Exercise 5

Submission: 17-11-2015

Problem 1 (4 Points). (*a*) Let $(X_t)_{t\geq 0}$ be a submartingale (respectively martingale) with respect to $\mathcal{F}_t$, and $\psi : \mathbb{R} \to \mathbb{R}$ be a convex function such that

$$\mathbb{E}[\psi(X_t)] < \infty \text{ holds for every } t \geq 0.$$

Show that, if $\psi(X_t)$ is nondecreasing, $\psi(X_t)$ is a submartingale with respect to $\mathcal{F}_t$.

(*b*) Let $(X_n)_{n=1,\cdots,N}$ be a supermartingale with respect to $\mathcal{F}_n$, and let τ, σ be two stopping times. i.e., $\tau, \sigma \leq N$. Show that

$$X_\sigma \geq \mathbb{E}[X_\tau | \mathcal{F}_\sigma] \quad \text{on } \{\tau \geq \sigma\} \quad P - a.s.$$

Problem 2 (4 Points). (*a*) Show that if $(X_n)_{n\in\mathbb{N}}$ is a non-negative supermartingale, then it converges to an integrable random variable.
Hint: To apply the Doob's martingale convergence theorem (see lecture note), all you need to verify is that the sequence X_n is bounded in L^1 i.e.,

$$\sup \mathbb{E}[|X_n|] < \infty.$$

(*b*) Suppose $X = (X_n)_{n\in\mathbb{N}}$ and $X' = (X'_n)_{n\in\mathbb{N}}$ are supermartingales with respect to $\mathcal{F}_n$. Show that $X \wedge X'$ is also a supermartingale.

Problem 3 (4 Points). Consider measurable $\psi > 0$ such that $\frac{\psi(x)}{x} \to \infty$ as $x \to \infty$ and a family of random variables $\mathcal{C}$. Suppose

$$\mathbb{E}[\psi(|X|)] \leq B < \infty \quad \forall X \in \mathcal{C}$$

and constant B. Show that $\mathcal{C}$ is uniformly integrable.

Problem 4 (4 Points). (*a*) Consider $X \in L^1(P)$ and define

$$\mathcal{C} = \{\mathbb{E}[X|\mathcal{G}] \quad \text{for sigma field } \mathcal{G} \subset \mathcal{F}\}.$$

Show that $\mathcal{C}$ is uniformly integrable.
Hint: Use the following Lemma: If $X \in L^1(P)$, there exist a convex function ψ such that $\frac{\psi(x)}{x} \to \infty$ as $x \to \infty$ and

$$\mathbb{E}[\psi(|X|)] < \infty.$$

(*b*) Show that a martingale $(X_t)_{0\leq t\leq T}$ for $0 < T < \infty$ is uniformly integrable.