Stochastische Prozesse **WS 15/16**

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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 9

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For a semimartingale X, $X_0 = 0$, the stochastic exponential of X, $\mathcal{E}(X)$, is the unique semimartingale Z that is the solution of

$$Z_t = 1 + \int_0^t Z_{s-} dX_s, \qquad t \geq 0. \tag{1}$$

Problem 1 (4 Points). (*a*) Let X and Y be two semimartingales with $X_0 = Y_0 = 0$.
Show that

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

(*b*) Let X be a continuous semimartingale, $X_0 = 0$.
Show that

$$\mathcal{E}(X)^{-1} = \mathcal{E}(-X + [X, X]).$$

Problem 2 (4 Points). Assume that $P' \overset{loc}{\ll} P$ and $Z = \mathcal{E}(X)$ is the density process. Let

$$Y_t = \sum_{i=1}^{N_t} \xi_i \qquad ((\xi_i)_{i\geq 1} \text{ are i.i.d random variables and } Y_0 = 0)$$

be a compound Poisson process where N is a Poisson process with intensity λ. We denote $X = H \cdot M$, where H is a constant and M given $M = Y - \langle Y \rangle$ is a local P-martingale.
If $\mathbb{E}|\xi_1| < \infty$;
Compute

$$M'' = M - \frac{1}{Z_-} \cdot \langle M, Z \rangle .$$

Problem 3 (4 Points). If $\mathbb{E}|\xi_1| < \infty$ (see Problem 2 for other conditions):

(*a*) Solves (1) with

$$X_t = \sum_{i=1}^{N_t} \xi_i - \lambda t \mathbb{E}(\xi_1).$$

(*b*) Show that

$$Z_t = \prod_{i=1}^{N_t} (1 + \xi_i) \exp(-\lambda t \mathbb{E}(\xi_1))$$

is a martingale.

Problem 4 (4 Points). Let $(\Omega, \mathcal{F}, P)$ be a probability space and $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a Brownian motion defined on it. Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a measurable and adapted process satisfying

$$P\left(\int_0^T X_t^2 dt < \infty\right) = 1, \quad 0 \leq T < \infty.$$

We set

$$Z_t = \exp\left(\int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds\right).$$

Note: The stochastic integral with respect to W is well defined and belongs to $\mathcal{M}_{loc}^c$.

(a) Assume that Z_t is a martingale. We define a process $W' = \{W'_t, \mathcal{F}_t; 0 \le t < \infty\}$ by

$$W'_t = W_t - \int_0^t X_s ds, \qquad 0 \le t < \infty.$$

For each fixed T, show that W' is a Brownian motion on $(\Omega, \mathcal{F}, P'_T)$.

(b) Recall

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

is the Black-Scholes stochastic differential equation under P.
Compute the new Black-Scholes stochastic differential equation under the change of measure.

Hint: Assuming Z_t is a martingale. If $M \in \mathcal{M}^c_{loc}$, then the process

$$M'_t = M_t - \int_0^t X_s d\langle M, W\rangle_s, \quad 0 \le t \le T$$

which is $\mathcal{F}_t$-measurable is in $\mathcal{M}'^c_{loc}$. If $G \in \mathcal{M}^c_{loc}$ and

$$G'_t = G_t - \int_0^t X_s d\langle G, W\rangle_s, \quad 0 \le t \le T,$$

then $\langle M', G'\rangle_t = \langle M, G\rangle_t; 0 \le t \le T$, a.s P and P'_T. Finally, use the Lévy 's characterization for Brownian motion: Let W be a local martingale, $W_0 = 0$. W is a Brownian motion implies $\langle W\rangle_t = t$.