UNI FREIBURG

Stochastische Prozesse

WS 15/16

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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 7

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A compound Poisson process is a process of the form:

$$X_t = \sum_{k=1}^{N_t} Y_k, \quad X_0 = 0$$

where N is a Poisson process with intensity λ, $(Y_k)_{k\geq 1}$ are i.i.d random variables with cumulative distribution function $F(y) = \mathbb{P}(Y_1 \leq y)$, independent of N. Let $F(dy)$ be the measure associated with F and F^n its n-th convolution. i.e.,

$$F^n(y) = \mathbb{P}\left(\sum_{k=1}^{n} Y_k \leq y\right).$$

Problem 1 (4 Points). (*a*) Show that X has independent and stationary increments.

(*b*) Show that the cumulative distribution function of the random variable X_t is

$$\mathbb{P}(X_t \leq x) = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F^n(x).$$

Problem 2 (4 Points). If $\mathbb{E}|Y_1| < \infty$;

(*a*) Show that

$$\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1).$$

(*b*) Show that the process

$$M_t = X_t - \lambda t \mathbb{E}(Y_1)$$

for $t \geq 0$ is a martingale.

Problem 3 (4 Points). Let $(N_t)_{t\geq 0}$ be a Poisson process with a constant intensity λ and M the associated compensated martingale, i.e., $M_t = N_t - \lambda t$.

(*a*) Let $(X_i, Y_i)_{i\geq 1}$ be i.i.d random variables, independent of N, and let

$$U_t = \sum_{i=1}^{N_t} X_i \quad \text{and} \quad V_t = \sum_{i=1}^{N_t} Y_i$$

be compound Poisson processes.
Compute $[U, V]_t$ and $\langle U, V\rangle_t$.

(*b*) Let p and q be two square integrable functions and

$$X_t = \int_0^t p(s) dM_s, \quad Y_t = \int_0^t q(s) dM_s.$$

Compute $[X, Y]_t$ and $\langle X, Y\rangle_t$.

Problem 4 (4 Points). An inhomogeneous Poisson process N with intensity λ is a counting process with independent increments which satisfies

$$\mathbb{P}(N_t - N_s = n) = \exp(-\Lambda(s,t))\frac{(\Lambda(s,t))^n}{n!}$$

where

$$\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$$

and

$$\Lambda(t) = \int_0^t \lambda(u)du \quad \text{for } t \geq s.$$

(a) Show that

$$M_t = N_t - \int_0^t \lambda(u)du$$

for $t \geq 0$ is a martingale.

(b) Give two examples of non-semimartingales.