1. **Properties of the Expected Value**
   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove the following assertions:
   (a) For a random variable $X \in L^0_+(\mathbb{P})$ the following are equivalent: (1)
   - $X = 0$
   - $\mathbb{E}[X] = 0$

   (b) For a random variable $X \in L^1(\mathbb{P})$ the following are equivalent: (2)
   - $X = 0$
   - $\mathbb{E}[X\mathbb{1}_A] = 0$ for every $A \in \mathcal{F}$

   Points for Question 1: 3

2. **The Radon-Nikodym Derivative**
   Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on a measurable space $(\Omega, \mathcal{F})$. We write $\mathbb{Q} \ll \mathbb{P}$ if $\mathbb{Q}(A) = 0$ for every $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$. In this case one can show that there exists a non-negative function $d\mathbb{Q}/d\mathbb{P} \in L^1(\mathbb{P})$ such that $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A(d\mathbb{Q}/d\mathbb{P})]$ for all $A \in \mathcal{F}$. We call $d\mathbb{Q}/d\mathbb{P}$ the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. If $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$, we denote this by $\mathbb{Q} \sim \mathbb{P}$. Now let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on a measurable space $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$.
   (a) Show that $d\mathbb{Q}/d\mathbb{P}$ is $\mathbb{P}$-a.s. unique. (1)
   (b) Show that $X \in L^1(\mathbb{Q})$ if and only if $X(d\mathbb{Q}/d\mathbb{P}) \in L^1(\mathbb{P})$. Show that in this case we have (2)
   $$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[X(d\mathbb{Q}/d\mathbb{P})],$$
   and that this equality also holds for every $X \geq 0$.
   (c) Show that $\mathbb{P} \ll \mathbb{Q}$ if and only if $d\mathbb{Q}/d\mathbb{P} > 0$ $\mathbb{P}$-a.s.. Show that in this case we have $d\mathbb{P}/d\mathbb{Q} = (d\mathbb{Q}/d\mathbb{P})^{-1}$. (2)

   Points for Question 2: 5

3. **A First Example**
   Let $\Omega = [90, 110] \subseteq \mathbb{R}$ and let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-field. We interpret $\omega \in \Omega$ as the outcome of a risky asset. Consider a call option with strike $K = 100$ given by
   $$X(\omega) := (\omega - K)^+ := (\omega - K)\mathbb{1}_{[100,110]}(\omega).$$
   (a) Determine $\sigma(X)$ and decide whether $[90, 95] \in \sigma(X)$. (2)
   (b) Equip $(\Omega, \mathcal{F})$ with the normalized Lebesgue measure $\mathbb{P} = \frac{1}{20}\lambda$. Suppose you paid 2 Euro for the payoff X. How high is the probability that you make profit with this investment? (2)

   Points for Question 3: 4

You can achieve a total of **12** points for this sheet.