Stochastic Filtering (SS2016) Exercise Sheet 12

Lecture and Exercises: JProf. Dr. Philipp Harms
Due date: July 20, 2016

Sequential tests

The following exercises are a guided tour of sequential tests—one of the oldest and most important application of filtering and optimal control theory. For a detailed treatment we refer to [1, Chapter VI.21].

We work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_{\pi_0})$ satisfying the usual conditions. The hidden state is a Bernoulli random variable $X \sim \mathrm{Ber}(\pi_0)$ for some $\pi_0 \in [0,1]$. The observation process $(Y_t)_{t\geq 0}$ is given by $Y_t = \mu t X + \sigma B_t$, where $(B_t)_{t\geq 0}$ a standard $(\mathscr{F}_t)$-Wiener process independent of X and $\mu \neq 0, \sigma > 0$ are given constants.

A sequential test for the hypothesis $X = 0$ versus $X = 1$ consists of an $\mathbb{F}(Y)$-stopping time T and an $\mathscr{F}^Y_T$-measurable random variable $\hat{X}$. The interpretation is that after stopping at time T, the random variable $\hat{X}$ indicates which hypothesis should be accepted under the test. The objective is to minimize the stopping time and the probabilities of type-I and type-II errors. More precisely, for given constants $a, b > 0$, one looks for minimizers $(T^*, \hat{X}^*)$ of

$$V(\pi_0) = \inf_{(T,\hat{X})} \mathbb{E}_{\pi_0}\left[T + a1_{\{X=1,\hat{X}=0\}} + b1_{\{X=0,\hat{X}=1\}}\right]. \tag{1}$$

12.1. Deriving the filtering equation

Let $(\pi_t)_{t\geq 0}$ be the $(\mathscr{F}^Y_t)$-optional projection of X.

a) Show that $(\pi_t)_{t\geq 0}$ satisfies

$$d\pi_t = \frac{\mu}{\sigma^2}\pi_t(1-\pi_t)\left(dY_t - \mu\pi_t dt\right), \quad \pi_0 = \pi_0. \tag{2}$$

Note: $(\pi_t)_{t\geq 0}$ takes values in $[0,1]$. Of course, the interval $[0,1]$ can identified with the set of probability measures on $\{0,1\}$.

b) Show that $f(\pi_t) - f(\pi_0) - \int_0^t \mathscr{A} f(\pi_s)ds$ is an $(\mathbb{F}(Y),\mathbb{P})$-martingale for each $f \in C^2([0,1])$, where

$$\mathscr{A} f(\pi) = \frac{\mu^2}{2\sigma^2}\pi^2(1-\pi)^2\frac{\partial^2 f(\pi)}{\partial \pi^2}.$$

12.2. Reduction to an optimal stopping problem

Show that

$$V(\pi_0) = \inf_T \mathbb{E}_{\pi_0}[T + g(\pi_T)], \qquad \text{where } g(\pi) = \min\{a\pi, b(1-\pi)\}. \tag{3}$$

Hint: For any fixed stopping time T, $\hat{X}^* = \mathbb{1}_{\{a\pi_T \geq b(1-\pi_T)\}}$ is optimal in (1).

Dynamic programming formulation

The Hamilton-Jacobi-Bellman equation associated to the stopping problem (3) is

$$\min\{\mathscr{A}W(\pi)+1, g(\pi)-W(\pi)\} = 0, \qquad \forall \pi \in [0,1].$$

The relation to the stopping problem will become clear in the following steps.

It can be shown using ODE methods that this equation has a unique[1] solution $W : [0,1] \to \mathbb{R}$. The function W is C^1 and piecewise C^2. Moreover, despite the possible

[1] The solution is unique in the viscosity sense.

singularities of W, Itō's formula in its standard form can be applied to the process $W(\pi_t)$. The function W has the following structure: there exist constants A, B satisfying $0 < A < B < 1$ such that $0 = \mathscr{A}W + 1$ holds on the interval (A, B), and $0 = g - W$ holds on the interval $[0, A] \cup [B, 1]$.

12.3. The function W is a lower bound for V

Show that $V(\pi_0) \geq W(\pi_0)$ holds for all $\pi_0 \in [0, 1]$.

Hint. Show for any $\mathbb{F}(Y)$-stopping time T that

$$\mathbb{E}[T + g(\pi_T)] \geq \mathbb{E}[T + W(\pi_T)] \geq W(\pi_0). \tag{4}$$

The first inequality follows directly from the HJB equation. To see the second inequality, use Itō's formula to express $W(\pi_T)$ as $W(\pi_0) + \int_0^T \mathscr{A}W(\pi_s)ds + M_T$, where M is an $(\mathscr{F}_t^Y)$-martingale. Then use the HJB equation to bound $\mathscr{A}W(\pi_s)$ from below.

12.4. The function W is equal to V

a) Show that $\mathbb{E}[T^* + g(\pi_{T^*})] = W(\pi_0)$, where $T^* = \inf\{t \geq 0 : \pi_t \notin (A, B)\}$.

   Hint. Show that equality holds in (4) with $T = T^*$.

b) Conclude that $V = W$ holds identically on $[0, 1]$ and that T^* is a minimizer of (1).

References

[1] Goran Peskir and Albert Shiryaev. *Optimal stopping and free-boundary problems*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, 2006.