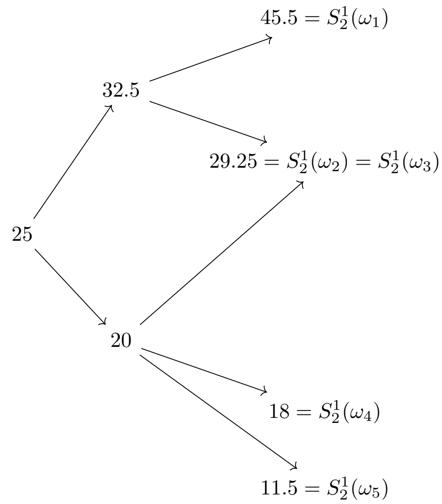
1. **A Discrete Market**
Let $\Omega = \{\omega_1, \dots, \omega_5\}$, $\mathbb{P}$ a strictly positive probability measure on Ω with $\mathcal{F} = 2^\Omega$. Consider the risky asset $(S^1_t)_{t=0,\dots,2}$ with the following evolution

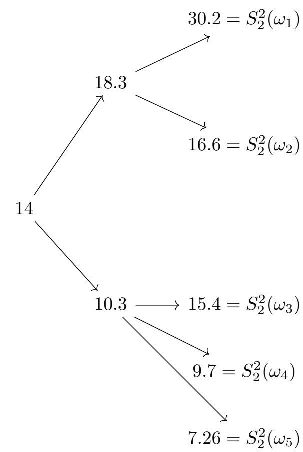


(a) Compute the filtration $(\mathcal{F}_t)$ generated by S^1. Compare $\mathcal{F}_2$ and $\sigma(S^1_2)$. (3)

(b) Compute the set of equivalent martingale measures. For this part, and the remaining exercise assume $S^0 = 1$. (3)

(c) Let $C = (S^1_2 - 25)^+$ be a European Call-option with strike 25. Compute the set of arbitrage-free prices $\{\mathbb{E}^{\mathbb{Q}}[C] : \mathbb{Q}$ equivalent martingale measure$\}$ for C. (3)

(d) We extend the market by a second risky asset S^2. Assume the following evolution for S^2: (3)



Construct an arbitrage for the extended market.

Points for Question 1: 12

2. **Characterization of Martingales**
Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space, and let $(X_t)_{t=0,\dots,T}$ be a process. Show that the following are equivalent:

- (X_t) is a martingale.
- (X_t) is a uniformly integrable martingale.
- There exists $X \in L^1(\mathcal{F}_T)$ such that $X_t = \mathbb{E}[X|\mathcal{F}_t]$ for $t = 0, \dots, T$.

Points for Question 2: 4

3. **Exponential**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Further, let $X_0, \dots, X_T$ be i.i.d. with law $\mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$. Define the process (M_t) by

$$M_t := \exp\Big(\sum_{0 \le k \le t} X_k \Big).$$

Show that M is a martingale if and only if $\mu = -\frac{1}{2}\sigma^2$. Points for Question 3: 5

4. **Predictable Martingales**
Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We call a martingale M predictable, if M_t is $\mathcal{F}_{t-1}$-measurable for $t \in \{1, \dots, T\}$ and $M_0 = 0$. Show that every predictable martingale is already zero. Points for Question 4: 3

You can achieve a total of **24** points for this sheet.