SPDEs 2016/17 Exercise Sheet 11

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Notation: see lecture notes (pdf)

11.1. Musiela parametrization and mild solutions

Let U be a separable Hilbert space, and let W be an I_U-cylindrical Wiener process. For every $T \in \mathbb{R}_+$ let

$$\alpha(\cdot,T) : [0,T] \times \Omega \to \mathbb{R}, \qquad \sigma(\cdot,T) : [0,T] \times \Omega \to L_2(U;\mathbb{R})$$

be predictable processes such that

$$\int_0^T |\alpha(s,T)| ds < \infty, \qquad \int_0^T \|\sigma(s,T)\|^2_{L_2(U;\mathbb{R})} ds < \infty.$$

Show that

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) ds + \int_0^t \sigma(s,T) dW_s,$$

holds for every $t \in [0,T]$ and every $T \in \mathbb{R}_+$ if and only if

$$f_t(x) = S_t f_0(x) + \int_0^t S_{t-s}\alpha_s(x) ds + \int_0^t S_{t-s}\sigma_s(x) dW_s$$

holds for every $t, x \in \mathbb{R}_+$ where $f_t(x) = f(t,t+x)$, $\alpha_t(x) = \alpha(t,t+x)$, and $\sigma_t(x) = \sigma(t,t+x)$. Give sufficient conditions such that this implies that f is a mild solution of a corresponding SPDE on a Hilbert space of real-valued functions on $\mathbb{R}_+$.

Hint: you can find the argument in [Fil01, Sections 4.1 and 4.2].

11.2. Spaces of forward rate curves

Let $\alpha > 3$, and let

$$H = \left\{ f \in H^1_{\mathrm{loc}} : \|f\|_H < \infty \right\},$$

where

$$\|f\|_H^2 = |f(0)|^2 + \int_0^\infty |f'(x)|^2 w(x) dx, \qquad w(x) = (1+x)^\alpha.$$

Show that H is a separable Hilbert space, that the shift semigroup is strongly continuous on H, and that the following mapping is bounded bilinear on H:

$$m : H \times H \to H, m(f,g)(x) = f(x) \int_0^x g(y) dy.$$

Hint: you can find the argument in [Fil01, Section 5].

11.3. Vasicek model

Let H be as in Exercise 11.2, let b be a constant, $\beta < 0$, $a > 0$, $U = \mathbb{R}$, W be I_U-Brownian motion, and let $(r_t)_{t \in \mathbb{R}_+}$ be the unique solution of

$$dr_t = (b + \beta r_t) dt + \sqrt{a} dW_t, \tag{1}$$

and define for each $t, x \in \mathbb{R}_+$

$$P_t(x) = \mathbb{E}\left[\exp\left(- \int_t^{t+x} r(s) ds \right) \middle| \mathscr{F}(t) \right],$$

$$f_t(x) = -\frac{d}{dx} \log P_t(x).$$

Show that f is a mild solution of the HJM equation with $\lambda = 0$ and

$$\sigma_t(f)(u)(x) = \sqrt{a} e^{\beta x} u, \qquad t \in \mathbb{R}_+, f \in H, u \in U, x \in \mathbb{R}_+.$$

Are the conditions of our existence and uniqueness result satisfied for this HJM equation?

Hint: use [Fil09, Section 5.4.1].

11.4. Cox-Ingersoll-Ross model

Let H be as in Exercise 11.2, let b be a constant and $b(x) \geq 0$ for each $x \in \mathbb{R}_+$, let $\beta < 0$, let $\alpha > 0$, let $U = \mathbb{R}$, let W be I_U-Brownian motion, let $(r_t)_{t\in\mathbb{R}_+}$ be the unique solution of

$$dr_t = (b + \beta r_t)dt + \sqrt{\alpha r_t}dW_t,$$

and define $P_t(x)$ and $f_t(x)$ as in Exercise 11.3. Show that f is a mild solution of the HJM equation with $\lambda = 0$ and

$$\sigma_t(f)(u)(x) = -\sqrt{\alpha f(0)}\Psi'(x)u, \qquad t \in \mathbb{R}_+, f \in H, u \in U, x \in \mathbb{R}_+,$$

where for each $x \in \mathbb{R}_+$,

$$\Psi(x) = \frac{-2\left(e^{\gamma x} - 1\right)}{\gamma(e^{\gamma x} + 1) - \beta\left(e^{\gamma x} - 1\right)}, \qquad \gamma = \sqrt{\beta^2 + 2\alpha}.$$

Are the conditions of our existence and uniqueness result satisfied for this HJM equation?

Hint: use [Fil09, Section 5.4.2].

References

[Fil01] Damir Filipović. *Consistency problems for Heath-Jarrow-Morton interest rate models*. Springer, 2001.

[Fil09] Damir Filipović. *Term-structure models. A graduate course.* Springer Finance. Berlin: Springer-Verlag, 2009.