SPDEs 2016/17 Exercise Sheet 2

Lecture and exercises: Philipp Harms, Tolulope Fadina
Due date: November 2, 2016

2.1. Injectivity of the Fourier transform

Show that E-valued random variables X_1 and X_2 are identically distributed if

$$\mathbb{E}\left[\exp\left(-i\langle X_1, x^*\rangle\right)\right] = \mathbb{E}\left[\exp\left(-i\langle X_2, x^*\rangle\right)\right], \qquad x^* \in E^*.$$

Note: you may use without proof that the Fourier transform is bijective on tempered distributions on $\mathbb{R}^n$, for each $n \in \mathbb{N}$.

2.2. Rotations of independent Gaussians

For iid centered Gaussian random variables X_1 and X_2, set $Y_1 := (X_1 + X_2)/\sqrt{2}$ and $Y_2 := (X_1 - X_2)/\sqrt{2}$. Show that Y_1 and Y_2 are iid and have the same distribution as X_1 and X_2.

Hint: use the previous exercise.

2.3. Convergence of Gaussians

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered Gaussian E-valued random variables, let X be an E-valued random variable, and assume that $\langle X_n, x^*\rangle \to \langle X, x^*\rangle$ in probability for each $x^* \in E^*$. Show that X is centered Gaussian.

2.4. Sazanov's theorem

Let H be a separable Hilbert space. Show that $Q \in L(H)$ is the covariance operator of a centered Gaussian H-valued random variable X if and only if Q is symmetric, non-negative definite, and $Tr(Q) < \infty$.