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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 2

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Problem 1 (4 Points). Let $X = (X_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$.

(a) Show that $Y_t = X_t - \lambda t$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the family of random variables $\{X_s : s \in [0,t]\}$.
(Hint: Y_t is a martingale implies (i) $Y_t \in L^1$, for all $t \geq 0$. (ii) $\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s$, for $0 \leq s \leq t$.

(b) Show that

$$\lim_{t\to\infty} \frac{X_t}{t} = \lambda \quad a.s.$$

Problem 2 (4 Points). (a) Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two independent Poisson processes with parameters $\lambda > 0$ and $\mu > 0$. Show that $(X_t + Y_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda + \mu$.

(b) Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two independent standard Brownian motions. Show that

$$\frac{X_t + Y_t}{\sqrt{2}}$$

is also a standard Brownian motion.

Problem 3 (4 Points). Let $(\xi_n)_{n\geq 0}$ be a martingale with a τ stopping time with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ such that the following conditions holds:

(i) $\tau < \infty \quad a.s.$

(ii) ξ_τ is integrable

(iii) $\mathbb{E}[\xi_n \mathbf{1}_{\tau > n}] \to 0$ as $n \to \infty$.

Then

$$\mathbb{E}[\xi_\tau] = \mathbb{E}[\xi_1]$$

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Problem 4 (4 Points). A real-valued stochastic process $X = (X_t)_{t\geq 0}$ is measurable if the mapping $[0,\infty) \times \Omega \to \mathbb{R}^d$; $(t,\omega) \mapsto X_t(\omega)$ is $\mathcal{B}([0,\infty]) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^d)$ measurable.

(a) Give an example of a predictable process.

(b) Give an example of an optional process which is not predictable

(c) Give an example of an adapted stochastic process X that is not measurable.

(d) Show that every progressively measurable stochastic process $X = (X_t)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is also measurable.