Stochastic Filtering (SS2016) Exercise Sheet 8

Lecture and Exercises: JProf. Dr. Philipp Harms
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8.1. Kushner-Stratonovich and Zakai equation

Let X solve the martingale problem associated to $A : \mathscr{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$ and let $dY_t = h(X_t)dt + dW_t$. Recall that for any bounded measurable function $f : \mathbb{X} \to \mathbb{R}$, $\rho(f)$ is defined as the $(\mathbb{F}(Y), \tilde{\mathbb{P}})$-optional projection of $\Lambda f(X)$, where

$$\mathbb{P}|_{\mathscr{F}_t} = \Lambda_t \tilde{\mathbb{P}}|_{\mathscr{F}_t}, \qquad \Lambda = \mathscr{E}(h(X) \bullet Y),$$
$$\tilde{\mathbb{P}}|_{\mathscr{F}_t} = Z_t \mathbb{P}|_{\mathscr{F}_t}, \qquad Z = \mathscr{E}(-h(X) \bullet W).$$

a) Assume that h is bounded. Show that $\rho(1) = \mathscr{E}(\pi(h) \bullet Y)$.

   Hint: Show that for any bounded stopping time T

$$\tilde{\mathbb{E}}\left[\mathbb{1}_{T<\infty}\Lambda_T\right] = \tilde{\mathbb{E}}\left[\mathbb{1}_{T<\infty}\int_0^T \pi_s(h)\rho_s(1)dY_s\right]$$

   by applying the martingale representation theorem to $\mathbb{1}_{T<\infty}$.

b) Deduce the Zakai equation from the Kushner-Stratonovich equation and a).

8.2. Change of measure approach for jump processes

Let X solve the martingale problem associated to $A : \mathscr{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$ and let Y be a Poisson process with rate $\lambda(X_-)$, i.e., $Y_t = N_{\int_0^t \lambda(X_{s-})ds}$, where N is a standard Poisson process independent of X and $\lambda : \mathbb{X} \to (0, \infty)$ is a measurable function. Assume that λ and λ^{-1} are bounded.

Define a change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ via

$$\mathbb{P}|_{\mathscr{F}_t} = \Lambda_t \tilde{\mathbb{P}}|_{\mathscr{F}_t}, \qquad d\Lambda_t = \Lambda_{t-}\big(\lambda(X_{t-}) - 1\big)\big(dY_t - dt\big),$$
$$\tilde{\mathbb{P}}|_{\mathscr{F}_t} = Z_t \mathbb{P}|_{\mathscr{F}_t}, \qquad dZ_t = Z_{t-}\big(\lambda^{-1}(X_{t-}) - 1\big)\big(dY_t - \lambda(X_{t-})dt\big).$$

It can be shown that the law of (X,Y) under $\tilde{\mathbb{P}}$ equals the law of (X,N) under $\mathbb{P}$.

a) Show that $[M^f, Y] = 0$, where

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_{s-})ds.$$

Sketch of proof: Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping times exhausting the jumps of M^f. Under the measure $\tilde{\mathbb{P}}$, the process Y is a standard Poisson process independent of X. Therefore, it has no fixed times of discontinuity, i.e., $\tilde{\mathbb{P}}[Y_t \neq 0] = 0$ holds for each deterministic time $t \in \mathbb{R}$. Together with the independence of T_n and Y this implies $\tilde{\mathbb{P}}[\Delta Y_{T_n} \neq 0] = 0$ for each $n \in \mathbb{N}$. Thus, M^f and Y have no common jumps and $[M^f, Y] = 0$.

b) Derive the Zakai equation, i.e.,

$$d\rho_t(f) = \rho_t(Af)dt + \big(\rho_{t-}(\lambda f) - \rho_{t-}(f)\big)(dY_t - dt).$$

Hint: You can use exactly the same steps as in the lecture, where Zakai's equation was derived for observations with additive Gaussian noise.

8.3. Singular filtering and stochastic volatility

This example shows why one typically assumes that the volatility of the observational noise does not depend on the signal process X.

We work on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. The signal is a càdlàg, $\mathbb{F}$-adapted, $\mathbb{X}$-valued process X.The observation process Y satisfies $Y_0 = 0$ and

$$dY_t = h(X_t)dt + \sigma(X_{t-})dW_t,$$

where W is a standard $\mathbb{F}$-Wiener process, $\sigma : \mathbb{X} \to [0,\infty)$ and $h : \mathbb{X} \to \mathbb{R}$ are both continuous.

a) Suppose $\mathbb{X} = [0,\infty)$ and σ is bijective. Derive an expression for the filter π.

   Hint: Show that X is $\mathbb{F}(Y)$-adapted.

b) Let $T > 0$ be fixed and consider an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $[0,T]$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in $[0,T]$. Suppose that the process Y from a) is observable only at the time-points $t \in A_n$. Let $\mathscr{G}_n$ be the (augmented) σ-algebra generated by $\{Y_t : t \in A_n\}$. Show that for any bounded f, we have

$$\lim_{n\to\infty} \mathbb{E}[f(X_T)|\mathscr{G}_n] = \pi_T(f) \quad \text{a.s.}$$

   Hint: Apply Doob's martingale convergence theorem.

c) Compare the results of this exercise to Exercise 7.4. Give a mathematically precise interpretation of the following sentence: "High-frequency data allows one to estimate volatilities, but not drifts."

8.4. Singular filtering of a two-dimensional process

This is an example of a singular filtering problem with an explicit solution. Consider the setting of Exercise 8.3 with $\mathbb{X} = \mathbb{R}^2$, $h(x) = 0$, $\sigma(x) = \|x\|$ for $x \in \mathbb{R}^2$, and X is a 2-dimensional standard Wiener process independent of W. Calculate the filter of X given Y.

Hint: Show that π_t is a spherical distribution on $\mathbb{R}^2$, i.e., $\pi_t(f) = \pi_t(f \circ U)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and U denotes multiplication by an orthogonal 2×2 matrix. Use that any spherical distribution on $\mathbb{R}^2$ can be represented as the law of RS, where R is a random variable with values in $[0,\infty)$ and S is uniformly distributed on the unit sphere in $\mathbb{R}^2$, independent of R.