1. SPDEs 2016/17 Exercise Sheet

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1.1. Pointwise convergence of measurable functions

Let $(A, \mathscr{A})$ be a measurable space, let E be a Banach space, and let $f : A \to E$. Show that the following are equivalent:

(i) f is measurable;

(ii) f is the pointwise limit of measurable functions.

1.2. Separability

Show that the following properties hold for each subset A of a Banach space E:

(i) If E is separable, then A is separable.

(ii) If A is separable, then the closure of A is separable.

1.3. Norming sequences

Let E be a separable Banach space. Show that there exists a norming sequence of dual elements $x_n^* \in E^*$, i.e., for every $x \in E$ one has

$$\|x\| = \sup_n |\langle x, x_n^* \rangle|.$$

1.4. Measurability, weak measurability, and strong measurability

Let $U = L^2(\mathbb{R})$ and let $L(U)$ be the set of all linear bounded operators on U. Note that U is separable by the Stone-Weierstrass theorem.

a) Let $(S_t)_{t\in\mathbb{R}}$ be the translation group on U, which is given by $(S_t f)(x) = f(t+x)$ for each $f \in U$ and $t,x \in \mathbb{R}$. Show that $\|S_t - S_s\|_{L(U)} \geq \sqrt{2}$ holds for each $s,t \in \mathbb{R}$. Show that this implies that $L(U)$ is not separable and that $S : \mathbb{R} \to L(U)$ is not strongly measurable.

   Note: S is weakly measurable because it is weakly continuous. S is, however, not measurable. This follows from the following surprisingly difficult result: every measurable function on a finite compact measure space is almost surely separably valued [Fre03, Lemma 451Q].

b) Give an example of a function which is measurable, but not strongly measurable.

References

[Fre03] Torres Fremlin. *Measure Theory*. Vol. 4. Torres Fremlin, 2003.