mathbb{1}_{\{Y \le t\}} \\
&\quad - \left(\int_{(0,t]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \right) \mathbb{1}_{\{Y > t\}}.
\end{aligned}$$

Therefore, by Proposition 2.3.23 we have to show

$$\begin{aligned}
&\left(\frac{A(Y)}{K(Y)} - \int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s) \right) \mathbb{1}_{\{Y \le t\}} \\
&= \left(\frac{A(Y)}{K(Y)} - \frac{V(Y)}{K(Y)} - \int_{]\!]0,Y]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \right) \mathbb{1}_{\{Y \le t\}} \\
\iff \quad &\left(\int_{[\![0,Y[\![} \frac{1}{K(s)} d\Pi(s) \right) \mathbb{1}_{\{Y \le t\}} = \left(\frac{V(Y)}{K(Y)} + \int_{]\!]0,Y]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \right) \mathbb{1}_{\{Y \le t\}}
\end{aligned}$$

and

$$\left(\frac{V(t)}{K(t)} - \int_{[0,t)} \frac{1}{K(s)} d\Pi(s) \right) \mathbb{1}_{\{Y > t\}} = -\left(\int_{(0,t]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u) \right) \mathbb{1}_{\{Y > t\}}$$

Thus, we have to show

$$\frac{V(Y \wedge t)}{K(Y \wedge t)} = \int_{[\![0,Y\wedge t[\![} \frac{1}{K(s)} d\Pi(s) - \int_{]\!]0,Y\wedge t]\!]} \frac{A(u) - V(u)}{K(u)} d\Lambda_Y(u).$$

By the Thiele integral equation (Theorem 1.2.34) we have

$$\frac{V(t)}{K(t)} = \int_{[0,t)} \frac{1}{K(s)} d\Pi(s) - \int_{(0,t]} \frac{A(u) - V(u)}{K(u)} d\Lambda(u), \quad t \in [0, \tau).$$

In case $\tau < \infty$ we also notice:

- By Proposition 2.3.13(b) the martingale M is continuous in τ. Therefore, the right-hand side of the claimed equation is continuous in τ.
- L is continuous in τ. This follows from the representation in Proposition 2.3.23 as well as Lemma 1.2.27.

Finally, the claimed right-continuity follows from the right-continuity of M; see Proposition 2.3.13(a). □

Definition 2.3.27. *Let $(t_i)_{i\in\mathbb{N}_0}$ be a sequence with $t_0 = 0$ and $t_{i-1} < t_i$ for all $i \in \mathbb{N}$. Here $t_{i-1} < t_i$ are the time points at the beginning and at the end of insurance periods. We set*

$$L^i := L(t_i) - L(t_{i-1}), \quad i \in \mathbb{N}$$

for the loss at the ith period of insurance.

Theorem 2.3.28 (Hattendorf's theorem)**.** *For the loss of a LIP under the equivalence principle we have*

$$\begin{aligned} \mathbb{E}[L(t)] &= 0 \quad \textit{for all } t \in \mathbb{R}_+ \textit{ and} \\ \mathbb{E}[L^i] &= 0 \quad \textit{for all } i \in \mathbb{N}. \end{aligned}$$

If furthermore

$$\int_{[0,t]} \left(\frac{A(u) - V(u)}{K(u)} \right)^2 (1 - \Delta\Lambda_Y(u)) dF_Y(u) < \infty \quad \textit{for all } t \in \mathbb{R}_+,$$

then the following statements are true:

(a) We have $\mathbb{E}[L^{j+1} \mid \mathscr{F}_{t_j}] = 0$.

(b) We have $\operatorname{Cov}(L^j, L^k) = 0$ for all $j, k \in \mathbb{N}$ with $j \neq k$.

(c) For the variance of the loss we have

$$\operatorname{Var}[L(t)] = \int_{[0,t]} \left(\frac{A(u) - V(u)}{K(u)} \right)^2 (1 - \Delta\Lambda_Y(u)) dF_Y(u), \quad t \in \mathbb{R}_+.$$

(d) For the variance of the present value we have

$$\operatorname{Var}[B] = \int_{[0,\infty)} \left(\frac{A(u) - V(u)}{K(u)} \right)^2 (1 - \Delta\Lambda_Y(u)) dF_Y(u) \in [0, \infty].$$

Proof. By Remark 2.3.25 the process L is a martingale, and hence the first two identities follow. Now assume that the integrability condition holds true as well.

(a) Follows, because L is a martingale.

(b) Follows with Lemma 2.2.5, because L is a square-integrable martingale.

(c) By Proposition 2.3.26 we have

$$L(t) = \int_{(0,t]} \frac{A(u) - V(u)}{K(u)} dM_u, \quad t \in \mathbb{R}_+.$$

Hence, the claimed formula follows from Proposition 2.3.19 with the function

$$f(u) = \frac{A(u) - V(u)}{K(u)} \mathbb{1}_{[0,t]}.$$

(d) By Proposition 2.3.26 and Remark 2.3.25 we have $\mathbb{P}$-almost surely

$$B = \lim_{t\to\infty} L(t) = \int_{\mathbb{R}_+} \frac{A(u) - V(u)}{K(u)} dM_u.$$

Hence, the claimed formula follows from Proposition 2.3.19 with the function

$$f(u) = \frac{A(u) - V(u)}{K(u)}.$$

□

Remark 2.3.29. *Hence the total variance of the loss can be decomposed as the sum of the variances of the single insurance periods.*

Chapter 3

Static models

When using static models, we model the total loss for a fixed period, say one year, from the point of view of the insurance company.

3.1 Models for the total loss in an insurance period

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

3.1.1 Individual model

We consider a portfolio with $n \in \mathbb{N}$ insured risks (insurance polices). Let $Y_1, \ldots, Y_n : \Omega \to \mathbb{R}_+$ be independent, non-negative random variables. Here Y_i is the (random) loss of the i-th police.

Definition 3.1.1. *The total loss of the portfolio in the individual model is given by*

$$S_{\text{ind}} := \sum_{i=1}^{n} Y_i.$$

Remark 3.1.2. *Because of the independence we have*

$$\mathbb{P} \circ (Y_1, \ldots, Y_n) = (\mathbb{P} \circ Y_1) \otimes \ldots \otimes (\mathbb{P} \circ Y_n).$$

3.1.2 Collective model

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of positive random variables $X_i : \Omega \to (0, \infty)$. Furthermore, let $N : \Omega \to \mathbb{N}_0$ be an integer-valued random variable.

Definition 3.1.3. *The total loss in the collective model is given by*

$$S_{\text{koll}} := \sum_{i=1}^{N} X_i.$$

Remark 3.1.4. *The random variables $X_1, \ldots, X_N > 0$ are the losses occurring in the insurance period; the number $N \in \mathbb{N}_0$ of losses is now random. Here, the losses are no longer associated to individual polices.*

Definition 3.1.5. *We speak of a standard model of collective risk theory if the random variables $(X_i)_{i\in\mathbb{N}}$ are independent and identically distributed with $X_1 \in \mathscr{L}^1$ and $F_{X_1}(0) = 0$, and the random variable N is independent of the sequence $(X_i)_{i\in\mathbb{N}}$.*

3.1.3 Models for the loss distribution

In order to model the distributions of the losses X_i one frequently uses absolutely continuous distributions with unimodal densities on $(0,\infty)$.

Definition 3.1.6. *A function $f : (0,\infty) \to \mathbb{R}_+$ is called unimodal with mode at $x \in (0,\infty)$ if f is strictly increasing on $(0,x)$, and strictly decreasing on (x,∞).*

Here are some important loss distributions.

Example 3.1.7. *The Gamma distribution $\Gamma(\alpha,\beta)$ for $\alpha,\beta > 0$ has the density*

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

We have:

- *For $\alpha \leq 1$ the density f is decreasing.*
- *For $\alpha > 1$ the density f is unimodal with mode at*

$$\frac{\alpha-1}{\beta}.$$

- $\Gamma(1,\beta) = \text{Exp}(\beta)$.

We call α the shape parameter and β the scale parameter.

Example 3.1.8. *The* *Weibull distribution* $\mathrm{WB}(c,\tau)$ *for* $c,\tau>0$ *has the density*

$$f(x) = c\tau x^{\tau-1}e^{-cx^\tau}, \quad x>0.$$

We have:

- *For* $\tau \le 1$ *the density* f *is decreasing.*
- *For* $\tau > 1$ *the density* f *is unimodal with mode at*

$$\left(\frac{\tau-1}{c\tau}\right)^{1/\tau}.$$

- $\mathrm{WB}(c,1) = \mathrm{Exp}(c)$.

We call τ *the shape parameter and* c *the scale parameter.*

Example 3.1.9. *The* *Log-Normal distribution* $\mathrm{LN}(\mu,\sigma^2)$ *for* $\mu\in\mathbb{R}$ *and* $\sigma^2>0$ *has the density*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}x}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x>0.$$

We have:

- f *is always unimodal with mode at* e^μ.
- *For* $X\sim\mathrm{LN}(\mu,\sigma^2)$ *we have* $\ln X\sim \mathrm{N}(\mu,\sigma^2)$.

Example 3.1.10. *The* *Log-Gamma distribution* $\mathrm{L}\Gamma(\alpha,\beta)$ *for* $\alpha,\beta>0$ *has the density*

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\ln x)^{\alpha-1}x^{-(\beta+1)}\mathbb{1}_{(1,\infty)}(x), \quad x>0.$$

We have:

- *For* $X\sim\mathrm{L}\Gamma(\alpha,\beta)$ *we have* $\ln X\sim\Gamma(\alpha,\beta)$.

Example 3.1.11. *The* *Burr distribution* $\mathrm{Burr}(\alpha,\tau,\sigma)$ *for* $\alpha,\tau,\sigma>0$ *has the density*

$$f(x) = \frac{\alpha\tau}{\sigma}\left(\frac{x}{\sigma}\right)^{\tau-1}\left(1+\left(\frac{x}{\sigma}\right)^\tau\right)^{-(\alpha+1)}, \quad x>0.$$

We have:

- *For* $\tau\ge 1$ *the density* f *is decreasing.*

- *For $\tau < 1$ the density f is unimodal.*
- *The distribution function is given by*

$$F(x) = 1 - \left(1 + \left(\frac{x}{\sigma}\right)^{\tau}\right)^{-\alpha}, \quad x > 0.$$

Example 3.1.12. *The* <u>*Pareto distribution*</u> *(type I)* $\mathrm{Par}(\kappa, \alpha)$ *for* $\kappa, \alpha > 0$ *has the density*

$$f(x) = \frac{\alpha \kappa^{\alpha}}{x^{\alpha+1}} \mathbb{1}_{[\kappa,\infty)}(x), \quad x > 0.$$

We have:

- *The distribution function is given by*

$$F(x) = \left(1 - \frac{\kappa^{\alpha}}{x^{\alpha}}\right) \mathbb{1}_{[\kappa,\infty)}(x), \quad x > 0.$$

- *For $\alpha > 1$ and $X \sim \mathrm{Par}(\kappa, \alpha)$ we have*

$$\mathbb{E}[X] = \frac{\alpha \kappa}{\alpha - 1}.$$

3.1.4 Models for the distribution of the number of losses

The following models are popular for the distribution of the number N of losses.

Example 3.1.13. *The* <u>*Binomial distribution*</u> $\mathrm{Bi}(n, p)$ *with parameters* $n \in \mathbb{N}$ *and* $p \in [0, 1]$ *is specified by the stochastic vector*

$$\pi(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$$

We have $\mathrm{Bi}(1, p) = \mathrm{Ber}(p)$.

Example 3.1.14. *The* <u>*Poisson distribution*</u> $\mathrm{Pois}(\lambda)$ *with parameter* $\lambda > 0$ *is specified by the stochastic vector*

$$\pi(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}_0.$$

Remark 3.1.15. *By the Poisson limit theorem we have*

$$\mathrm{Bi}\Big(n, \frac{\lambda}{n}\Big) \xrightarrow{w} \mathrm{Pois}(\lambda) \quad \textit{for every } \lambda > 0.$$

Therefore, we have

$$\mathrm{Bi}(n,p) \approx \mathrm{Pois}(np) \quad \textit{for large } n \in \mathbb{N} \textit{ and small } p \in (0,1).$$

Hence, the Poisson distribution is suitable for large portfolios with small probabilities of losses.

Example 3.1.16. *The negative Binomial distribution* $\mathrm{NB}(\beta, p)$ *with parameters* $\beta > 0$ *and* $p \in (0,1)$ *is specified by the stochastic vector*

$$\pi(k) = \binom{\beta + k - 1}{k} p^{\beta}(1-p)^k, \quad k \in \mathbb{N}_0,$$

where

$$\binom{\beta + k - 1}{k} := \frac{(\beta + k - 1)(\beta + k - 2) \cdot \ldots \cdot \beta}{k!}.$$

We have $\mathrm{NB}(1,p) = \mathrm{Geo}(p)$.

If $\beta \in \mathbb{N}$, then $\pi(k)$ is the probability that for independent Bernoulli experiments k failures occur before the first β successes.

3.2 Computation of the loss distribution

3.2.1 Convolutions and generating functions

Definition 3.2.1. *For two probability measures* μ *and* ν *on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *the convolution* $\mu * \nu$ *is defined as*

$$(\mu * \nu)(B) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(x+y)\mu(dx)\nu(dy), \quad B \in \mathcal{B}(\mathbb{R}).$$

Definition 3.2.2. *Let* F *and* G *be two distribution functions. Then the function* $F * G$ *given by*

$$(F * G)(x) = \int_{\mathbb{R}} F(x-t)G(dt), \quad x \in \mathbb{R}$$

is called the convolution of F *and* G.

Lemma 3.2.3. *Let μ and ν be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with distribution functions F and G. Then the distribution function of $\mu * \nu$ is given by $F * G$.*

Proof. We have

$$(\mu * \nu)((-\infty, t]) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]}(x+y) F(dx) G(dy) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{x+y \leq t\}} F(dx) G(dy)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{x \leq t-y\}} F(dx) G(dy) = \int_{\mathbb{R}} F(t-y) G(dy).$$

□

Proposition 3.2.4. *Let X and Y be two independent random variables with distribution functions F and G.*

(a) The distribution function of $X+Y$ is given by $F * G$.

(b) If X is absolutely continuous with density f, then $X+Y$ is absolutely continuous with density

$$h : \mathbb{R} \to \mathbb{R}_+, \quad h(x) = \int_{\mathbb{R}} f(x-t) G(dt).$$

(c) If X and Y are absolutely continuous with densities f and g, then we have

$$h(x) = \int_{\mathbb{R}} f(x-t) g(t) dt = \int_{\mathbb{R}} f(t) g(x-t) dt, \quad x \in \mathbb{R}.$$

Proof. Exercise. □

Definition 3.2.5. *Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

(a) We set $\mu^{*0} := \delta_0$.

(b) For every $n \in \mathbb{N}$ we set

$$\mu^{*n} := \underbrace{\mu * \ldots * \mu}_{n \text{ times}}.$$

Accordingly we introduce:

Definition 3.2.6. *Let F be a distribution function.*

(a) We set $F^{*0} := \mathbb{1}_{\mathbb{R}_+}$.

(b) For every $n \in \mathbb{N}$ we set

$$F^{*n} := \underbrace{F * \ldots * F}_{n \text{ times}}.$$

Definition 3.2.7. *Let X be a random variable with distribution function F.*

(a) The function defined on $\mathscr{M}_X := \{t \in \mathbb{R} : \mathbb{E}[e^{tX}] < \infty\}$ and given by

$$\psi_X : \mathscr{M}_X \to \mathbb{R}_+, \quad \psi_X(t) := \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} F(dx)$$

is called the moment generating function of X and F respectively.

(b) The function defined on $\mathscr{M}_X^p := \{t > 0 : \mathbb{E}[t^X] < \infty\}$ and given by

$$\phi_X : \mathscr{M}_X^p \to \mathbb{R}_+, \quad \phi_X(t) := \mathbb{E}[t^X] = \int_{\mathbb{R}} t^x F(dx)$$

is called the probability generating function of X and F respectively.

(c) The function

$$\chi_X : \mathbb{R} \to \mathbb{C}, \quad \chi_X(t) := \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} F(dx)$$

is called the characteristic function of X and F respectively.

Remark 3.2.8.

(a) One also calls $t \mapsto \psi_X(-t)$ the Laplace transform of X and F respectively.

(b) The characteristic function is also frequently called the Fourier transform.

Proposition 3.2.9 (Uniqueness theorem)**.**

(a) If $\mathscr{M}_X$ has an inner point, then ψ_X uniquely determines F.

(b) If $\mathscr{M}_X^p$ has an inner point, then ϕ_X uniquely determines F.

(c) χ_X uniquely determines F.

Proof. We prove part (b) in case that X is $\mathbb{N}_0$-valued. Then we have

$$\phi_X(t) = \sum_{k=0}^{\infty} t^k \mathbb{P}(X = k) \quad \text{for all } t \in (0, 1).$$

By the geometric series we can extend ϕ_X to the interval $(-1,1)$, and obtain

$$\phi_X^{(n)}(0) = n!\mathbb{P}(X=n) \quad \text{for all } n \in \mathbb{N}_0,$$

and hence

$$\mathbb{P}(X=n) = \frac{\phi_X^{(n)}(0)}{n!} \quad \text{for all } n \in \mathbb{N}_0,$$

proving the uniqueness of the distribution of X. □

The proof shows why we call ϕ_X the probability generating function of X.

Proposition 3.2.10. *Let $X_1, \dots, X_n$ be independent random variables, and set*

$$S_n := \sum_{k=1}^n X_k.$$

(a) We have

$$\psi_{S_n}(t) = \prod_{k=1}^n \psi_{X_k}(t), \quad t \in \bigcap_{k=1}^n \mathscr{M}_{X_k}.$$

(b) We have

$$\phi_{S_n}(t) = \prod_{k=1}^n \phi_{X_k}(t), \quad t \in \bigcap_{k=1}^n \mathscr{M}_{X_k}^p.$$

(c) We have

$$\chi_{S_n}(t) = \prod_{k=1}^n \chi_{X_k}(t), \quad t \in \mathbb{R}.$$

Proof.

(a) By the independence of the random variables $X_1, \dots, X_n$ we have

$$\psi_{S_n}(t) = \mathbb{E}[e^{tS_n}] = \mathbb{E}\left[\prod_{k=1}^n e^{tX_k}\right] = \prod_{k=1}^n \mathbb{E}[e^{tX_k}] = \prod_{k=1}^n \psi_{X_k}(t).$$

(b) Exercise.

(c) Exercise.

□

Proposition 3.2.11. *For $X \sim \Gamma(\alpha, \beta)$ we have*

$$\chi_X(t) = \left(\frac{\beta}{\beta - it} \right)^{\alpha}, \quad t \in \mathbb{R}.$$

Proposition 3.2.12. *For independent random variables $X \sim \Gamma(\alpha, \beta)$ and $Y \sim \Gamma(\bar{\alpha}, \beta)$ we have $X + Y \sim \Gamma(\alpha + \bar{\alpha}, \beta)$.*

Proof. By Proposition 3.2.10(c) and Proposition 3.2.11 we have for all $t \in \mathbb{R}_+$

$$\chi_{X+Y}(t) = \chi_X(t)\chi_Y(t) = \left(\frac{\beta}{\beta - it} \right)^{\alpha} \left(\frac{\beta}{\beta - it} \right)^{\bar{\alpha}} = \left(\frac{\beta}{\beta - it} \right)^{\alpha + \bar{\alpha}}.$$

By the uniqueness theorem (Proposition 3.2.9(c)) we deduce $X+Y \sim \Gamma(\alpha+\bar{\alpha}, \beta)$. □

Proposition 3.2.13. *Let N be a $\mathbb{N}_0$-valued random variable.*

(a) If $\mathbb{P} \circ N = \mathrm{Bi}(n, p)$, then we have $\mathscr{M}_N^p = (0, \infty)$ and

$$\phi_N(t) = (1 - p + pt)^n.$$

(b) If $\mathbb{P} \circ N = \mathrm{Pois}(\lambda)$, then we have $\mathscr{M}_N^p = (0, \infty)$ and

$$\phi_N(t) = e^{-\lambda(1-t)}.$$

(c) If $\mathbb{P} \circ N = \mathrm{NB}(\beta, p)$, then we have $\mathscr{M}_N^p = (0, \infty)$ and

$$\phi_N(t) = \left(\frac{1 - (1-p)t}{p} \right)^{-\beta}.$$

Proof. Exercise. □

3.2.2 Formulas for the distribution of the total loss

We consider a standard model of collective risk theory. Thus, the random variables $(X_i)_{i \in \mathbb{N}} \subset \mathscr{L}^1$ are independent and identically distributed, and N is independent of the sequence $(X_i)_{i \in \mathbb{N}}$. We denote by F the distribution function of X_1. The total loss is given by

$$S_{\mathrm{koll}} = \sum_{i=1}^{N} X_i.$$

We denote by G the distribution function of S_{koll}.

Lemma 3.2.14. *We have*

$$G(x) = \sum_{n=0}^{\infty} F^{*n}(x)\mathbb{P}(N = n) \quad \textit{for all } x \in \mathbb{R}.$$

Proof. By Proposition 3.2.4(a) we have

$$\begin{aligned} G(x) = \mathbb{P}(S_{\text{koll}} \le x) &= \mathbb{P}\biggl(\sum_{i=1}^{N} X_i \le x\biggr) = \sum_{n=0}^{\infty} \mathbb{P}\biggl(\sum_{i=1}^{n} X_i \le x, N = n\biggr) \\ &= \sum_{n=0}^{\infty} \mathbb{P}\biggl(\sum_{i=1}^{n} X_i \le x\biggr)\mathbb{P}(N = n) = \sum_{n=0}^{\infty} F^{*n}(x)\mathbb{P}(N = n). \end{aligned}$$

□

Corollary 3.2.15. *If $X_1 \sim \Gamma(\alpha, \beta)$, then we have*

$$G(x) = \sum_{n=0}^{\infty} \Gamma_{n\alpha,\beta}(x)\mathbb{P}(N = n) \quad \textit{for all } x \in \mathbb{R},$$

where $\Gamma_{0,\beta} = \mathbb{1}_{\mathbb{R}_+}$ and $\Gamma_{n\alpha,\beta}$ denotes the distribution function of $\Gamma(n\alpha, \beta)$ for each $n \in \mathbb{N}$.

Proof. This is a consequence of Lemma 3.2.14 and Proposition 3.2.12. □

Proposition 3.2.16. *We have*

$$\psi_{S_{\text{koll}}}(t) = \phi_N(\psi_{X_1}(t)) \quad \textit{for all } t \in \mathscr{M}_{\text{koll}},$$

where

$$\mathscr{M}_{\text{koll}} := \{t \in \mathbb{R} : t \in \mathscr{M}_{X_1} \textit{ and } \psi_{X_1}(t) \in \mathscr{M}_N^p\}.$$

Proof. We set $S_n := \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}_0$. Using Proposition 3.2.10(a), we obtain

$$\begin{aligned} \phi_N(\psi_{X_1}(t)) = \mathbb{E}\bigl[\psi_{X_1}(t)^N\bigr] &= \sum_{n=0}^{\infty} \psi_{X_1}(t)^n \mathbb{P}(N = n) = \sum_{n=0}^{\infty} \psi_{S_n}(t)\mathbb{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}[e^{tS_n}]\mathbb{E}[\mathbb{1}_{\{N=n\}}] = \sum_{n=0}^{\infty} \mathbb{E}[e^{tS_n}\mathbb{1}_{\{N=n\}}] \\ &= \mathbb{E}[e^{tS_N}] = \mathbb{E}[e^{tS_{\text{koll}}}] = \psi_{S_{\text{koll}}}(t). \end{aligned}$$

□

Proposition 3.2.17. *We have*

$$\phi_{S_{\text{koll}}}(t) = \phi_N(\phi_{X_1}(t)) \quad \textit{for all } t \in \mathscr{M}^p_{\text{koll}},$$

where

$$\mathscr{M}^p_{\text{koll}} := \{t \in \mathbb{R} : t \in \mathscr{M}^p_{X_1} \textit{ and } \phi_{X_1}(t) \in \mathscr{M}^p_N\}.$$

Proof. Exercise. □

The following result shows why we call ψ_X the moment generating function of X.

Proposition 3.2.18. *Let X be a random variable such that the moment generating function ψ_X exists on some neighborhood of 0.*

(a) ψ_X is infinitely often differentiable in 0, and we have

$$\psi_X^{(n)}(0) = \mathbb{E}[X^n] \quad \textit{for all } n \in \mathbb{N}_0.$$

(b) We have $\mathbb{E}[X] = \psi'_X(0)$.

(c) We have $\operatorname{Var}[X] = \psi''_X(0) - (\psi'_X(0))^2$.

Proof.

(a) For all $n \in \mathbb{N}_0$ we have

$$\psi_X^{(n)}(0) = \frac{d^n}{dt^n}\mathbb{E}[e^{tX}]\bigg|_{t=0} = \mathbb{E}\left[\frac{d^n}{dt^n}e^{tX}\right]\bigg|_{t=0} = \mathbb{E}[X^n e^{tX}]|_{t=0} = \mathbb{E}[X^n],$$

where the interchange of differentiation and integration is valid by Lebesgue's dominated convergence theorem.

(b) Follows from part (a).

(c) Follows from (a) and the identity $\operatorname{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

□

Lemma 3.2.19. *Let N be a random variable with values in $\mathbb{N}_0$. Then we have*

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}(N \geq k).$$

Proof. Exercise. □

Theorem 3.2.20 (First Wald's Equation)**.** *Let $(X_k)_{k\in\mathbb{N}} \subset \mathscr{L}^1$ be independent, identically distributed random variables, and let $N \in \mathscr{L}^1$ be a random variable with values in $\mathbb{N}_0$. We assume that $\{N = n\}$ and $(X_k)_{k\geq n+1}$ are independent for all $n \in \mathbb{N}_0$, and set*

$$S_n := \sum_{k=1}^{n} X_k \quad \text{for all } n \in \mathbb{N}_0.$$

Then we have $S_N \in \mathscr{L}^1$ and

$$\mathbb{E}[S_N] = \mu\mathbb{E}[N],$$

where $\mu = \mathbb{E}[X_1]$.

Proof. For each $k \in \mathbb{N}$ the random variable X_k and the event $\{N \geq k\}$ are independent. For this, we show that X_k and $\{N < k\}$ are independent. Indeed, for every Borel set $B \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned}\mathbb{P}(X_k \in B, N < k) &= \sum_{n=0}^{k-1} \mathbb{P}(X_k \in B, N = n) = \sum_{n=0}^{k-1} \mathbb{P}(X_k \in B)\mathbb{P}(N = n) \\ &= \mathbb{P}(X_k \in B)\mathbb{P}(N < k).\end{aligned}$$

Next, we show that $S_N \in \mathscr{L}^1$. Indeed, by the monotone convergence theorem and Lemma 3.2.19 we have

$$\begin{aligned}\mathbb{E}[|S_N|] &= \mathbb{E}\left[\left|\sum_{k=1}^{N} X_k\right|\right] \leq \mathbb{E}\left[\sum_{k=1}^{N} |X_k|\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} |X_k|\mathbb{1}_{\{N\geq k\}}\right] = \sum_{k=1}^{\infty} \mathbb{E}[|X_k|\mathbb{1}_{\{N\geq k\}}] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[|X_k|]\mathbb{E}[\mathbb{1}_{\{N\geq k\}}] = \mathbb{E}[|X_1|] \sum_{k=1}^{\infty} \mathbb{P}(N \geq k) = \mathbb{E}[|X_1|]\mathbb{E}[N] < \infty.\end{aligned}$$

Now, using Fubini's theorem and Lemma 3.2.19 we obtain

$$\begin{aligned}\mathbb{E}[S_N] &= \mathbb{E}\left[\sum_{k=1}^{N} X_k\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} X_k\mathbb{1}_{\{N\geq k\}}\right] = \sum_{k=1}^{\infty} \mathbb{E}[X_k\mathbb{1}_{\{N\geq k\}}] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[X_k]\mathbb{E}[\mathbb{1}_{\{N\geq k\}}] = \mathbb{E}[X_1] \sum_{k=1}^{\infty} \mathbb{P}(N \geq k) = \mathbb{E}[X_1]\mathbb{E}[N].\end{aligned}$$

□

Lemma 3.2.21. *Let $\mathbb{F} = (\mathscr{F}_n)_{n\in\mathbb{N}_0}$ be a filtration. Furthermore, let $(Y_k)_{k\in\mathbb{N}}$ be a $\mathbb{F}$-adapted process (that is Y_k is $\mathscr{F}_k$-measurable for each $k \in \mathbb{N}$) and let $(Z_k)_{k\in\mathbb{N}}$ be a $\mathbb{F}$-predictable process (that is Z_k is $\mathscr{F}_{k-1}$-measurable for each $k \in \mathbb{N}$) such that for each $k \in \mathbb{N}$ we have:*

- *$Y_k \in \mathscr{L}^2$ with $\mathbb{E}[Y_k] = 0$, and Y_k and $\mathscr{F}_{k-1}$ are independent.*
- *Z_k is bounded.*

We define the process $M = (M_n)_{n \in \mathbb{N}_0}$ as

$$M_n := \sum_{k=1}^{n} Y_k Z_k.$$

Then M is a square-integrable $\mathbb{F}$-martingale with $M_0 = 0$ and

$$\mathbb{E}[M_n^2] = \sum_{k=1}^{n} \mathbb{E}[(Y_k Z_k)^2] \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Exercise. □

Theorem 3.2.22 (Second Wald's Equation)**.** *In addition to the hypotheses of Theorem 3.2.20 we assume that $(X_k)_{k \in \mathbb{N}} \subset \mathscr{L}^2$. Then we have $S_N - N\mu \in \mathscr{L}^2$ and*

$$\mathbb{E}[(S_N - N\mu)^2] = \sigma^2 \mathbb{E}[N],$$

where $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathrm{Var}[X_1]$.

Proof. We define the filtration $\mathbb{F} = (\mathscr{F}_n)_{n \in \mathbb{N}_0}$ as

$$\mathscr{F}_n := \sigma(X_1, \ldots, X_n, \mathbb{1}_{\{N=0\}}, \ldots, \mathbb{1}_{\{N=n\}}).$$

Furthermore, we define the process $M = (M_n)_{n \in \mathbb{N}_0}$ as

$$M_n := \sum_{k=1}^{n} (X_k - \mu) \mathbb{1}_{\{N \geq k\}}.$$

Then M is a square-integrable $\mathbb{F}$-martingale. Indeed, we set

$$Y_k := X_k - \mu \quad \text{and} \quad Z_k := \mathbb{1}_{\{N \geq k\}} \quad \text{for all } k \in \mathbb{N}.$$

Then we have

$$M_n = \sum_{k=1}^{n} Y_k Z_k \quad \text{for all } n \in \mathbb{N}.$$

Moreover, for each $k \in \mathbb{N}$ we have:

(1) Y_k is $\mathscr{F}_k$-measurable. Thus $(Y_k)_{k \in \mathbb{N}}$ is a $\mathbb{F}$-adapted process. Furthermore, we have $Y_k \in \mathscr{L}^2$ with $\mathbb{E}[Y_k] = 0$. Moreover Y_k and $\mathscr{F}_{k-1}$ are independent, since by assumption X_k and $\{N = n\}$ are independent for each $n \in \{0, 1, \ldots, k-1\}$.

(2) Z_k is bounded. Furthermore, we have

$$\{N < k\} = \bigcup_{j=0}^{k-1} \{N = j\} \in \mathscr{F}_{k-1},$$

and hence $\{N \geq k\} \in \mathscr{F}_{k-1}$. Therefore $(Z_k)_{k \in \mathbb{N}}$ is a $\mathbb{F}$-predictable process.

Consequently, by Lemma 3.2.21 the process M is a square-integrable martingale, and for all $n \in \mathbb{N}$ we have

$$\mathbb{E}[M_n^2] = \sum_{k=1}^{n} \mathbb{E}[(Y_k Z_k)^2] = \sum_{k=1}^{n} \mathbb{E}[(X_k - \mu)^2 \mathbb{1}_{\{N \geq k\}}].$$

In the proof of Theorem 3.2.20 we have shown that for each $k \in \mathbb{N}$ the random variable X_k and the event $\{N \geq k\}$ are independent. Therefore, we obtain for all $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}[M_n^2] &= \sum_{k=1}^{n} \mathbb{E}[(X_k - \mu)^2] \mathbb{E}[\mathbb{1}_{\{N \geq k\}}] \\ &= \sum_{k=1}^{n} \mathrm{Var}[X_k] \mathbb{P}(N \geq k) = \sigma^2 \sum_{k=1}^{n} \mathbb{P}(N \geq k). \end{aligned}$$

Using Lemma 3.2.19, it follows

$$\lim_{n \to \infty} \mathbb{E}[M_n^2] = \sigma^2 \mathbb{E}[N].$$

Therefore, we obtian

$$\sup_{n \in \mathbb{N}} \mathbb{E}[M_n^2] < \infty,$$

and hence the martingale M is uniformly integrable. By the convergence theorem for uniformly integrable martingales there exists a limit $M_\infty \in \mathscr{L}^2$ such that $M_n \stackrel{f.s.}{\to} M_\infty$ and $M_n \stackrel{\mathscr{L}^2}{\to} M_\infty$. Hence, $\mathbb{P}$-almost surely we have

$$M_\infty = \lim_{n \to \infty} M_n = \sum_{k=1}^{\infty} (X_k - \mu) \mathbb{1}_{\{N \geq k\}} = \sum_{k=1}^{N} X_k - N\mu = S_N - N\mu,$$

and hence $S_N - N\mu \in \mathscr{L}^2$. Since $M_n \stackrel{\mathscr{L}^2}{\to} M_\infty$, we obtain

$$\mathbb{E}[(S_N - N\mu)^2] = \mathbb{E}[M_\infty^2] = \lim_{n \to \infty} \mathbb{E}[M_n^2] = \sigma^2 \mathbb{E}[N].$$

$\square$

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