1. **Complete Markets**

For each part of this exercise, you may use all previous parts without proof.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let $(\overline{S}_t)_{t=0,\dots,T}$ be a $(d+1)$-dimensional price process (including the numeraire).

(a) Let $A \in \mathcal{F}$ be an atom and $H \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. Show that H is constant on A. (1)
A set $A \in \mathcal{F}$ is called atom, if $\mathbb{P}(A) > 0$ and for every measurable subset $B \subseteq A$ one has either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = \mathbb{P}(A)$.

(b) Let $(A_k)_{k=1,\dots,n}$ be pairwise disjoint subsets of Ω with $\mathbb{P}(A_k) > 0$ for $k = 1, \dots, n$. Show that $(\mathbb{1}_{A_k})_{k=1,\dots,n} \subseteq L^p(\Omega, \mathcal{F}, \mathbb{P})$ are linearly independent for every $p \in [0, \infty]$. (1)

(c) Let $(A_k)_{k=1,\dots,n}$ be a partition of Ω into atoms. Show that $(\mathbb{1}_{A_k})_{k=1,\dots,n} \subseteq L^p(\Omega, \mathcal{F}, \mathbb{P})$ forms a basis for every $p \in [0, \infty]$. (1)

(d) Conclude that, for $p \in [0, \infty]$, we have (2)

$$\dim L^p(\Omega, \mathcal{F}, \mathbb{P}) = \sup\{n \in \mathbb{N} : \exists \text{ partition } (A_k)_{k=1,\dots,n} \text{ of } \Omega : \mathbb{P}(A_k) > 0\}.$$

(e) For $T = 1$, assume the market $(\overline{S}_t)_{t=0,\dots,T}$ is complete. Show that $\dim L^0(\Omega, \mathcal{F}, \mathbb{P}) \leq d + 1$. (1)

(f) For $T \geq 2$, assume the market $(\overline{S}_t)_{t=0,\dots,T}$ is complete. Show that the restricted market $(\overline{S}_t)_{t=0,\dots,T-1}$ is also complete. (2)

(g) For $T \geq 2$, assume the market $(\overline{S}_t)_{t=0,\dots,T}$ is complete. Show that $\dim L^\infty(\Omega, \mathcal{F}, \mathbb{P}(\cdot \mid A)) \leq d + 1$, for every atom A of $\mathcal{F}_{T-1}$. Here, $\mathbb{P}(\cdot \mid A)$ is the (elementary) conditional probability of $\mathbb{P}$ given A. (1)

(h) Let $(A_k)_{k=1,\dots,n}$ be a partition of Ω with $\mathbb{P}(A_k) > 0$ for $k = 1, \dots, n$. Show, for $p \in \{0, \infty\}$, that the map (1)

$$L^p(\Omega, \mathcal{F}, \mathbb{P}) \ni X \mapsto (X \mathbb{1}_{A_k})_k \in \prod_k L^p(\Omega, \mathcal{F}, \mathbb{P}(\cdot \mid A_k))$$

is well-defined and injective.
Hint. Let $\mathbb{Q} \ll \mathbb{P}$. Then, for $p \in \{0, \infty\}$, the map $L^p(\Omega, \mathcal{F}, \mathbb{P}) \ni X \mapsto X \in L^p(\Omega, \mathcal{F}, \mathbb{Q})$ is well-defined.

(i) Assume the market $(\overline{S}_t)_{t=0,\dots,T}$ is complete. Show that $\dim L^0(\Omega, \mathcal{F}, \mathbb{P}) \leq (d+1)^T$. (2)
Hint. You may use without proof the following fact: let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be finite dimensional vector spaces over $\mathbb{R}$. Then, $\dim(\prod_k \mathcal{X}_k) = \sum_k \dim \mathcal{X}_k$.

Points for Question 1: 12

You can achieve a total of **12 Bonus** points for this sheet. This means, they are not relevant for the total number of points achievable, but they do add to the number of achieved points.