UNI FREIBURG

Stochastische Prozesse **WS 15/16**

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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015SoSeStochastische prozesse

Exercise 1

Submission: 22-10-2015

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ be a measurable space. Let $X = (X_n)_{n\geq 0}$ be a sequence of random variables taking value in E. We call X a stochastic process in E.

A filtration $(F_n)_n$ is an increasing family of sub $\sigma-$algebras of $\mathcal{F}$. i.e., $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. We can think of $\mathcal{F}_n$ as the information available to us at time n. Every process has a natural filtration $(\mathcal{F}_n^X)_n$ given by $F_n^X = \sigma(X_k, k \leq n)$.
The process X is called adapted to the filtration $(\mathcal{F}_n)_n$, if X_n is $\mathcal{F}_n$-measurable for all n. Every process is adapted to a natural filtration. We say X is integrable if X_n is integrable for all n.

Definition 1 (Martingale). A sequence $\xi_1, \xi_2, \cdots$, of random variables is called a martingale with respect to the filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$, if

(1) ξ_n is integrable for each $n = 1, 2, \cdots$

(11) $\xi_1, \xi_2, \cdots$, is adapted to $\mathcal{F}_1, \mathcal{F}_2, \cdots$,

(111) $\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = \xi_n \quad a.s.$ for each $n = 1, 2, \cdots$

Definition 2. Let $(\xi_k, k \geq 1)$ be i.i.d. (independent and identically distributed) random variables. Then

$$S_n = \sum_{k=1}^{n} \xi_k, \quad n \in \mathbb{N},$$

is a random walk. Random walks have stationary and independent increments,

$$\xi_k = S_k - S_{k-1} \quad k \geq 1.$$

Stationary simply implies that the $(\xi_k)_{k\geq 1}$ have identical distribution.

Definition 3. A process X_n, $n \in \mathbb{N}$ with stationary independent increments is called a Lévy process. i.e., the increment $X_{n_k} - X_{n_{k-1}}$ are independent and $X_{n_k} - X_{n_{k-1}} \sim X_{n_k - n_{k-1}}$, for $k = 1, \cdots, n$

Definition 4 (Markov chain). A discrete process $\{X_n, n = 0, 1, \cdots\}$ with discrete state space $X_n \in \{0, 1, 2, \cdots\}$ is a Markov chain if it has the Markov property

$$\mathbb{P}[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0] = \mathbb{P}[X_{n+1} = j | X_n = i].$$

Problem 1. Let $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H} \subset \mathcal{G}$ be $\sigma-$algebras. Show that

$$\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[\xi|\mathcal{H}] \quad a.s. \tag{1}$$

Problem 2. Show that if $\xi = (\xi_n)_{n\geq 1}$ is a martingale with respect to $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$, then

$$\mathbb{E}(\xi_1) = \mathbb{E}(\xi_2) = \cdots .$$

Hint: What is the expectation of $\mathbb{E}(\xi_{n+1}|\mathcal{F}_n)$?

Problem 3. Suppose that $\xi = (\xi_n)_{n\geq 1}$ is a martingale with respect to the filtration $\mathcal{G} = (\mathcal{G}_n)_{n\geq 1}$. Show that ξ is a martingale with respect to the filtration

$$\mathcal{H}_n = \sigma(\xi_1, \cdots, \cdots, \xi_n)$$

Hint: Observe that $\mathcal{H}_n \subset \mathcal{G}_n$ and use the tower property of conditional expectation, (1).

Problem 4. Let $\xi = (\xi_k)_{k\geq 1}$ be independent and in L^1 (see Definition (2)) show that

$$S'_n = \sum_{k=1}^{n} (\xi_k - \mathbb{E}[\xi_k])$$

satisfies the Martingale property.

Problem 5. Given a martingale $(S_n)_{n\geq 1}$, show that

$$\mathbb{E}[S_n|\mathcal{F}_m] = \mathbb{E}[S_n|S_m], \quad \text{for } m < n$$

which implies the Markov property.