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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 8

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We say that the adapted cadlág process X solves the Stochastic differential equation

$$dX(t) = a(X(t))dt + b(X(t))dW(t)$$

if for all $t \geq 0$ it holds that

$$X(t) = X(0) + \int_0^t a(X(s))ds + \int_0^t b(X(s))dW(s), \qquad t \geq 0.$$

(see Problem 1 and so on.)

Problem 1 (4 Points)**.** Show that the exponential martingale

$$X(t) = \exp(W(t)) \exp(-t/2), \qquad t \geq 0$$

is an Itô process and verify that it satisfies the equation

$$dX(t) = X(t)dW(t).$$

Hint: Use the Itô formula with $f(t,x) = \exp(x)\exp(-t/2)$. Recall $\mathbb{E}[\exp(W_t)] = \exp(t/2)$.

Problem 2 (4 Points)**.** Applying the Itô formula to $f(t,x) = x^n$, for $t \geq 0$, show that

$$dW(t)^n = \frac{n(n-1)}{2} W(t)^{n-2}dt + nW(t)^{n-1}dW(t) \qquad .$$

Hint: This is a direct application of the Itô formula. We assume that $nW(t)^{n-1}$ is square integrable.

Problem 3 (4 Points)**.** (Ornstein-Uhlenbeck Process) Suppose that $\alpha > 0$ and $\sigma \in \mathbb{R}$ are fixed. Let $Y(t)$, $t > 0$ be an adapted modification of the Itô integral

$$Y(t) = \sigma \exp(-\alpha t) \int_0^t \exp(\alpha s)dW(s)$$

with almost surely continuous paths.
Show that $Y(t)$ satisfies

$$dY(t) = -\alpha Y(t)dt + \sigma dW(t).$$

Hint: $Y(t) = f(t, \xi(t))$ with

$$\xi(t) = \sigma \int_0^t \exp(\alpha s)dW(s)$$

and $f(t,x) = \exp(-\alpha t)x$.

Problem 4 (4 Points)**.** Suppose that $a, \sigma \in \mathbb{R}$ are fixed.

(a) Show that $S(t)$ defined by

$$S(t) = S_0 \exp\left(at + \sigma W(t)\right), \qquad t \geq 0 \tag{1}$$

is a solution of the linear stochastic differential equation

$$dS(t) = \left(a + \frac{\sigma^2}{2}\right) S(t)dt + \sigma S(t) dW(t), \tag{2}$$

with initial condition $X(0) = X_0$.

Hint: Use the Itô formula with

$$f(t, x) = \exp\left(at + \sigma x\right).$$

(b) If $a = \mu - \sigma^2/2$ in (1), then,

$$S(t) = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma W(t)\right), \qquad t \geq 0$$

is a solution of the Black-Scholes stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

Show that $S(t)$ is a martingale if and only if $\mu = 0$.

Hint: Set $S_0 = 1$, $\sigma = 1$ and W is a Brownian motion.