SPDEs 2016/17 Exercise Sheet 4

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4.1. Regularity of diagonal operators on Hilbert spaces

Let H be a Hilbert space with orthonormal basis B, let $\lambda : B \to \mathbb{R}$ be a function, and let $T : D(T) \subseteq H \to H$ be the diagonal linear operator given by

$$Tb = \lambda_b b, \qquad D(T) = \left\{ h \in H : \sum_{b \in B} |\lambda_b|^2 \langle b, h \rangle_H^2 < \infty \right\}.$$

We consider B as a measure space with the counting measure #. Show the following statements hold:

a) $T \in L(H)$ iff $\lambda \in L^\infty(B)$, and $\|T\|_{L(H)} = \|\lambda\|_{L^\infty(B)}$.

b) $T \in L_1(H)$ iff $\lambda \in L^1(B)$, and $\|T\|_{L_1(H)} = \|\lambda\|_{L^1(B)}$.

c) $T \in L_2(H)$ iff $\lambda \in L^2(B)$, and $\|T\|_{L_2(H)} = \|\lambda\|_{L^2(B)}$.

Hint: While this is not necessary, you might find it convenient to represent elements of the completed tensor product not by equivalence classes of Cauchy sequences, but as in [Rya02, Proposition 6.10]: any $u \in E \hat{\otimes}_{g_p} F$ can be represented as a convergent series $u = \sum_{n=1}^\infty x_n \otimes y_n$ such that $\|(x_n)\|_{p'}^w \|(y_n)\|_p$ is finite and arbitrarily close to $\|u\|_{E \hat{\otimes}_{g_p} F}$.

References

[Rya02] Raymond A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer Monographs in Mathematics. Springer, 2002.