1. **Useful Facts**

   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ σ-algebras. Show:

   (a) For $Z \in L^1(\mathcal{F})$ with $\sigma(Z, \mathcal{A})$ independent of $\mathcal{B}$ we have (2)

   $$\mathbb{E}[Z|\sigma(\mathcal{A}, \mathcal{B})] = \mathbb{E}[Z|\mathcal{A}]\,.$$

   (b) Let $Z_1, \dots, Z_n \in L^1$ be i.i.d. random variables for some $n \in \mathbb{N}$. Then (2)

   $$\mathbb{E}\left[Z_j \middle| \sum_{k=1}^n Z_k\right] = \frac{1}{n}\sum_{k=1}^n Z_k$$

   holds for every $j = 1, \dots, n$.

   Points for Question 1: 4

2. **Insider Information**

   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $Y_1, \dots, Y_T$ i.i.d. random variables in L^1 which are not $\mathbb{P}$-a.s. constant and satisfy $\mathbb{E}[Y_1] = 0$. We observe the filtration defined by $\mathcal{F}_t := \sigma(Y_1, \dots, Y_t)$ for $t = 0, \dots, T$ and the price processes $X_t^0 = 1$, $X_0^1 := 1$,

   $$X_t^1 := X_0^1 + \sum_{s=1}^t Y_s$$

   for $t = 1, \dots, T$. Furthermore, we enlarge the filtration with the insider information X_T^1, i.e. for $t = 0, \dots, T$ let

   $$\hat{\mathcal{F}}_t := \sigma(\mathcal{F}_t, X_T^1)\,.$$

   (a) Show that X^1 is a martingale with respect to $(\mathcal{F}_t)$, but not with respect to $(\hat{\mathcal{F}}_t)$. (3)

   (b) Show that the process defined by (3)

   $$\hat{X}_t := X_t^1 - \sum_{s=0}^{t-1} \frac{X_T^1 - X_s^1}{T - s}$$

   for $t = 0, \dots, T$ is a martingale with respect to the enlarged filtration.

   (c) Compute a self-financing strategy $\bar{\xi}$, which maximizes the expected gain $\mathbb{E}[G_T]$ for (2)

   $$G_T = \sum_{s=1}^T \xi_s^1 \Delta X_s^1$$

   over all self-financing strategies with respect to $(\hat{\mathcal{F}}_t)$ with $\left|\xi_t^1\right| \leq 1$ for every $t = 1, \dots, T$.

   Points for Question 2: 8

You can achieve a total of **12** points for this sheet.