Supermodular and directionally convex comparison results for general factor models

Jonathan Ansari[a,*], Ludger Rüschendorf[b]

[a] *Department of Artificial Intelligence & Human Interfaces, University of Salzburg, Hellbrunnerstraße 34, 5020 Salzburg, Austria*
[b] *Department of Mathematical Stochastics, University of Freiburg, Ernst-Zermelo-Straße 1, 79104 Freiburg, Germany*

Abstract

This paper provides comparison results for general factor models with respect to the supermodular and directionally convex order. These results extend and strengthen previous ordering results from the literature concerning certain classes of mixture models as mixtures of multivariate normals, multivariate elliptic and exchangeable models to general factor models. For the main results, we first strengthen some known orthant ordering results for the multivariate $*$-product of the specifications, which represents the copula of the factor model, to the stronger notion of the supermodular ordering. The stronger comparison results are based on classical rearrangement results and in particular are rendered possible by some involved constructions of transfers as arising from mass transfer theory. The ordering results for $*$-products are then extended to factor models with general conditional dependencies. As a consequence of the ordering results, we derive worst case scenarios in relevant classes of factor models allowing, in particular, interesting applications to deriving sharp bounds in financial and insurance risk models. The results and methods of this paper are a further indication of the particular effectiveness of Sklar's copula notion.

Keywords: conditional independence, factor model, positive dependence, product of copulas, rearrangements, Schur order, supermodular order
2020 MSC: 60E15, 62H05, 62P05

1. Introduction

Factor models are an important device for modeling multivariate distributions in nearly all fields of science, such as statistics, psychology, social sciences, finance, econometrics, operations research, and risk management; see, e.g., [9, 12, 19, 21, 25, 26, 36, 53, 56]. The basic idea of a factor model is to reduce the complexity of a multivariate model by means of low-dimensional factor variables explaining and describing in a simple functional way the randomness and dependence of the model. In general, factor models appear in statistics in the form of regression models and in the form of mixture models where the mixing variable can be seen as a common factor. For instance, normal mean-variance mixture models generate many well-known distributions such as the generalized hyperbolic, the variance Gamma, and the normal inverse Gaussian distribution; see [5].

In the general setting of a factor model, it is assumed that the underlying random vector $X = (X_1, \dots, X_d)$ defined on a probability space $(\Omega, \mathcal{A}, P)$ takes a functional form

$$X_i = f_i(Z, \varepsilon_i)\,, \quad i \in \{1, \dots, d\}\,,$$

with a common random factor Z and some random vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ of idiosyncratic (individual) random influences. In the standard factor model, the random variables $Z, \varepsilon_1, \dots, \varepsilon_d$ are assumed to be mutually independent while, in the more general frame of a conditionally independent factor model, it is assumed that $X_1, \dots, X_d$ are conditionally independent given Z. Typically, the variables $X_1, \dots, X_d$ and the common factor Z are observable and their distributions can be estimated. In many applications, the copula of (X_i, Z) can also be specified or partially specified.

[*]Corresponding author. Email address: `jonathan.ansari@plus.ac.at`

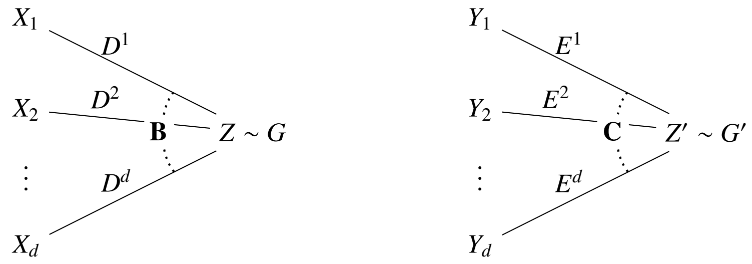

Fig. 1 The setting for the factor model comparison results: The left-hand graph illustrates a factor model $X = (X_1, \dots, X_d)$, $X_i = g_i(Z, \varepsilon_i)$, with bivariate copula specifications $D^i = C_{X_i,Z}$ for $i \in \{1, \dots, d\}$, with univariate factor distribution function $G = F_Z$, and with d-variate conditional copula specifications $\mathbf{B} = (B_t)_{t\in[0,1]}$ such that $B_t^G = C_{X|Z=G^{-1}(t)}$ for almost all $t \in [0, 1]$; see (A.5) for the definition of B_t^G. The right-hand graph illustrates a factor model $Y = (Y_1, \dots, Y_d)$, $Y_i = h_i(Z', \varepsilon_i')$, $i \in \{1, \dots, d\}$, with the corresponding specifications $E^1, \dots, E^d$, G', and $\mathbf{C} = (C_t)_{t\in[0,1]}$.

However, the copula of X conditionally on $Z = z$ can be estimated only partially if at all. This situation implies that ordering results in general factor models may be useful to solve optimization problems or identify worst/best cases in classes of models of this type. The aim of our paper is to derive ordering results in general factor models that are strong enough to be useful for the solution of a general class of optimization problems as described above.

We establish various general conditions on the specifications $D^1, \dots, D^d, \mathbf{B}, G$ and $E^1, \dots, E^d, \mathbf{C}, G'$ (see Fig. 1) of factor models $X = (g_i(Z, \varepsilon_i))_{1\leq i\leq d}$ and $Y = (h_i(Z', \varepsilon_i'))_{1\leq i\leq d}$ which allow for large classes of functions $f\colon \mathbb{R}^d \to \mathbb{R}$ a pointwise or an averaged comparison of the conditional expectations such that

$$\mathbb{E}f(X) = \int \mathbb{E}[f(X) \mid Z] \,\mathrm{d}P \leq \int \mathbb{E}[f(Y) \mid Z'] \,\mathrm{d}P = \mathbb{E}f(Y), \tag{1}$$

considering general conditional dependencies. In particular, we develop such integral inequalities for classes of supermodular and directionally convex functions, extending particular known ordering results for copula products in factor models (see [4]) to the substantially larger class of supermodular functions. The proofs are based on various combinatorial arguments for mass transfers which characterize several integral stochastic orderings by duality; see [43] and [41].

An important consequence of the supermodular and directionally convex ordering is its applicability to an interesting range of functions. In particular, each of them implies the convex ordering of the component sums, i.e.,

$$X \leq_{sm} Y \text{ or } X \leq_{dcx} Y \qquad \Longrightarrow \qquad \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i \qquad \Longrightarrow \qquad \Psi\Big(\sum_{i=1}^d X_i\Big) \leq \Psi\Big(\sum_{i=1}^d Y_i\Big) \tag{2}$$

where $\leq_{cx}$ denotes the univariate convex order and where Ψ is any convex, law-invariant risk measure on a proper probability space such as the space of integrable or the space of bounded random variables; see [23, Chapter 4] and [7, 11, 28]. Hence, the supermodular and the directionally convex comparison of random vectors yield a comparison of the risk of the component sums which may stand for a portfolio risk in finance or for the risk of total damages in the insurance framework.

As a consequence, we determine for relevant subclasses of factor models least and greatest elements corresponding to best- and worst-case distributions with respect to various stochastic orderings. In particular, we study the conditionally independent factor model in detail. Since the supermodular and directionally convex ordering imply by (2) the convex ordering of the component sums, important applications can be given in finance and risk management in the context of price and risk bounds where the underlying random variables often exhibit positive dependencies; see [46] and [51]. Further, see [6] for applications of the supermodular ordering to other stochastic models.

Modeling multi-dimensional dependence structures is a challenging issue; see, e.g., [17, 27, 47]. Since the univariate marginal distributions are often inferred from data and thus can be assumed to be given, it is by Sklar's theorem sufficient to analyze copulas for modeling dependencies. Promising approaches have been investigated in form of vine copula models which leverage from bivariate copulas, satisfy a high flexibility, and enable extensions to arbitrary dimension; see [8, 14, 34] for an overview. General comparison results in vine copula models with respect to integral stochastic orderings are possible for C-vine structures, which correspond to the setting of a factor model; see [3].

The most widely-used factor model is the standard factor model; see, e.g., [25, 26, 39, 40]. Single and multi-factor copula models have been investigated in [33, 35] under the conditional independence assumption; see also [39, Chapter 12.2] for several applications. The $*$-product of two bivariate copulas introduced in [15], corresponding to a Markov structure, has been generalized in [16] in the bivariate case allowing for arbitrary conditional dependencies. An extension to arbitrary dimension is given in [4]. The main property of this product is that it describes the copula of a general factor model; see Theorem 2.7 in [4].

The structure of conditionally independent factor models has been used in a relevant part of the literature on ordering of stochastic models and, in particular, on the construction of positive dependence models and related probability inequalities as given in [38] and [61] as well as in [18, 24, 29, 30]. Tong [60] gives general majorization ordering conditions to imply ordering properties in the sense of positive dependence ordering in conditionally independent factor models under some structural assumptions. The model examples for these type of comparison results are the exchangeable or positive dependence by mixture models; see [54, 59, 60] which are particular cases of the conditionally independent factor models. In our paper, we state, in particular, related comparison results for conditionally independent random variables without further distributional assumptions.

Since the conditional independence assumption is often not grounded in data, see, e.g., [13], partially specified factor models have been introduced in [9] where only the distributions of (X_i, Z)–the specifcations– are assumed to be known for all $i \in \{1, \ldots, d\}$ and, thus, the conditional dependence structure of X given $Z = z$ is not specified. In the context of risk bounds, these classes of factor models yield a considerable improvement of the dependence uncertainty spread compared to the pure marginal model. Further, the upper risk bound is described by a conditionally comonotonic random vector whose copula corresponds to the upper product of the copula specifications; see [1]. However, conditional comonotonicity may be an unrealistic scenario, and also flexibility in the specifications of (X_i, Z) may be desirable. This motivates to investigate various ordering results for general factor models with respect to all of its specifications.

In Section 2, we recall the necessary notation, the relevant integral stochastic orderings, and the $*$-product of copulas which describes the copula of the factor model with given specifications. Section 3 considers various important supermodular ordering results for $*$-products using a new general supermodular criterion for $*$-products which we give in Appendix B and which is based on a lower orthant comparison of approximating sequences of $*$-products on finite grids. The proof of the general supermodular ordering criterion is given in Appendix C and makes use of the dual characterization of integral stochastic orders by mass transfers.

Combining the supermodular ordering results for $*$-products with the ordering of marginal distributions, we present in Section 4 the main comparison results for factor models with easy to handle and interpretable ordering conditions on the specifications of the models. Due to its particular meaning, we discuss the conditionally independent factor model in detail. In Section 5, we give applications to upper bounds in several classes of conditionally independent factor models. The main results of our paper are substantial extensions of [1] and [4] concerning model assumptions and sharpenings to the stronger supermodular order respectively.

Our paper builds on several results from [4] concerning the representation of a conditional distribution function as generalized copula derivative as well as approximation and continuity properties of $*$-products, the Schur-order for copula derivatives, and rearranged copulas.

2. Preliminaries

In this section, we introduce the notation and provide the necessary tools for the factor model comparison results in Section 4. First, we state for the stochastic orderings considered in this paper some of their most important properties and relations. Then, we define the $*$-product of copulas which describes the copula of a factor model.

2.1. Stochastic orderings

Denote by $\triangle_i^\varepsilon f(x) := f(x + \varepsilon e_i) - f(x)$ the difference operator of length $\varepsilon > 0$, where e_i is the ith unit vector with respect to the canonical base in $\mathbb{R}^d$. Then f is said to be supermodular and directionally convex, respectively, if

$\triangle_i^{\varepsilon_i}\triangle_j^{\varepsilon_j} f(x) \geq 0$ for all $x \in \mathbb{R}^d$, for all $\varepsilon_i, \varepsilon_j > 0$ as well as for all $1 \leq i < j \leq d$ and $1 \leq i \leq j \leq d$, respectively. Further, f is said to be Δ-monotone and Δ-antitone, respectively, if $\triangle_{i_1}^{\varepsilon_1} \cdots \triangle_{i_k}^{\varepsilon_k} f(x) \geq 0$ and $(-1)^k \triangle_{i_1}^{\varepsilon_1} \cdots \triangle_{i_k}^{\varepsilon_k} f(x) \geq 0$, respectively, for all $x \in \mathbb{R}^d$, for all $k \in \{1, \ldots, d\}$, for all $\varepsilon_1, \ldots, \varepsilon_k > 0$, and for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$. Denote by $\mathcal{F}_\uparrow$, $\mathcal{F}_{sm}$, $\mathcal{F}_{dcx}$, $\mathcal{F}^\Delta$, and $\mathcal{F}^\Delta_-$ the class of increasing, supermodular, directionally convex, Δ-monotone, and Δ-antitone functions, respectively. Denote by F_X and $\overline{F}_X$ the distribution function and survival function, respectively, associated with the random vector X.

Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be random vectors. Then the supermodular order and directionally convex order are defined by

$$\begin{aligned} X \leq_{sm} Y \quad &\text{if} \quad \mathbb{E} f(X) \leq \mathbb{E} f(Y) \text{ for all } f \in \mathcal{F}_{sm}\,, \\ X \leq_{dcx} Y \quad &\text{if} \quad \mathbb{E} f(X) \leq \mathbb{E} f(Y) \text{ for all } f \in \mathcal{F}_{dcx}\,, \end{aligned}$$

respectively, whenever the expectations exist. The lower orthant order and the upper orthant order are defined by the pointwise comparison of the distribution and survival functions, respectively, i.e., $X \leq_{lo} Y$ if $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}^d$, and $X \leq_{uo} Y$ if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}^d$. The concordance order $X \leq_c Y$ is defined by $X \leq_{lo} Y$ and $X \leq_{uo} Y$. The stochastic order $X \leq_{st} Y$ is defined by $\mathbb{E} f(X) \leq \mathbb{E} f(Y)$ for all $f \in \mathcal{F}_\uparrow$,

All these orderings are integral stochastic orderings comparing by (1) for respective classes $\mathcal{F}$ of functions f the expectations of random vectors $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ defined on the probability space $(\Omega, \mathcal{A}, P)$, which we generally assume to be non-atomic. Note that the lower orthant order and the upper orthant order are generated by the classes $\mathcal{F}^\Delta_-$ and $\mathcal{F}^\Delta$ of Δ-antitone and Δ-monotone functions, respectively; see [49].

The basic relations between the above considered orderings are

$$X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_c Y \quad \Longleftrightarrow \quad X_i \stackrel{\mathrm{d}}{=} Y_i \text{ for all } i,\ X \leq_{lo} Y,\ X \leq_{uo} Y\,. \tag{3}$$

If $d = 2$ and $X_i \stackrel{\mathrm{d}}{=} Y_i$ for $i \in \{1, 2\}$, then the orderings $\leq_{lo}$, $\leq_{uo}$, $\leq_c$, $\leq_{sm}$, and $\leq_{dcx}$ are equivalent.

Due to (3), the supermodular order and the concordance order are pure dependence orderings. Both orderings have the important property that they are invariant under increasing transformations of the components, i.e., for all increasing functions $k_1, \ldots, k_d \colon \mathbb{R} \to \mathbb{R}$, one has

$$(X_1, \ldots, X_d) \leq_{sm} (Y_1, \ldots, Y_d) \quad \Longrightarrow \quad (k_1(X_1), \ldots, k_d(X_d)) \leq_{sm} (k_1(Y_1), \ldots, k_d(Y_d))\,, \tag{4}$$

similarly for the concordance order. Since, for $d \geq 3$, $X \leq_c Y$ does not imply $\sum_i X_i \leq_{cx} \sum_i Y_i$, see [44, Theorem 2.6], we focus on comparison results with respect to the stronger notion of the supermodular order.

For an overview of stochastic orderings, see [46, 51, 55].

2.2. *Copulas and positive dependence concepts*

A d-copula is a distribution function $C \colon [0,1]^d \to [0,1]$ with uniform univariate marginals. Due to Sklar's Theorem, every d-variate distribution function F can be decomposed into its univariate marginal distribution functions $F_1, \ldots, F_d$ and a d-copula C such that

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))\,, \quad (x_1, \ldots, x_d) \in \mathbb{R}^d\,, \tag{5}$$

where C is uniquely determined on $Ran(F_1) \times \cdots \times Ran(F_d)$, where $Ran(f)$ denotes the range of a function f. Conversely, for all univariate distribution functions $F_1, \ldots, F_d$ and for all d-copulas, the right-hand side in (5) defines a d-variate distribution function. Denote by C_d the class of d-copulas and by $\mathcal{F}^d$ $(\mathcal{F}^d_c)$ the class of d-dimensional (continuous) distribution functions. For an overview of copula theory, see, e.g., [17, 47].

Some important copulas are the lower Fréchet bound W^2, the independence copula Π^d, and the upper Fréchet bound M^d, where

$$W^d(u) := \max\left\{\sum_{i=1}^d u_i - d + 1\,, 0\right\}, \qquad \Pi^d(u) := \prod_{i=1}^d u_i\,, \qquad M^d(u) := \min_{1 \leq i \leq d}\{u_i\} \tag{6}$$

for $u = (u_1, \ldots, u_d) \in [0,1]^d$. Note that W^d is not a copula for $d \geq 3$. The independence copula models independence, i.e., for a random vector $U = (U_1, \ldots, U_d)$, it holds that $F_U = \Pi^d$ if and only if $U_1, \ldots, U_d \sim \mathcal{U}(0,1)$ are independent, where $\mathcal{U}(0,1)$ denotes the uniform distribution on $[0,1]$. The upper Fréchet copula corresponds to a comonotonic random vector, i.e., $F_{U_1,\ldots,U_d} = M^d$ if and only if there exists $V \sim \mathcal{U}(0,1)$ such that $U_i = V$ P-almost surely for all $i \in \{1, \ldots, d\}$. Further, the lower Fréchet copula corresponds to a countermonotonic random vector, i.e., $F_{U_1,U_2} = W^2$ if and only if there exists $V \sim \mathcal{U}(0,1)$ such that $U_1 = V$ and $U_2 = 1 - V$ P-almost surely. Note that the conditional independence product, the upper product, and the lower product in (11) are defined with respect to these copulas.

For modeling positive dependencies, we make use of the several positive dependence concepts on the class $\mathcal{C}_d$.

A copula $C \in \mathcal{C}_d$ is said to be positive supermodular dependent (PSMD) if $\Pi^d \leq_{sm} C$. Further, C is conditionally increasing (CI) if there exists a random vector $(U_1, \ldots, U_d)$ with distribution function C such that

$$U_i \uparrow_{st} (U_j\,,\ j \in J) \tag{7}$$

for all $i \in \{1, \ldots, d\}$ and $J \subseteq \{1, \ldots, d\} \setminus \{i\}$, where (7) means that the conditional distribution $U_i \mid (U_j = u_j, j \in J)$, is stochastically increasing in u_j for all $j \in J$. C is conditionally increasing in sequence (CIS) if there exists a random vector $(U_1, \ldots, U_d)$ with distribution function C such that (7) holds for all $J = \{i+1, \ldots, d\}$ and $i \in \{1, \ldots, d-1\}$. Lastly, C is totally positive of order 2 (MTP$_2$) if C has a log-supermodular density, i.e., if

$$\log\left(\frac{\partial^d}{\partial x_1 \cdots \partial x_d} C(x_1, \ldots, x_d)\right)$$

is supermodular. Due to their invariance under increasing transformations, these concepts also apply to the class $\mathcal{F}^d$.

Note that a bivariate copula $D \in \mathcal{C}_2$ is CIS (CI) if and only if, for all $u \in [0,1]$, $\partial_2 D(u,t)$ is decreasing in t outside a Lebesgue null set (and if, for all $v \in [0,1]$, $\partial_1 D(t,v)$ is deceasing in t outside a Lebesgue null set). The above defined positive dependence concepts are related as follows,

$$\text{MTP}_2 \implies \text{CI} \implies \text{CIS} \implies \text{PSMD}\,, \tag{8}$$

where each implication is strict for $d \geq 2$; see, e.g., [46, page 146] for an overview of these concepts.

2.3. *Products of copulas for factor models*

For deriving supermodular ordering results for random vectors following a factor model structure, we need to analyse their dependence structure. Since the supermodular order is a pure dependence ordering and invariant under increasing transformations, we consider without loss of generality the case of uniform univariate marginal distributions, which leads to the well-known concept of copula.

For analyzing dependence structures in factor models, the copula C of $X = (g_i(Z, \varepsilon_i))_i$ is given by the $*$-product of the copula specifications. This copula product has been introduced in [4] and extends the $*$-products considered in [15, 16] to the multivariate case, to general conditional copulas, and to a general factor distribution function $G \in \mathcal{F}^1$. For a general factor distribution function G , the definition of the $*$-product requires the use of a generalized differential operator. We refer to Appendix A for the technically involved definition which is in particular needed for the proofs of our main supermodular comparison results in Section 3.

For a continuous distribution function G of the factor variable Z , the definition of the $*$-product of $D^1, \ldots, D^d \in \mathcal{C}_2$ with respect to $\mathbf{B} := (B_t)_{t\in[0,1]}$, $B_t \in \mathcal{C}_d$ for all $t \in [0,1]$, simplifies to

$$*_{\mathbf{B},G} D^i\,(u) = *_{i=1,\mathbf{B},G}^{d} D^i\,(u) = \int_0^1 B_t\left(\partial_2 D^1(u_1, t), \ldots, \partial_2 D^d(u_d, t)\right)\,\mathrm{d}t \tag{9}$$

for $u = (u_1, \ldots, u_d) \in [0,1]^d$, where ∂_i denotes the operator that takes the partial derivative with respect to the i-th component of a function of several arguments. The number d of bivariate copulas is typically clear from the context and therefore the simplified notation in (9) is used. By definition of the integral in (9), it is implicitly assumed that the family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas is measurable, i.e., that the mapping $(t,u) \mapsto B_t(u)$, $(t,u) \in [0,1]\times[0,1]^d$, is Borel-measurable.

By Sklar's Theorem for factor models (see [4, Theorem 2.7]), the distribution function of a factor model $X = (X_1,\dots,X_d)$, $X_i = g_i(Z,\varepsilon_i)$, $i \in \{1,\dots,d\}$, can be decomposed into the composition of its univariate marginal distribution functions $F_i := F_{X_i}$, $i \in \{1,\dots,d\}$, and the $*$-product of the copula specifications of the factor model by

$$F_{X_1,\dots,X_d}(x) = \ast_{\mathbf{B},G} D^i\,(F_1(x_1),\dots,F_d(x_d)), \quad x = (x_1,\dots,x_d)\in\mathbb{R}^d, \tag{10}$$

where $D^i := C_{X_i,Z}$ is a copula of (X_i,Z), where $\mathbf{B} = (B_t)_{t\in[0,1]}$ is a measurable family of d-copulas such that $C_{(X_1,\dots,X_d)|Z=G^{-1}(t)} = B_t^G$ for Lebesgue-almost all $t\in[0,1]$, and where G is the distribution function of the real-valued random variable Z which is the common factor variable. Note that, by (A.5), $B_t^G = B_t$ if G is continuous. In consequence, the $*$-product is a copula of the factor model. Vice versa, for all bivariate copulas $D^1,\dots,D^d$, for all measurable families $\mathbf{B}$ of d-copulas, and for all univariate distribution functions $F_1,\dots,F_d,G$, the right-hand side of (10) defines the distribution of a factor model with these specifications. It particular, one has that $\ast_{\mathbf{B},G}D^i$ is a d-copula.

Some specific $*$-products are defined in the case where the conditional copula is given by the independence copula Π^d, the upper Fréchet copula M^d, and lower Fréchet bound W^d, respectively. Denote by

$$\Pi_G D^i := D^1 \,\Pi_G \cdots \Pi_G D^d := \ast_{\Pi^d,G} D^i, \qquad \bigvee\nolimits_G D^i := \ast_{M^d,G} D^i, \qquad \bigwedge\nolimits_G D^i := \ast_{W^d,G} D^i \tag{11}$$

the conditional independence product, the upper product, and the lower product of the bivariate copulas $D^1,\dots,D^d$ with respect to $G\in\mathcal{F}^1$. Since W^d is a copula only if $d\le 2$, we clarify that for $d\ge 3$, the lower product is defined in the sense of (9) and (A.4), respectively. For $d=2$, we write $D^1 \wedge_G D^2$ for the lower product of D^1 and D^2.

As a consequence of (10), the conditional independence product, the upper product, and the lower product is characterized by conditional independence, conditional comonotonicity, and conditional countermonotonicity, respectively, i.e., for a $(d+1)$-dimensional random vector $(X_1,\dots,X_d,Z)$ with $X_i\sim F_i$, $C_{X_i,Z} = D^i$, $i\in\{1,\dots,d\}$, and $Z\sim G$, one has for a G-null set $N\subset\mathbb{R}$ that

$$F_{X_1,\dots,X_d} = \Pi_G D^i\,(F_1,\dots,F_d) \iff X_1,\dots,X_d \mid Z=z \text{ are independent for all } z\in N^c, \tag{12}$$

$$F_{X_1,\dots,X_d} \sim \bigvee\nolimits_G D^i\,(F_1,\dots,F_d) \iff X_1,\dots,X_d \mid Z=z \text{ are comonotonic for all } z\in N^c, \tag{13}$$

$$F_{X_1,X_2} = D^1 \wedge_G D^2\,(F_1,F_2) \iff X_1,X_2 \mid Z=z \text{ are countermonotonic for all } z\in N^c. \tag{14}$$

Note that the conditional independence product $\Pi_G D^i$ describes the dependence structure of a conditionally independent factor model. Several properties of the $*$-product $\ast_{\mathbf{B},G}D^i$ are given in [4].

3. Main comparison results for $*$-products of copulas

In this section, we derive various supermodular ordering results for $\ast_{\mathbf{B},G}D^i$ with respect to the specifications $\mathbf{B}$ and $D^1,\dots,D^d$. By the Sklar-type representation in (10), these results imply meaningful supermodular and directionally convex ordering results for factor models, which we provide in Section 4.

Proving supermodular ordering results for $*$-products is much more difficult than proving lower and upper orthant ordering results which are based on the application of well-known integral inequalities. As a new general supermodular comparison result, we provide through Theorem 5 in Appendix B a sufficient lower orthant ordering criterion on the specfciations $D^1,\dots,D^d$ for the supermodular ordering of $\ast_{\mathbf{B},G}D^i$ with respect to $D^1,\dots,D^d$. This criterion is based on a discrete approximation of $*$-products by grid copulas defined on a finite grid. For its proof, which we give in Appendix C, we make use of several results from mass transfer theory characterizating various integral stochastic orderings for distributions with finite support by mass transfers; see [41, 43]. Making use of the general comparison result, we establish in Sections 3.1 and 3.2 several relevant and easily interpretable criteria on the specifications of $*$-products implying important supermodular ordering results for $*$-products.

In contrast to the ordering of $*_{\mathbf{B},G}D^i$ with respect to D^i, an ordering with respect to the copula family $\mathbf{B}$ is a simple task and given in the following proposition which extends [16, Proposition 3] and [4, Proposition 3.2] to the supermodular order.

Proposition 1 (Ordering with respect to conditional copulas)**.**
Let $\mathbf{B} = (B_t)_{0\le t\le 1}, \mathbf{C} = (C_t)_{0\le t\le 1}$ be measurable families of d-copulas. If $B_t \prec C_t$ for almost all t, where $\prec$ is one of the orderings $\le_{lo}$, $\le_{uo}$, $\le_c$, and $\le_{sm}$, then it follows that

$$*_{\mathbf{B},G}D^i \prec *_{\mathbf{C},G}D^i$$

for all $G \in \mathcal{F}^1$ and for all copulas $D^i \in \mathcal{C}_2$ with $i \in \{1,\dots,d\}$.

Proof: The statement follows from the closure of these orderings under mixtures (see [55, Theorems 6.G.3.(e) and 9.A.9(d)]). □

In the sequel, we prove several supermodular ordering results for $*_{\mathbf{B},G}D^i$ with respect to the specifications $D^1,\dots,D^d$ applying the grid approximation criterion in Theorem 5.

3.1. Lower orthant ordering criterion

Denote by $\mathcal{C}_d^{ccx}$ the class of componentwise convex d-copulas. Further, we call a family $(\Phi_t)_{t\in[0,1]}$ of functions $\Phi_t\colon \Theta \to \mathbb{R}$, $\Theta = \mathbb{R}^d$ or $\Theta = [0,1]^d$, continuous, if the mapping $(t,x) \mapsto \Phi_t(x)$ is continuous for all $(t,x) \in [0,1]\times\Theta$. We make use of the following submodularity condition.

Assumption 1 (Submodularity)**.** A continuous family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas satisfies for all $u = (u_1,\dots,d) \in [0,1]^d$, for all $t \in [0,1]$, for all $\varepsilon \in (0,1-t]$, for all $i \in \{1,\dots,d\}$, and for all $h \in (0,1-u_i]$ the inequality

$$B_{t+\varepsilon}(u+he_i) + B_t(u) - B_t(u+he_i) - B_{t+\varepsilon}(u) \le 0\,,$$

where e_i denotes the i-th unit vector.

An important subclass of copulas modeling positive dependencies are CIS copulas (see Section 2.2 for a definition). The following main result provides an important characterization of the supermodular ordering of $*$-products in the case where the bivariate specifications are CIS copulas. Several examples of well-known bivariate copula families that are CIS and increasing with respect to the lower orthant order are given in [4, Examples 3.18 and 3.19] regarding elliptical and Archimedean copulas. Note also that, due to [4, Theorem 3.17], the uniquely determined increasing rearranged copula associated with a bivariate copula is CIS.

Theorem 1 ($\le_{\mathbf{sm}}$-ordering of componentwise convex $*$-products)**.**
Let $\mathbf{B} = (B_t)_{t\in[0,1]}$ be a continuous family of d-copulas. Then the following statements are equivalent:

(i) *For all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ with $D^i \le_{lo} E^i$ for all $i \in \{1,\dots,d\}$, one has*

$$*_{\mathbf{B},G}D^i \le_{sm} *_{\mathbf{B},G}E^i\,. \tag{15}$$

(ii) $\mathbf{B}$ *satisfies the submodularity Assumption 1 and $B_t \in \mathcal{C}_d^{ccx}$ for all $t \in [0,1]$.*

The technical proof deferred to Appendix D is based on Theorem 5 and on the classical Ky Fan-Lorentz integral inequality; see [22]. Note that the condition $D^i \le_{lo} E^i$ in the above theorem is equivalent to $D^i \le_{sm} E^i$ because D^i and E^i are bivariate copulas.

Denote by $\overline{C}$ the survival function of a copula $C \in \mathcal{C}_d$. Then the survival copula $\widehat{C}$ of C is defined by $\widehat{C}(u_1,\dots,u_d) := \overline{C}(1-u_1,\dots,1-u_d)$ for $(u_1,\dots,u_d) \in [0,1]^d$.

As an interesting byproduct of Theorem 1, we obtain that a copula is componentwise convex if and only if its survival copula is componentwise convex. This statement is only in the bivariate case trivial where $\widehat{C}(u,v) = 1-u-v+C(1-u,1-v)$ for all $u,v \in [0,1]$.

Corollary 1. *For $C \in \mathcal{C}_d$, one has $C \in \mathcal{C}_d^{ccx}$ if and only if $\widehat{C} \in \mathcal{C}_d^{ccx}$.*

Proof: From Theorem 1, we obtain that componentwise convexity of $C \in \mathcal{C}_d$ is equivalent to $\ast_{C,G} D^i \leq_{sm} \ast_{C,G} E^i$ for all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ such that $D^i \leq_{lo} E^i$ for all $i \in \{1, \ldots, d\}$. Since the supermodular order implies the upper orthant order, it follows that $\ast_{C,G} D^i \leq_{uo} \ast_{C,G} E^i$ for all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ such that $D^i \leq_{lo} E^i$, $i \in \{1, \ldots, d\}$. Due to the characterization of the upper orthant ordering of $\ast$-products in [4, Theorem 3.8], this implies that the survival copula $\widehat{C}$ is componentwise convex.

The reverse direction follows in the same way using that the survival copula associated with $\widehat{C}$ is C . □

Remark 1. In the literature, componentwise convex copulas are discussed in the context of constructing copula; see [31, 32] and [52], where Corollary 1 is given in the bivariate case. Since copulas are supermodular functions, a componentwise convex copula is directionally convex, which is also referred to as ultramodularity or Wright-convexity.

3.2. Schur-ordering criterion

For an extension of the lower orthant ordering criterion for the supermodular ordering of $\ast$-products with componentwise convex conditional copulas given by Theorem 1, we make use of the following dependence ordering on the class of bivariate copulas.

Denote by $\prec_S$ the Schur-order for integrable functions on $[0,1]$, i.e., for $f, g\colon [0,1] \to \mathbb{R}$ integrable, the Schur-order $f \prec_S g$ is defined by $\int_0^x f^*(t)\,\mathrm{d}t \leq \int_0^x g^*(t)\,\mathrm{d}t$ for all $x \in (0,1)$ and $\int_0^1 f(t)\,\mathrm{d}t = \int_0^1 g(t)\,\mathrm{d}t$ where h^* denotes the decreasing rearrangement of an integrable function h , i.e., the (essentially with respect to the Lebesgue measure λ) uniquely determined decreasing function h^* such that $\lambda(h^* \leq t) = \lambda(h \leq t)$ for all $t \in \mathbb{R}$.

Definition 1 ($\leq_{\partial_2 S}$-order, [4], cf. [57])**.**
For $D, E \in \mathcal{C}_2$ and for $G \in \mathcal{F}^1$, the Schur-order $D \leq_{\partial_2 S, G} E$ for copula derivatives is defined by

$$\partial_2^G D(v, \cdot) \prec_S \partial_2^G E(v, \cdot), \quad v \in [0,1] . \tag{16}$$

If G is continuous, we abbreviate $\leq_{\partial_2 S, G}$ by $\leq_{\partial_2 S}$. In this case the generalized differential operator ∂_2^G defined by (A.2) reduces to ∂_2 .

The $\leq_{\partial_2 S, G}$-order compares bivariate copulas with respect to their strength of dependence. The independence copula Π^2 is the least element with respect to $\leq_{\partial_2 S}$, whereas the lower and upper Fréchet copula W^2 and M^2 , respectively, as well as all of their shuffles are maximal elements. Further, (a version of) the asymmetric dependence measure ζ_1 , introduced in [63], is increasing with respect to the $\leq_{\partial_2 S}$-order; see [4, Proposition 2.14]. Denote by E^* the reflected copula of $E \in \mathcal{C}_2$ defined by

$$E^*(u, v) = u - E(u, 1 - v), \tag{17}$$

for $(u, v) \in [0,1]^2$. For CIS copulas, the $\leq_{\partial_2 S}$-order and the $\leq_{lo}$-order coincide; see [4, Lemma 3.16 (ii)].

The following main result provides a general supermodular comparison of $\ast$-products based on a Schur-ordering criterion for the bivariate copula specifications.

Theorem 2 (Schur-ordering criterion)**.**
Let $G \in \mathcal{F}^1$ and let $D^i, E^i \in \mathcal{C}_2$ be bivariate copulas with E^i CIS and $D^i \leq_{\partial_2 S, G} E^i$ for all $i \in \{1, \ldots, d\}$. Assume that $\mathbf{B} = (B_t)_{t \in [0,1]}$ is continuous, satisfies the submodularity Assumption 1, and $B_t \in \mathcal{C}_d^{ccx}$ for all t . Then

$$\ast_{\mathbf{B},G} D^i \leq_{sm} \ast_{\mathbf{B},G} E^i . \tag{18}$$

The technical proof given in Appendix D is based on Theorem 5 and the classical rearrangement inequalities of Lorentz [37].

Remark 2.

(a) Theorems 1 and 2 are based on classical rearrangement inequalities due to [22, 37] and strengthen [4, Theorems 3.8/3.10 and 3.20] from the lower and upper orthant order to the supermodular order. Further, Theorem 2 implies the supermodular ordering of the $*$-products in Theorem 1 because $D^i \leq_{\partial_2 S,G} E^i$ is equivalent to $D^i(u,t) \leq E^i(u,t)$ for all $(u,t) \in [0,1] \times Ran(G)$ whenever D^i and E^i are CIS; cf. [4, Lemma 3.16 (ii)].

(b) Theorems 1 and 2 do not intersect with the upper product ordering results in [1, 2] because the upper product is defined via the conditional copulas $B_t = M^d$ for all t, see (11), and the upper Fréchet copula M^d is componentwise concave and not componentwise convex.

If $B_t = \Pi^d$ for all $t \in [0,1]$, then it follows trivially that $B_t \in \mathcal{C}_d^{ccx}$ for all t and that $\mathbf{B} = (B_t)_{t\in[0,1]}$ satisfies Assumption 1 on submodularity. Hence, Theorem 1 provides general ordering conditions for the conditional independence product as follows.

Corollary 2 (Ordering conditionally independent products)**.**
Let $D^1, \ldots, D^d, E^1, \ldots, E^d \in \mathcal{C}_2$. If E^i is CIS and if $D^i \leq_{\partial_2 S,G} E^i$ for all $i \in \{1, \ldots, d\}$, then

$$\Pi_G D^i \leq_{sm} \Pi_G E^i \,. \tag{19}$$

Since the lower Fréchet bound W^d is componentwise convex, Theorem 1 also provides general ordering conditions for the lower product as follows.

Corollary 3 (Ordering lower products)**.**
If E^i is CIS and if $D^i \leq_{\partial_2 S,G} E^i$, $i \in \{1, \ldots, d\}$, then

$$\bigwedge\nolimits_G D^i \leq_{sm} \bigwedge\nolimits_G E^i \,, \tag{20}$$

whenever both products induce signed measures.

Remark 3.

(a) If $D^i = \Pi^2$ for all i, then the supermodular comparison in (19) is given in [46, Corollary 8.3.18]; see also [33, Proposition 1] for a continuous factor distribution function G and $d = 2$.

(b) If $d = 2$, then the lower products in (20) are copulas and thus measure inducing. If $d = 3$ and $D^1 = E^1 = M^2$, then the lower products in (20) are 3-copulas if and only if G is continuous; see [4, Proposition 2.25]. Note that, for $d \geq 3$, the lower product $\bigwedge_G D^i$ is in general not a copula because the lower Fréchet bound W^d is a proper quasi-copula and does not induce a signed measure; see, e.g., [48, Theorem 2.4].

4. Comparison results for factor models

In this section, we combine the comparison results on $*$-products of copulas established in Chapter 3 with the ordering of the univariate marginal distributions in order to derive various comparison results for random vectors $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ having a factor model structure, where

$$X_i = g_i(Z, \varepsilon_i)\,,\ Y_i = h_i(Z', \varepsilon_i')\,, \quad i \in \{1, \ldots, d\}\,, \tag{21}$$

for some measurable functions $g_i, h_i \colon \mathbb{R}^2 \to \mathbb{R}$, $i \in \{1, \ldots, d\}$, for random variables Z, Z', and for random vectors $(\varepsilon_1, \ldots, \varepsilon_d)$ and $(\varepsilon_1', \ldots, \varepsilon_d')$. Note that $Z, \varepsilon_1, \ldots, \varepsilon_d$ as well as $Z', \varepsilon_1', \ldots, \varepsilon_d'$ are not assumed to be independent and thus the setting in (21) does not restrict generality.

For the representation (21), denote by G and G' the distribution function of the common factor variable Z and Z', respectively. We generally assume that $\overline{Ran(G)} = \overline{Ran(G')}$. Then, due to Sklar's theorem for factor models (see (10)), the copula of X and Y is given by the $*$-products

$$C_X = \ast_{\mathbf{B},G} D^i \quad \text{and} \quad C_Y = \ast_{\mathbf{C},G} E^i \,,$$

respectively, where $D^i = C_{X_i,Z}$ and $E^i = C_{Y_i,Z}$ are the copulas of (X_i, Z) and (Y_i, Z), respectively, for $i \in \{1, \dots, d\}$, and where $\mathbf{B} = (B_t)_{t\in[0,1]}$ and $\mathbf{C} = (C_t)_{t\in[0,1]}$ are measurable families of d-copulas such that $B_t^G = C_{X|Z=G^{-1}(t)}$ and $C_t^{G'} = C_{Y|Z'=G'^{-1}(t)}$ for Lebesgue-almost all $t \in [0, 1]$; see Fig. 1. The corresponding distribution functions of X and Y are given by

$$F_X = \ast_{\mathbf{B},G} D^i(F_{X_1}, \dots, F_{X_d}), \qquad\qquad F_Y = \ast_{\mathbf{C},G} E^i(F_{Y_1}, \dots, F_{Y_d}).$$

As a main contribution of our paper, we obtain as a consequence of Proposition 1 and Theorem 2 general conditions for the supermodular and directionally convex ordering of factor models with respect to the bivariate dependence specifications, the marginal distributions, and the conditional copula families as follows. This strengthens corresponding lower and upper orthant ordering results in [4, Theorem 4.2]. Note that for bivariate CIS copulas D and E, $D \le_{lo} E$ implies $D \le_{\partial_2 S,G} E$ for all factor distribution functions G; cf. [4, Lemma 3.16 (ii)].

Theorem 3 (Comparison results for general factor models)**.**
Let $\mathbf{B}' = (B'_t)_{t\in[0,1]}$, $B'_t \in \mathcal{C}_d^{ccx}$, $t \in [0, 1]$, be a continuous family of copulas that satisfies the submodularity Assumption 1. Assume that E^i is CIS and that $D^i \le_{\partial_2 S,G} E^i$, $i \in \{1, \dots, d\}$.

(i) *If $\mathbf{B} \le_{sm} \mathbf{B}' \le_{sm} \mathbf{C}$ and $X_i \overset{\mathrm{d}}{=} Y_i$ for all $i \in \{1, \dots, d\}$, then $X \le_{sm} Y$.*

(ii) *If $\mathbf{B} \le_{sm} \mathbf{B}' \le_{sm} \mathbf{C}$, $X_i \le_{cx} Y_i$ for all $i \in \{1, \dots, d\}$, and if one of the products $\ast_{\mathbf{B},G} D^i$, $\ast_{\mathbf{B}',G} D^i$, $\ast_{\mathbf{C},G} D^i$, $\ast_{\mathbf{B},G} E^i$, $\ast_{\mathbf{B}',G} E^i$, or $\ast_{\mathbf{C},G} E^i$, is CI, then $X \le_{dcx} Y$.*

Proof: Statement (i) follow from Proposition 1, Theorem 2, and the invariance of the supermodular order under increasing transformations in (4). Statement (ii) is a consequence of (i) and [45, Theorem 4.5]. □

Examples of several well-known families of bivariate copulas that exhibit positive dependencies and are ordered with respect to $\le_{lo}$-order and thus also with respect to the $\le_{\partial_2 S}$-order are given in [4, Examples 3.18 and 3.19]. These families are highly relevant for applications of Theorem 3; see also Remark 5(a).

Remark 4 (Upper/lower bounds in classes of partially specified factor models)**.**
In a partially specified factor model, $X_i = f_i(Z, \varepsilon_i)$, $i \in \{1, \dots, d\}$, the joint distributions of (X_i, Z) are specified for all i, but the conditional distribution of $(X_1, \dots, X_d) \mid Z = z$ is unspecified for any z; see [9]. The copula of a worst/best case distribution is given by the upper/lower product of the bivariate dependence specifications. Several ordering criteria for the upper product $\bigvee_G D^i$ and the lower product $\bigwedge_G D^i$ of bivariate copulas $D^1, \dots, D^d$ are given in [1, Theorem 3.10], [2, Theorem 1], and [4, Theorems 3.27 and 3.28]. It can be shown by an application of Theorem 5 that these results are still valid for a general factor distribution function G and in particular also with respect to the stronger notion of the supermodular order. As a consequence of these upper/lower product ordering results, the worst/best case bounds in classes of partially specified factor models as given in [4, Theorems 4.5 - 4.8] are still valid for a general factor model distribution function G and also hold true with respect to the stronger supermodular order.

The most commonly used one-factor model in practical applications is the standard factor model. The copula of $(X_1, \dots, X_d)$ is then given by the conditional independence product $\Pi_G D^i$, where $D_i = C_{X_i,Z}$, $i \in \{1, \dots, d\}$, and where $F_Z = G$. Often, the variables $X_1, \dots, X_d$ exhibit positive dependencies which motivates to consider positive dependence properties for the conditional independence product; see Section 2.2 for the concepts PSMD, CIS, CI, and MTP_2.

The following result establishes several positive dependence properties of the conditional independence product based on positive dependence conditions on the bivariate copula specifications.

Proposition 2 (Positive dependence concepts for conditional independence product)**.**
Let $E^1, \dots, E^d \in \mathcal{C}_2$ be bivariate copulas and $G \in \mathcal{F}^1$.

(i) *If $E^1, \dots, E^d$ are CIS, then $\Pi_G E^i$ is PSMD.*

(ii) *If E^1 and E^2 are CI, then $E^1 \Pi_G E^2$ is CI.*

(iii) *If $E^1, \dots, E^d$ are MTP_2, then $\Pi_G E^i$ is MTP_2.*

Proof: For all i, since E^i is CIS, it follows that $E^i \geq_{lo} \Pi^2$; see, e.g., [45]. Hence, Theorem 1 implies

$$\Pi^d = \Pi_{i=1,G}^d \Pi^2 \leq_{sm} \Pi_G E^i ,$$

which proves statement (i). For continuous G, statement (ii) is given by [33, Proposition 2(b)] by an application of the integration by parts formula. The general case follows similarly with an argument as in (C.24). Statement (iii) is a consequence of [20, Proposition 7.1] using the closure of the MTP_2-property under marginalization. □

As a direct consequence of Theorem 3 (and Proposition 2), we obtain for the conditionally independent factor model, which includes the widely-used standard factor model, the following comparison result.

Corollary 4 (Ordering results for conditionally independent factor models)**.**
Assume that $\mathbf{B} = \mathbf{C} = \mathbf{\Pi^d} = (\Pi^d)$ *and assume that* $D^i \leq_{\partial_2 S,G} E^i$, $i \in \{1, \ldots, d\}$.

(i) *If* E^i *is CIS and* $X_i \stackrel{\mathrm{d}}{=} Y_i$ *for all* $i \in \{1, \ldots, d\}$, *then* $X \leq_{sm} Y$.

(ii) *If* E^i *is* MTP_2 *and* $X_i \leq_{cx} Y_i$ *for all* $i \in \{1, \ldots, d\}$, *then* $X \leq_{dcx} Y$.

Remark 5.

(a) If both D^i and E^i are CIS, then the dependence condition $D^i \leq_{\partial_2 S,G} E^i$ in Theorem 3 and Corollary 4 is satisfied whenever $D^i \leq_{lo} E^i$, compare [4, Lemma 3.16(ii)]. Hence, Corollary 4 provides in particular simple conditions for a supermodular or directionally convex comparison of positively dependent distributions. Further, Corollary 4 includes the cases of independence and comonotonicity: If $D^1 = \cdots = D^d = \Pi^2$, then $\Pi_G D^i = \Pi^d$ and thus $X_1, \ldots, X_d$ are independent. If $E^1 = \cdots = E^d = M^2$ and if G is continuous, then $\Pi_{i=1}^d E^i = M^d$ and $Y_1, \ldots, Y_d$ are comonotonic.

(b) Our comparison results for factor models in Theorem 3 and Corollary 4 are based on the general supermodular comparison criterion for $*$-products in Theorem 5 (in Appendix B) extending comparison results for factor models known from the literature in several respects. For conditionally independent factor models, a supermodular comparison result is given in [6, Theorem 3.1] which follows from Corollary 4 (i) by a mixing argument and choosing $D^i = \Pi^2$ for all i; see also [46, Section 8.3.2].
Integral inequalities for conditionally independent factor models have also been studied in the literature under some structural assumptions. For exchangeable and for conditionally i.i.d. random variables, integral inequalities are given in [54] and [59], respectively. Ordering results for positively dependent random variables with common marginals are derived in [60] applying a conditional independence argument. In Corollary 4, the setting is more general because we compare conditionally independent random variables without further distributional assumptions. Note that Corollary 4 also allows to consider negatively depend distributions, for example, choosing $D^1 = W^2$ and $D^2 = E^1 = E^2 = M^2$ yields $W^2 = \Pi_{i=1}^2 D^i \leq_{sm} \Pi_{i=1}^2 E^i = M^2$ applying Corollary 4 (i).

5. Improved risk bounds in classes of conditionally independent factor models

In this section, we determine for various classes of conditionally independent factor models improved upper bounds with respect to the supermodular order. This yields by (2) improved portfolio risk bounds for applications in finance and insurance. The improvement refers to a comonotonic random vector denoted by

$$X^c := (X_1^c, \ldots, X_d^c) := (F_{X_1}^{-1}(U), \ldots, F_{X_d}^{-1}(U)), \quad U \sim \mathcal{U}(0,1), \tag{22}$$

which is the greatest element with respect to $\leq_{sm}$ in the pure marginal model, where the univariate marginal distributions are specified, but the dependence structure is not specified; see [58] and [46, Theorem 3.9.8]. Note that the supermodular order is invariant under increasing transformations, see (4), and thus all the examples with respect to the supermodular order apply for any choice of marginal distributions.

For $E^1, \ldots, E^d \in \mathcal{C}_2$, for $F_1, \ldots, F_d, G \in \mathcal{F}^1$, and for a random variable Z with $F_Z = G$, consider the class of conditionally independent factor models

$$\mathcal{M}^{ci} := \{X = (X_1, \ldots, X_d) \mid F_{X_i} = F_i\,,\ C_{X_i,Z} \leq_{\partial_2 S,G} E^i\,,\ i \in \{1, \ldots, d\},\ C_{X|Z=z} = \Pi^d \text{ for } P^Z\text{-almost all } z\} \tag{23}$$

with upper bounds on the copula specifications in the Schur-order and with fixed marginal distributions.

Allowing also for flexibility with respect to the marginal distribution, consider similarly the class of conditionally independent factor models

$$\mathcal{M}^{ci}_{cx} := \{X = (X_1,\dots,X_d) \mid F_{X_i} \leq_{cx} F_i\,,\ C_{X_i,Z} \leq_{\partial_2 S,G} E^i\,,\ i \in \{1,\dots,d\}\,,\ C_{X|Z=z} = \Pi^d \text{ for } P^Z\text{-almost all } z\} \tag{24}$$

with upper bounds on the copula specifications in the Schur-order and, now, with marginal distributions that are upper bounded in the convex order.

As an immediate consequence of Corollary 4 and (4), we obtain improved risk bounds for the classes $\mathcal{M}^{ci}$ and $\mathcal{M}^{ci}_{cx}$ of conditionally independent factor models. In particular, the improved bounds are the greatest elements in these classes.

Theorem 4 (Improved risk bounds).
Let Y be a d-dimensional random vector with $F_Y = \Pi_G E^i \circ (F_1,\dots,F_d)$.

(i) *If E^i is CIS for all $i \in \{1,\dots,d\}$, then $Y \in \mathcal{M}^{ci}$ and $X \leq_{sm} Y \leq_{sm} X^c$ for all $X \in \mathcal{M}^{ci}$.*

(ii) *If E^i is MTP_2 for all $i \in \{1,\dots,d\}$, then $Y \in \mathcal{M}^{ci}$ and $X \leq_{dcx} Y \leq_{sm} Y^c$ for all $X \in \mathcal{M}^{ci}_{cx}$.*

Note that the comonotonic vector X^c in statement (i) of the above result has the same distribution as the comonotonic vector Y^c because all elements in $\mathcal{M}^{ci}$ have the same univariate marginal distributions.

6. Conclusions

In this paper, we extend and strengthen several ordering results from the literature to general factor models and to the stronger notion of the supermodular and directionally convex order. The strengthened ordering results are rendered possible by some new general supermodular comparison conditions for copula products. The proof of this result is based on mass transfer theory in [43] which characterizes various integral stochastic orderings in duality. As consequence we obtain several simple ordering conditions for general factor models and, in particular, for conditionally independent factor models. As illustrated, our novel results are of considerable relevance in risk analysis allowing to derive sharp risk bounds for various classes of risk models.

Acknowledgement

We thank the Associate Editor and an anonymous reviewer for carefully reading and for their very detailed and helpful comments on our paper, which led to a restructuring of our manuscript. The first author gratefully acknowledges the support of the Austrian Science Fund (FWF) project P 36155-N *ReDim: Quantifying Dependence via Dimension Reduction* and the support of the WISS 2025 project 'IDA-lab Salzburg' (20204-WISS/225/197-2019 and 20102-F1901166-KZP).

Appendix A. Definition of $*$-product for general G

For a general factor distribution function G, the definition of the $*$-product requires the use of the generalized differential operator ∂^G which has the important property that it provides a representation of the conditional distribution function as a copula derivative also for general marginal distribution functions. More precisely, denote by

$$\begin{aligned} \iota_G\colon [0,1] &\to Ran(G)\,, & t &\mapsto G \circ G^{-1}(t)\,, \\ \iota_G^-\colon [0,1] &\to Ran(G^-)\,, & t &\mapsto G^- \circ G^{-1}(t) \end{aligned} \tag{A.1}$$

the transformation of the identity with respect to to G and G^-, respectively, where $G^{-1}\colon [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ defined by $G^{-1}(u) := \inf\{x \mid G(x) \geq u\}$, $\inf \emptyset = \infty$, is the generalized inverse of G, and G^- denotes the left-continuous version of G. Several properties of the transformations ι_G and ι_G^- are given in [4, Lemma A.1]. For a function $f\colon [0,1] \to \mathbb{R}$, the generalized differential operator ∂^G is then defined by the left-hand limit

$$\partial^G f(t_0) := \lim_{t \nearrow t_0} \frac{f(\iota_G(t_0)) - f(\iota_G^-(t))}{\iota_G(t_0) - \iota_G^-(t)}\,, \tag{A.2}$$

$t_0 \in (0,1]$, if the limit exists. As usual, denote by ∂_i^G the operator ∂^G which is applied to the ith coordinate of a function of several arguments.

Now, for a bivariate random vector (X_i, Z) with copula $C \in \mathcal{C}_2$, there exists for all $x \in \mathbb{R}$ a G-null set $N_x \subset \mathbb{R}$ such that the conditional distribution function of $X_i \mid Z = z$ is given by

$$F_{X_i|Z=z}(x) = \partial_2^G C(F_{X_i}(x), G(z)) \tag{A.3}$$

for all $z \in \mathbb{R} \setminus N_x$; see [4, Theorem 2.7].

The $*$-product of bivariate copulas $D^1, \ldots, D^d$ is defined in dependence on a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas and on a distribution function $G \in \mathcal{F}^1$ as follows.

Definition 2 ($*$-product of copulas, [4]).

(i) Let $\mathbf{B} := (B_t)_{t\in[0,1]}$ be measurable, $B_t \in \mathcal{C}_d$ for all $t \in [0,1]$, and let $G \in \mathcal{F}^1$. For bivariate copulas $D^1, \ldots, D^d \in \mathcal{C}_2$, the ($d$-dimensional) $*$-product of $D^1, \ldots, D^d$ with respect to $\mathbf{B}$ and G is defined by

$$\ast_{\mathbf{B},G} D^i\,(u) := \ast_{i=1,\mathbf{B},G}^d D^i\,(u) := \int_0^1 B_t^G\left(\partial_2^G D^1(u_1,t), \ldots, \partial_2^G D^d(u_d,t)\right) \mathrm{d}t \tag{A.4}$$

for $u = (u_1, \ldots, u_d) \in [0,1]^d$ where B_t^G is defined by

$$B_t^G := \begin{cases} B_t\,, & \text{if } \iota_G^-(t) = \iota_G(t)\,, \\ \frac{1}{\iota_G(t) - \iota_G^-(t)} \int_{\iota_G^-(t)}^{\iota_G(t)} B_s \,\mathrm{d}s\,, & \text{if } \iota_G^-(t) \neq \iota_G(t)\,, \end{cases} \tag{A.5}$$

for the transformations ι_G and ι_G^- given by (A.1).

(ii) If there exists a copula $B \in \mathcal{C}_d$ such that $B_t^G = B$ for Lebesgue-almost all t, then we write $\ast_{B,G} D^i := \ast_{\mathbf{B},G} D^i$ and call it simplified $*$-product of $D^1, \ldots, D^d$ with respect to B and G.

If G is continuous, then $\partial_2^G = \partial_2$ and $\iota_G^-(t) = \iota_G(t)$ for all t and thus the $*$-product in the above definition coincides with (9). We sometimes use the notation $D^1 \ast_{\mathbf{B},G} \cdots \ast_{\mathbf{B},G} D^d := \ast_{\mathbf{B},G} D^i$ for the $*$-product of d bivariate copulas $D^1 \ldots, D^d$ with respect to to $\mathbf{B}$ and G.

Appendix B. General supermodular comparison result for copula products

In this section, we formulate a general comparison result which provides sufficient conditions for the supermodular ordering of $\ast_{\mathbf{B},G} D^i$ with respect to the bivariate dependence specifications D^i. These sufficient conditions are given in terms of the lower orthant order for approximating sequences of $*$-products with finite support. In contrast to the supermodular order, such lower orthant ordering conditions can often easily be verified. As application of this theorem, we establish in Sections 3.1 and 3.2 several relevant supermodular ordering results for $*$-products.

The formulation of the supermodular ordering criterion requires a discretization of $*$-products. We introduce the necessary notation as follows.

For $n \in \mathbb{N}$ and any integer $d \geq 1$ denote by

$$\mathbb{G}_n^d : = \left\{(\tfrac{i_1}{n}, \ldots, \tfrac{i_d}{n}) \mid i_k \in \{1, \ldots, n\}, k \in \{1, \ldots, d\}\right\}, \qquad \mathbb{G}_{n,0}^d : = \left\{(\tfrac{i_1}{n}, \ldots, \tfrac{i_d}{n}) \mid i_k \in \{0, \ldots, n\}, k \in \{1, \ldots, d\}\right\}$$

the (extended) uniform n-grid of dimension d with edge length $1/n$. Denote by $\mathbb{M}_d^1(\mathbb{G}_{n,0}^d)$ the class of signed measures μ on the Borel σ-algebra on $\mathbb{R}^d$ with support in $\mathbb{G}_{n,0}^d$ such that $\mu(\mathbb{G}_{n,0}^d) = 1$.

The following notion of an n-grid d-copula is related to an d-subcopula with domain $\mathbb{G}_n^d$, see, e.g., [47, Definition 2.10.5]. For our purposes, we also need a signed version. Denote by $\lfloor \cdot \rfloor$ the componentwise floor function.

Definition 3 (Grid copula)**.**
For $d \in \mathbb{N}$, a (signed) n-grid d-copula (shortly, grid copula) is the measure generating function $D\colon [0,1]^d \to \mathbb{R}$ of a (signed) measure $\mu \in \mathbb{M}^1_d(\mathbb{G}^d_{n,0})$ with uniform univariate marginals, i.e., one has

(i) $D(u) = D(\lfloor nu \rfloor / n) = \mu([0, \lfloor nu \rfloor / n])$, $u \in [0,1]^d$, and

(ii) for all $i \in \{1,\dots,d\}$, it holds $D(u_1,\dots,u_d) = k/n$, $k \in \{0,\dots,n\}$, whenever $u_i = k/n$ and $u_j = 1$, $j \neq i$.

Denote by $C_{d,n}$ ($C^s_{d,n}$) the set of (signed) n-grid d-copulas.

Define by Δ^i_n the difference operator of length $1/n$ applied to the i-th argument of a function $f\colon [0,1]^d \to \mathbb{R}$, i.e.,

$$\Delta^i_n f(u) := f(u) - f(u_1,\dots,u_{i-1},\max\{u_i - \tfrac{1}{n}, 0\}, u_{i+1},\dots,u_d) \tag{B.1}$$

for $u = (u_1,\dots,u_d) \in [0,1]^d$. We make use of two types of approximations of a copula $D \in C_d$:

For an approximation by distributions with finite support, denote by $\mathbb{G}_n(D)\colon [0,1]^d \to [0,1]$ the n-grid d-copula associated with D defined by

$$\mathbb{G}_n(D)(u) := D(\lfloor nu \rfloor / n) \quad \text{for all } u \in [0,1]^d . \tag{B.2}$$

For an approximation by copulas, denote by $\mathfrak{Ch}_n(D)$ the checkerboard copula associated with D that distributes for each cube $I^{\mathbf{k}}_n := (\frac{k_1-1}{n}, \frac{k_1}{n}] \times \cdots \times (\frac{k_d-1}{n}, \frac{k_d}{n}]$, $\mathbf{k} := (k_1,\dots,k_d) \in \{1,\dots,n\}^d$, the mass of D on $I^{\mathbf{k}}_n$ uniformly on $I^{\mathbf{k}}_n$, i.e., for all $x \in [0,1]^d$, we have

$$\mathfrak{Ch}_n(D)(x) := \int_{[0,x]} \beta(v)\,\mathrm{d}v\,, \quad \beta(v) := \sum_{\mathbf{k} \in \{1,\dots,d\}^n} \frac{\Delta^1_n \cdots \Delta^d_n D_n(\mathbf{k})}{n^d} \mathbb{1}_{I^{\mathbf{k}}_n}(v)\,; \tag{B.3}$$

see [42]. For an n-grid copula $D \in C_{d,n}$, the associated checkerboard copula $\mathfrak{Ch}_n(D)$ is defined in the same way by (B.3). Note that $\mathfrak{Ch}_n(D) = \mathfrak{Ch}_n(\mathbb{G}_n(D))$ for all $D \in C_d$.

Denote by $\mathcal{F}^1_n := \{G \in \mathcal{F}^1 \mid Ran(G) \subseteq \mathbb{G}^1_{n,0}\}$ the set of univariate distribution functions with finite range contained in the extended n-grid $\mathbb{G}^1_{n,0} = \{0, 1/n, \dots, (n-1)/n, 1\}$. We define the $*$-product for signed grid copulas as follows.

Definition 4 ($*$-product for signed grid copulas)**.**
For $G \in \mathcal{F}^1_n$ and for all $i \in \{1,\dots,d\}$, let $D^i_n \in C^s_{2,n}$ be a signed n-grid copula with $0 \leq \partial^G_2 D^i_n(u_i, \frac{k}{n}) \leq 1$ for all $u_i \in [0,1]$ and $k \in \{1,\dots,n\}$. Then, define for a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas the discrete $*$-product of $D^1_n,\dots,D^d_n$ with respect to $\mathbf{B}$ and G by

$$*_{\mathbf{B},G} D^i_n(u) := \int_0^1 B^G_t\left(\left(\partial^G_2 D^i_n(u_i,t)\right)_{1\leq i\leq d}\right) \mathrm{d}t\,, \quad u = (u_1,\dots,u_d) \in [0,1]^d, \tag{B.4}$$

where B^G_t is defined by (A.5).

Similar to [4, Proposition 2.6], the $*$-product of (signed) grid copulas is a (signed) grid copula. Note that the integrand in (B.4) is piecewise constant in t, because $G \in \mathcal{F}^1_n$.

For bivariate copulas D, D_n, $n \in \mathbb{N}$, denote by $D_n \xrightarrow{\partial_2} D$ the ∂_2-convergence of bivariate copulas defined by

$$\int_0^1 |\partial_2 D_n(x,t) - \partial_2 D(x,t)|\,\mathrm{d}t \xrightarrow{n\to\infty} 0 \quad \text{for all } \ x \in [0,1]\,; \tag{B.5}$$

see [4, Section 2.4]. Note that the ∂_2-convergence corresponds to the D_1-convergence considered in [62].

To formulate a general criterion for the supermodular ordering of $*$-products, we state without proof the following simple lemma on bivariate grid copula approximations. Since bivariate grid copulas correspond to doubly stochastic matrices, it roughly states that, for grid approximations $D_n, E_n \in C_{2,n}$, $n \in \mathbb{N}$, of $D, E \in C_2$, the bivariate copula D can be transformed into the bivariate copula E by successively transforming for all $n \in \mathbb{N}$ two columns of the matrix associated with D_n into the corresponding columns of the matrix associated with E_n for all $n \in \mathbb{N}$.

For signed grid copulas $D, E \in C^s_{d,n}$, the lower orthant order $D \leq_{lo} E$ is defined pointwise by $D(u) \leq E(u)$ for all $u \in [0,1]^d$.

Lemma 1. *Let $D^i, E^i \in C_2$ be bivariate copulas with $D^i \leq_{lo} E^i$ for all $i \in \{1, \ldots, d\}$. Then, for all $n \in \mathbb{N}$, there exist finite sequences $(D^i_{j,n})_{0 \leq j \leq m_n}$ of signed n-grid copulas in $C^s_{2,n}$ such that*

$$\mathfrak{Ch}(D^i_{0,n}) \xrightarrow{\partial_2} D^i, \quad \mathfrak{Ch}(D^i_{m_n,n}) \xrightarrow{\partial_2} E^i, \qquad \text{for all } i\,, \tag{B.6}$$

$$\Delta_n^2 D^i_{j-1,n}(u_i, t) = \Delta_n^2 D^i_{j,n}(u_i, t) \qquad \text{for all } t \in \mathbb{G}_n^1 \setminus \{t_*, t^*\}\,, u_i \in [0,1]\,, i\,, j\,, \tag{B.7}$$

$$0 \leq \Delta_n^2 D^i_j(u_i, t) \leq 1\,, \qquad \text{for all } t \in \mathbb{G}_n^1\,, u_i\,, i\,, j \tag{B.8}$$

$$D^i_{j-1,n} \leq_{lo} D^i_{j,n}\,, \qquad \text{for all } i\,, j\,, \tag{B.9}$$

for $t_*, t^* \in \mathbb{G}_n^1$ depending only on j and n.

Note that the sequences $(D^i_{j,n})_{0 \leq j \leq m_n}$ in the above lemma are far from being uniquely determined.

The following result is a main result of this paper. It provides a general $\leq_{lo}$-ordering criterion for the supermodular ordering of $*_{\mathbf{B},G} D^i$ with respect to the specifications D^i.

Theorem 5 (General supermodular ordering criterion for $*$-products)**.**
For $G \in \mathcal{F}^1$, let $(G_n)_{n \in \mathbb{N}}$ be an approximation of G with $G_n \in \mathcal{F}^1_n$ for all n such that $\iota_{G_n} \to \iota_G$ almost surely pointwise. Let $D^i, E^i \in C_2$ with $D^i \leq_{lo} E^i$ be copulas with approximating sequences $(D^i_{j,n})_{0 \leq j \leq m_n}$, $n, m_n \in \mathbb{N}$, of signed grid copulas in $C^s_{2,n}$ satisfying (B.6) - (B.9) *for all $n \in \mathbb{N}$ and $1 \leq i \leq d$.*
Assume for a measurable family $\mathbf{B} = (B_t)_{t \in [0,1]}$ of d-copulas that

$$*_{\mathbf{B},G_n} D^i_{j-1,n} \leq_{lo} *_{\mathbf{B},G_n} D^i_{j,n}\,, \quad j \in \{1, \ldots, m_n\} \text{ and } n \in \mathbb{N}\,. \tag{B.10}$$

Then, it follows that

$$*_{\mathbf{B},G} D^i \leq_{sm} *_{\mathbf{B},G} E^i\,.$$

The proof of the above theorem is given in Appendix C.2 and requires several tools from mass transfer theory provided in Appendix C.1. Note that we make use of Theorem 5 to establish several supermodular comparison results for copula products in Section 3. Corresponding comparison results for factor models are provided in Section 4.

Appendix C. Proof of the general supermodular ordering criterion for $*$-products

In this appendix, we prove Theorem 5 which requires several tools from mass transfer theory.

Appendix C.1. Construction of mass transfers for grid copulas

For the proof of Theorem 5, we need some tools from mass transfer theory which characterizes several integral stochastic orders for distributions with finite support by duality; see [43], cf. [41].

Denote by δ_x, $x \in \mathbb{R}^d$, the one-point probability measure in x. Further, denote for a finite signed measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by (μ^+, μ^-) the Jordan decomposition of μ, i.e., μ^+ and μ^- are the uniquely determined finite measures such that $\mu = \mu^+ - \mu^-$.

Then, for two finite signed measures P, Q supported on a finite subset of $\mathbb{R}^d$, the signed measure $Q - P$ is called a transfer from P to Q. If

$$(Q - P)^- = \sum_{i=1}^n \alpha_i \delta_{x_i}\,, \qquad (Q - P)^+ = \sum_{j=1}^m \beta_j \delta_{y_j}$$

then the transfer $Q - P$ removes (probability) mass $\alpha_i > 0$ from point $x_i \in \mathbb{R}^d$ for $i \in \{1, \ldots, n\}$ and adds (probability) mass $\beta_j > 0$ to the point $y_j \in \mathbb{R}^d$ for $j \in \{1, \ldots, m\}$. The transfer $Q - P$ is indicated by writing

$$\sum_{i=1}^n \alpha_i \delta_{x_i} \quad \to \quad \sum_{j=1}^m \beta_j \delta_{y_j}\,.$$

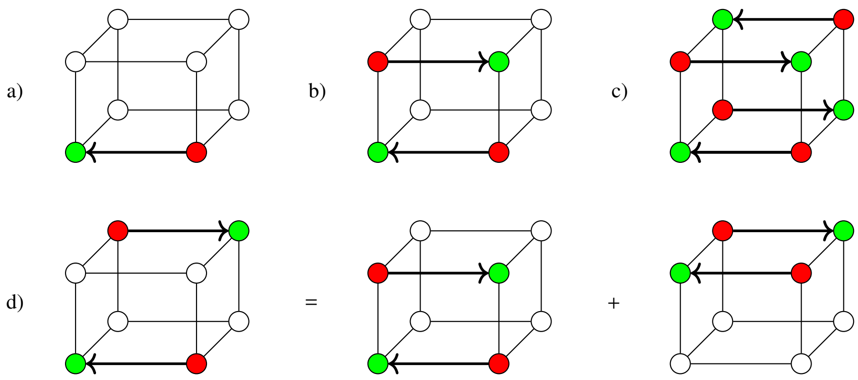

Fig. C.2 Every arrow illustrates a mass transfer of size $\eta > 0$. a) illustrates a one-dimensional Δ-antitone or decreasing transfer; b) illustrates a 2-dimensional Δ-antitone transfer; c) illustrates a 3-dimensional Δ-antitone transfer, which consists of a 2-dimensional Δ-antitone and a reverse 2-dimensional Δ-antitone transfer; d) illustrates a (3-dimensional) supermodular transfer, which is decomposed into two 2-dimensional Δ-antitone transfers, which are also supermodular transfers.

The reverse transfer corresponding to $Q - P$ is defined as $P - Q$.

We make use of the following transfers. Denote by $\wedge$ and $\vee$ the component-wise minimum and maximum, respectively, of vectors having the same length.

Definition 5 (Δ-antitone/increasing/supermodular transfer)**.**
Let $\eta > 0$ and $x, y \in \mathbb{R}^d$ such that $x \leq y$ with strict inequality in k components, i.e., $x_i < y_i$, $i \in I \subseteq \{1, \dots, d\}$ with $|I| = k$, and $x_j = y_j$, $j \in I^c = \{1, \dots, d\} \setminus I$. Denote by $\mathcal{V}_o(x, y)$ and $\mathcal{V}_e(x, y)$ the set of all vertices z of the k-dimensional hyperbox $[x, y]$ such that the number of components with $z_i = x_i$, $i \in \{i_1, \dots, i_k\}$ is odd and even, respectively.

(i) A transfer indicated by

$$\sum_{z \in \mathcal{V}_o(x,y)} \eta\, \delta_z \to \sum_{z \in \mathcal{V}_e(x,y)} \eta\, \delta_z\,, \quad \text{if } k \text{ is even,} \qquad\qquad \sum_{z \in \mathcal{V}_e(x,y)} \eta\, \delta_z \to \sum_{z \in \mathcal{V}_o(x,y)} \eta\, \delta_z\,, \quad \text{if } k \text{ is odd,}$$

is called (k-dimensional) Δ-antitone transfer.

(ii) A transfer indicated by

$$\eta\, \delta_x \to \eta\, \delta_y \quad \text{resp.} \quad \eta\, \delta_y \to \eta\, \delta_x \tag{C.1}$$

is called (k-dimensional) increasing and decreasing transfer, respectively.

(iii) For $v, w \in \mathbb{R}^d$ such that $v \wedge w = x$ and $v \vee w = y$, a transfer indicated by

$$\eta\, (\delta_v + \delta_w) \to \eta\, (\delta_{v \wedge w} + \delta_{v \vee w})$$

is called (k-dimensional) supermodular transfer.

Denote by $\mathcal{T}_\Delta^-$, $\mathcal{T}^\uparrow$, $\mathcal{T}^\downarrow$, and $\mathcal{T}_{sm}$ the set of Δ-antitone, increasing, decreasing, and supermodular transfers, respectively. For $x, y \in \mathbb{R}^1$, define by

$$\mathcal{T}_x^\downarrow := \left\{\mu \mid \mu = \eta(\delta_{(x,u^1)} - \delta_{(x,u^2)})\,,\ u^1, u^2 \in \mathbb{R}^d\,,\ u^1 \leq u^2\,,\ \eta \geq 0\right\},$$
$$\mathcal{T}_y^\uparrow := \left\{\mu \mid \mu = \eta(\delta_{(y,u^2)} - \delta_{(y,u^1)})\,,\ u^1, u^2 \in \mathbb{R}^d\,,\ u^1 \leq u^2\,,\ \eta \geq 0\right\}$$

the set of conditionally on the first variable decreasing and increasing transfers, respectively. Then, for $x < y$, denote by

$$\mathcal{T}_{x,y}^{\downarrow\uparrow} := \left\{\mu^1 + \mu^2 \mid \mu^1 \in \mathcal{T}_x^{\downarrow}\,,\ \mu^2 \in \mathcal{T}_y^{\uparrow}\right\} \tag{C.2}$$

the set of decreasing-increasing transfers conditional on x and y.

The following lemma is the key of the proof of Theorem 5. It states that a set of decreasing-increasing transfers that corresponds to a set of ($k \geq 2$)-dimensional Δ-antitone transfers can be decomposed into a set of supermodular transfers.

Lemma 2. *Let P, Q be finite (signed) measures with finite support in $\mathbb{R}^{d+1}$. Assume that P and Q have the same univariate marginals. For fixed $x < y$, let $\{\nu_\ell\}_\ell \subset \mathcal{T}_{x,y}^{\downarrow\uparrow}$ and $\{\mu_\ell\}_\ell \subset \mathcal{T}_\Delta^-$ be transfers. If $Q - P = \sum_\ell \nu_\ell = \sum_\ell \mu_\ell$, then there exists a set $\{\gamma_\ell\}_\ell \subset \mathcal{T}_{sm}$ such that $Q - P = \sum_\ell \gamma_\ell$.*

Proof: We show that the transfers $\{\mu_\ell\} \subset \mathcal{T}_\Delta^-$ with $Q - P = \sum_\ell \nu_\ell = \sum_\ell \mu_\ell$ can be chosen two-dimensional. Then the statement is proved because every two-dimensional Δ-antitone transfer is supermodular.

Since P and Q have the same univariate marginal measures and $Q - P = \sum_\ell \mu_\ell$, no one-dimensional Δ-antitone transfer μ_ℓ is possible.

Assume for $k \geq 3$ that a k-dimensional Δ-antitone transfer μ_ℓ is necessary. Since for such k, any k-dimensional Δ-antitone transfer consists of 2^{k-3} two-dimensional Δ-antitone and 2^{k-3} two-dimensional reverse Δ-antitone transfers, this implies that a two-dimensional reverse Δ-antitone transfer is necessary; see Fig. C.2.
By definition of the class $\mathcal{T}_{x,y}^{\downarrow\uparrow}$ in (C.2), every transfer ν_ℓ is decreasing-increasing and, thus, of the form $\nu_\ell = \nu_\ell^1 + \nu_\ell^2$ where ν_ℓ^1 and ν_ℓ^2 are indicated by

$$\eta_\ell^1 \delta_{(x,v_\ell)} \to \eta_\ell^1 \delta_{(x,u_\ell)}, \qquad\qquad \eta_\ell^2 \delta_{(y,u_\ell)} \to \eta_\ell^2 \delta_{(y,v_\ell)}\,,$$

respectively, with $u_\ell \leq v_\ell$ and $\eta_\ell^1, \eta_\ell^2 \geq 0$. Since $Q - P = \sum_\ell \nu_\ell$, this means that no 2-dimensional reverse Δ-antitone transfer is necessary. $\square$

Appendix C.2. Proof of Theorem 5

We make use of the following result which states that for random vectors with identical copula, the lower orthant order coincides with the converse stochastic order; cf. [50, Proposition 7] and [45, Theorem 4.1].

Proposition 3. *Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be random vectors with identical copula $C \in \mathcal{C}_d$, i.e., $C = C_X = C_Y$. Then, $X \leq_{lo} Y$ and $X \geq_{st} Y$ are equivalent.*

Proof: Assume $X \leq_{lo} Y$. For a random vector $(U_1, \ldots, U_d) \sim C$ holds

$$X = \left(F_i^{-1}(U_i)\right)_{1 \leq i \leq d}, \quad Y = \left(G_i^{-1}(U_i)\right)_{1 \leq i \leq d} \tag{C.3}$$

almost surely where $F_i = F_{X_i}$, $G_i = F_{Y_i}$, $i \in \{1, \ldots, d\}$. Then, $X \leq_{lo} Y$ implies $X_i \leq_{lo} Y_i$ which means that $F_i(x) \leq G_i(x)$ for all $x \in \mathbb{R}$. But this is equivalent to $F_i^{-1}(t) \geq G_i^{-1}(t)$ for all $t \in [0,1]$, which implies $X \geq_{st} Y$ due to the representation in (C.3).

The reverse direction follows from choosing functions of the form $f(x) = -\prod_{i=1}^d \mathbb{1}_{\{x_i \leq t_i\}}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, for $t_i \in \mathbb{R}$, $i \in \{1, \ldots, d\}$. $\square$

For the proof of Theorem 5, we need a version of the above statement for signed grid distributions. Similar to the case of probability measures, there is a one-to-one correspondence between a signed measure μ on $\mathbb{G}_n^d$ and a measure generating function F by $F(x) = \mu([0,x])$ for all $x \in [0,1]^d$.For $i \in \{1, \ldots, d\}$, denote by F_i the i-marginal function defined by $F_i(x_i) = F(x)$ for $x \in [0,1]^d$ such that $x_j = 1$ for all $j \neq i$. If $0 \leq F_i(x) \leq 1$ for all x_i and i, then similar to Sklar's theorem, F may be decomposed into $F(x) = C(F_1(x_1), \ldots, F_d(x_d))$ for all $x = (x_1, \ldots, x_d) \in [0,1]^d$ and for a copula $C\colon [0,1]^d \to [0,1]$. The lower orthant and the stochastic order for generating functions of signed measures on $\mathbb{G}_n^d$ are defined in the canonical way.

Lemma 3. *Let F, G be signed distribution functions on $\mathbb{G}_n^d$ with $0 \leq F(x), G(x) \leq 1$ for all $x \in [0,1]^d$. Assume that F and G have a common copula, i.e. $C_F = C_G$ on $[0,1]^d$. Then, $F \leq_{lo} G$ implies $F \geq_{st} G$.*

Proof: Consider the measure $\nu_n := n^d \nu$, where ν follows the uniform distribution on the grid $\mathbb{G}_n^d$. Then ν_n has mass of size 1 at each grid point of $\mathbb{G}_n^d$. Denote by μ_F and μ_G the signed measure associated with F and G, respectively. Then, the measures $\nu_F := (\mu_F + \nu_n)/(n^d+1)$ and $\nu_G := (\mu_G + \nu_n)/(n^d+1)$ are probability measures on $\mathbb{G}_n^d$ with $\nu_F \leq_{lo} \nu_G$ and with a common copula. Hence, Proposition 3 implies $\nu_F \geq_{st} \nu_G$. This yields

$$\int f \,\mathrm{d}\mu_F = (n^d+1)\int f\,\mathrm{d}\nu_F - \int f\,\mathrm{d}\nu_n \geq (n^d+1)\int f\,\mathrm{d}\nu_G - \int f\,\mathrm{d}\nu_n = \int f\,\mathrm{d}\mu_G$$

for all increasing functions f, which implies that $F \geq_{st} G$. □

Denote by $\rightsquigarrow$ convergence in distribution and recall the ∂_2-convergence $\xrightarrow{\partial_2}$ in (B.5). By the following lemma, the $*$-product of copulas can be approximated by discretized $*$-products.

Lemma 4 (Grid copula approximation of $*$-products)**.**
Let $(G_n)_{n\in\mathbb{N}}$ be an approximation of $G \in \mathcal{F}^1$ with $G_n \in \mathcal{F}_n^1$ such that $\iota_{G_n} \to \iota_G$ almost surely pointwise. For all $i \in \{1,\ldots,d\}$, let $D^i \in \mathcal{C}_2$ and $(D_n^i)_{n\in\mathbb{N}}$ be a sequence of n-grid copulas in $\mathcal{C}_{2,n}$ such that $\mathfrak{Ch}(D_n^i) \xrightarrow{\partial_2} D^i$ as $n\to\infty$. Then for all measurable families $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas, it holds that

$$*_{\mathbf{B},G_n} D_n^i \rightsquigarrow *_{\mathbf{B},G} D^i \quad \text{for } n\to\infty\,. \tag{C.4}$$

Proof: Since the discretized $*$-product coincides at the grid points $\mathbb{G}_n^d$ with the $*$-product of the associated checkerboard copulas, we obtain

$$*_{\mathbf{B},G_n} D_n^i(u) = *_{\mathbf{B},G_n} \mathfrak{Ch}(D_n^i)(\tfrac{\lfloor nu\rfloor}{n}) \quad\longrightarrow\quad *_{\mathbf{B},G} D^i(u) \quad \text{for } n\to\infty$$

for all $u\in[0,1]^d$. For the convergence, we apply [4, Theorem 2.23] and use the uniform continuity of the class $\mathcal{C}_d$ of d-copulas. □

Remark 6. Let $D^1,\ldots,D^d$ be bivariate copulas. The grid approximation of D^i in (B.2) defines a sequence of n-grid 2-copulas that satisfies the assumptions of Lemma 4, i.e.,

$$\mathfrak{Ch}_n(\mathbb{G}_n(D^i)) = \mathfrak{Ch}_n(D^i) \xrightarrow{\partial_2} D^i, \quad n\to\infty \tag{C.5}$$

for all i; see [42, Theorem 5] for the convergence. Hence, we obtain for the $*$-product of the canonical grid copulas $D_n^i = \mathbb{G}_n(D^i)$, $i\in\{1,\ldots,d\}$, the convergence in (C.4), i.e.,

$$*_{\mathbf{B},G_n} \mathbb{G}_n(D^i) \rightsquigarrow *_{\mathbf{B},G} D^i, \quad n\to\infty \tag{C.6}$$

for $(G_n), G$, and $\mathbf{B}$ as in Lemma 4.

We are now able to provide the proof of the general supermodular ordering criterion.

Proof of Theorem 5: We show for all $n\in\mathbb{N}$ that

$$*_{\mathbf{B},G_n} D_{n,0}^i \leq_{sm} *_{\mathbf{B},G_n} D_{m_n,n}^i\,. \tag{C.7}$$

Then, the statement follows from the convergence of the grid copula approximations in Lemma 4 and from the closure of the supermodular order under weak convergence; see [44, Theorem 3.5]. Note that both $*$-products in (C.7) are probability distribution functions.

In order to prove (C.7), we define for $G'\in\mathcal{F}_n^1$, for a measurable family $\mathbf{C} = (C_t)_{t\in[0,1]}$ of d-copulas, and for $C_n^i \in \mathcal{C}_{2,n}^s$ with $0\leq \partial_2^{G'} C_n^i \leq 1$ for all $i\in\{1,\ldots,d\}$, the extended $*$-product $\bigstar_{i=1,\mathbf{B},G'}^d C_n^i\colon [0,1]^{d+1}\to[0,1]$ by

$$\bigstar_{\mathbf{C},G'} C_n^i(u) := \bigstar_{i=1,\mathbf{C},G'}^d C_n^i(u) := \int_0^{\iota_{G'}^+(u_0)} C_s^{G'}\left((\partial_2^{G'} C_n^i(u_i,s))_{1\leq i\leq d}\right)\mathrm{d}s,$$

for $u = (u_0, u_1, \dots, u_d) \in [0,1]^{d+1}$, where $C_t^{G'}$ is given as in (A.5) and where ι_G^+ denotes the right-continuous version of ι_G. Note that ι_G^- is left-continuous, but ι_G is in general neither left- nor right-continuous; see [4, Lemma A.1 (vi) and (viii)]. Similar to (B.4), $C_n := \bigstar_{\mathbf{C},G'} C_n^i$ is a signed n-grid $(d+1)$-copula on $\mathbb{G}_n^{d+1}$. Denote by $C_n^{[t]}$ where

$$C_n^{[t]}(u_1, \dots, u_d) := \partial_1^{G'} C_n(t, u_1, \dots, u_d) = \frac{1}{\iota_{G'}^+(t) - \iota_{G'}^-(t)} \int_{\iota_{G'}^-(t)}^{\iota_{G'}^+(t)} C_s^{G'}\left((\partial_2^{G'} C_n^i(u_i, s))_{1\le i\le d}\right) \mathrm{d}s, \tag{C.8}$$

for $(u_1, \dots, u_d) \in [0,1]^d$, the conditional signed measure generating function of C_n under $u_0 = t \in \mathbb{G}_n^1$. Note that both $\iota_{G'}^+(t)$ and $\iota_{G'}^+(t) - \iota_{G'}^-(t)$ take values in $\{1/n, 2/n, \dots, 1\}$, because $G' \in \mathcal{F}_n^1$; see Lemma [4, Lemma A.1].

Now, for fixed $j \in \{1, \dots, m_n\}$ and $n \in \mathbb{N}$, we abbreviate the bivariate grid specifications from the assumptions of Theorem 5 by $D_n^i := D_{j-1,n}^i$ and $E_n^i := D_{j,n}^i$ for $i \in \{1, \dots, d\}$, and consider the extended $*$-products $D_n := \bigstar_{\mathbf{B},G_n} D_n^i$ and $E_n := \bigstar_{\mathbf{B},G_n} E_n^i$, which are well-defined due to condition (B.8). Using that $Ran(G_n) \subset \mathbb{G}_{n,0}^1$, conditions (B.7) and (B.9) imply that

$$\begin{aligned}
&\partial_2^{G_n} D_n^i(\cdot, s) = \partial_2^{G_n} E_n^i(\cdot, s), && s \in (t_{r-1}, t_r),\ t_r \in Ran(G_n) \setminus \{t_*, t^*, 0\}, \\
&\partial_2^{G_n} D_n^i(\cdot, t_*) \le_{lo} \partial_2^{G_n} E_n^i(\cdot, t_*), && s \in (t_{r-1}, t_r),\ t_r = t_*, \\
&\partial_2^{G_n} D_n^i(\cdot, t^*) \ge_{lo} \partial_2^{G_n} E_n^i(\cdot, t^*), && s \in (t_{r-1}, t_r),\ t_r = t^*.
\end{aligned}$$

This yields for the conditional signed measure generating functions that

$$D_n^{[t]} = E_n^{[t]} \quad \text{for all } t \in Ran(G_n) \setminus \{t_*, t^*, 0\}, \qquad D_n^{[t_*]} \le_{lo} E_n^{[t_*]}, \qquad D_n^{[t^*]} \ge_{lo} E_n^{[t^*]}, \tag{C.9}$$

using that $B_s^{G_n}$, $s \in [0,1]$, is a copula and, thus, componentwise increasing. For the extended $*$-products, this implies with condition (B.10) that

$$D_n \le_{lo} E_n. \tag{C.10}$$

Hence, from a version of Theorem 2.5.7 in [43] for finite signed measures, we obtain the existence of a finite set $\{\mu_l\}_l \subset \mathcal{T}_\Delta^-$ of Δ-antitone transfers such that

$$P_{E_n} - P_{D_n} = \sum_l \mu_l. \tag{C.11}$$

Further, the inequalities in (C.9) yield $D_n^{[t_*]} \ge_{st} E_n^{[t_*]}$ and $D_n^{[t^*]} \le_{st} E_n^{[t^*]}$; see Lemma 3. Since the stochastic order for signed measures with finite support is characterized by increasing transfers, there exists a finite set $\{\nu_l^{\downarrow}\}_l \subset \mathcal{T}^{\downarrow}$ of decreasing transfers and a finite set $\{\nu_l^{\uparrow}\}_l \subset \mathcal{T}^{\uparrow}$ of increasing transfers such that

$$P_{E_n^{[t_*]}} = P_{D_n^{[t_*]}} + \sum_l \nu_l^{\downarrow}, \quad P_{E_n^{[t^*]}} = P_{D_n^{[t^*]}} + \sum_l \nu_l^{\uparrow},$$

see [43, Theorem 2.5.1], which also holds true for finite signed measures. This implies

$$P_{E_n} - P_{D_n} = \sum_l \nu_l, \tag{C.12}$$

for some set $\{\nu_l\}_l \subset \mathcal{T}_{t_*,t^*}^{\downarrow\uparrow}$ of decreasing-increasing transfers.

Finally, (C.11) and (C.12) yield by Lemma 2 a set $\{\gamma_l\}_l \subset \mathcal{T}_{sm}$ of supermodular transfers such that $P_{E_n} - P_{D_n} = \sum_l \gamma_l$. This implies $D_n \le_{sm} E_n$; cf. Theorem 2.5.4 in [43]. Then, (C.7) follows from the closure under marginalization and the transitivity of the supermodular order. □

Appendix D. Proofs of Theorems 1 and 2

In this section, we provide the proofs of Theorems 1 and 2 applying the supermodular ordering criterion in Theorem 5.

Proof of Theorem 1: '(i) $\Longrightarrow$ (ii)': Since the supermodular order implies the lower orthant order, we have that $*_{\mathbf{B},G}D^i \leq_{lo} *_{\mathbf{B},G}E^i$ for all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ with $D^i \leq_{lo} E^i$, $i \in \{1,\ldots,d\}$. Due to Theorem 3.7 in [4], this is equivalent to (ii).

To show '(ii) $\Longrightarrow$ (i)', we apply Theorem 5 as follows. Let $(G_n)_{n\in\mathbb{N}}$ be an approximation of G such that $G_n \in \mathcal{F}_n^1$ for all n and $\iota_{G_n} \to \iota_G$ almost surely pointwise, for example, take the approximation in [4, Example 2.18(a)] assuming without loss of generality that G is supported on a finite interval (see [4, Proposition 2.14]). Without loss of generality, we assume that $D^1 \neq E^1$ and $D^i = E^i$, $i \in \{2,\ldots,d\}$. For all $n \in \mathbb{N}$ and $1 \leq i \leq d$, we construct sequences $(D^i_{j,n})_{0\leq j\leq m_n}$ of (signed) n-grid copulas in $\mathcal{C}^s_{2,n}$ that fulfill conditions (B.6) – (B.10) for the discretized $*$-product with respect to $\mathbf{B}$ and G_n.

Consider the following algorithm which determines for fixed $n \in \mathbb{N}$ a sequence $(D^1_{j,n})_{0\leq j\leq m_n}$ of signed n-grid copulas that transforms the discretized n-grid copula of D^1 into the discretized n-grid copula of E^1 by modifying in each step only four entries in two columns and two rows of the associated mass matrices.

(I) Set $j = 0$ and $k = 1$. For $i \in \{1,\ldots,d\}$, define the n-grid copulas $D^i_{0,n}$ and E^i_n by $D^i_{0,n} := \mathbb{G}_n(D^i)$ and $E^i_n := \mathbb{G}_n(E^i)$.

(II) Mass compensation in line $n-k+1$: If $\Delta_{n,2}D^1_{j,n}(k/n,t) = \Delta_{n,2}E^1_n(k/n,t)$ for all $t \in \mathbb{G}^1_n$, go to step (III). Otherwise fix the columns $\ell_* := \min\{\ell \in \{1,\ldots,n\} \mid \Delta_{n,2}D^1_{j,n}(k/n,\ell/n) < \Delta_{n,2}E^1_n(k/n,\ell/n)\}$ and $\ell^* := \max\{\ell \in \{1,\ldots,d\} \mid \Delta_{n,2}D^1_{j,n}(k/n,\ell/n) > \Delta_{n,2}E^1_n(k/n,\ell/n)\}$ of the mass matrix associated with $D^1_{j,n}$. Define the transferred mass η by

$$\eta := \min\left\{\Delta_{n,2}E^1_n(k/n,\ell_*/n) - \Delta_{n,2}D^1_{j,n}(k/n,\ell_*/n)\,,\ \Delta_{n,2}D^1_{j,n}(k/n,\ell^*/n) - \Delta_{n,2}E^1_n(k/n,\ell^*/n)\right\}.$$

Define the signed n-grid copula $D^1_{j+1} \in \mathcal{C}^s_{2,n}$ by

$$D^1_{j+1,n}(r/n,\ell/n) := \begin{cases} D^1_{j,n}(k/n,\ell/n) + \eta, & \text{if } r = k \text{ and } \ell \in \{\ell_*,\ell_*+1,\ldots,\ell^*-1\}\,,\\ D^1_{j,n}(r/n,\ell/n), & \text{else,}\end{cases} \tag{D.1}$$

for $r,\ell \in \{1,\ldots,n\}$. Set $j = j+1$ and repeat step (II).

(III) If $k = n-1$, set $m_n := j$, define $D^i_{j,n} := D^i_{0,n}$ for all $2 \leq i \leq d$ and $1 \leq j \leq m$, and stop the algorithm. Otherwise set $k = k+1$ and go to step (II).

Since $D^i = E^i$ for $2 \leq i \leq d$, the algorithm produces in step (III) a constant sequence $(D^i_{j,n})_{0\leq j\leq m}$ for each of these indices i.

We show for $i = 1$ that

$$E^i_n = D^i_{m_n,n}, \tag{D.2}$$

$$\mathfrak{Ch}(D^i_{0,n}) \xrightarrow{\partial_2} D^i,\quad \mathfrak{Ch}(D^i_{m_n,n}) \xrightarrow{\partial_2} E^i, \tag{D.3}$$

$$\Delta^2_n D^i_{j-1,n}(u,t) = \Delta^2_n D^i_{j,n}(u,t), \qquad j \in \{1,\ldots,m_n\},\ t \in \mathbb{G}^1_n \setminus \{t_*,t^*\},\ u \in [0,1], \tag{D.4}$$

$$0 \leq \Delta^2_n D^i_{j,n}(\cdot,t) \leq 1, \qquad j \in \{0,\ldots,m_n\},\ t \in \mathbb{G}^1_n, \tag{D.5}$$

$$D^i_{j-1,n} \leq_{lo} D^i_{j,n}, \qquad j \in \{1,\ldots,m_n\}, \tag{D.6}$$

$$D^i_{j,n} \text{ is CIS}, \qquad j \in \{0,\ldots,m_n\}, \tag{D.7}$$

where $t_* = \ell_*/n$ and $t^* = \ell^*/n$ depend on j. For $i \in \{2,\ldots,d\}$, the above equations are trivially fulfilled because the sequence $(D^i_{j,n})_{0\leq j\leq m}$ is constant and D^i is CIS.

From steps (II) and (III), we observe that only a finite number of iterations is possible which yields $m_n \in \mathbb{N}_0$. Note that the algorithm already stops when the lower $n-1$ rows of the matrix associated with $D^1_{m_n,n}$ are adjusted to the matrix associated with E^1_n, because in this case also the first row given by $\Delta_{n,2} D^1_{m_n,n}(1,\cdot)$ equals $\Delta_{n,2} E^1_n(1,\cdot)$ due to the uniform marginal property of signed grid copulas on $\mathbb{G}^d_{n,0}$. This yields (D.2). Then, (D.3) follows with (C.5).

We show (D.4) - (D.7) inductively for $j \in \{0,\dots,m_n\}$. Note that $D^1_{0,n} \leq_{lo} E^1_n$ because $D^1 \leq_{lo} E^1$. If $D^1_{0,n} = E^1_n$, we have $m_n = 0$ and all the properties are trivially fulfilled. Otherwise, it follows from $D^1_{0,n} \leq_{lo} E^1_n$ by construction of the sequence $(D^1_{j,n})_{1\leq j\leq m_n}$ that $\ell_* < \ell^*$ which implies $\eta > 0$. Note also that $D^1_{j,n}$ is constructed from $D^1_{j-1,n}$ by adding positive mass η to each of the two grid points $(k/n, \ell_*/n)$ and $((k+1)/n, \ell^*/n)$, which lie in diagonal direction, and by subtracting positive mass η from each of the two grid points $(k/n, \ell^*/n)$ and $((k+1)/n, \ell_*/n)$, which lie in off-diagonal direction. This does not affect the uniform marginal property. Hence, each $D^1_{j,n}$ is a bivariate signed n-grid copula with (potentially negative) mass distributed on $\mathbb{G}^2_n$.

For the base case $j = 0$, properties (D.4), (D.5), and (D.7) are trivially fulfilled because D^1 and E^1 are bivariate CIS copulas with $D^1 \leq_{lo} E^1$. For the induction step, fix $j \in \{1,\dots,m_n\}$ and let k such that $D^1_{j-1,n}(k/n,\cdot) \neq D^1_{j,n}(k/n,\cdot)$ and $D^1_{j-1,n}(r/n,\cdot) = D^1_{j,n}(r/n,\cdot)$ for $1 \leq r \leq k-1$. Since $\ell_* < \ell^*$ it follows by construction of $D^1_{j,n}$ that

$$
\begin{aligned}
&\Delta^2_n D_{j-1,n}(\tfrac{k}{n}, t_*) < \Delta^2_n D_{j,n}(\tfrac{k}{n}, t_*)\,,\\
&\Delta^2_n D_{j-1,n}(\tfrac{k}{n}, t^*) > \Delta^2_n D_{j,n}(\tfrac{k}{n}, t^*)\,,\\
&\Delta^2_n D_{j-1,n}(\tfrac{k}{n}, t) = \Delta^2_n D_{j,n}(\tfrac{k}{n}, t), \quad t \in \mathbb{G}^1_n \setminus \{t_*, t^*\}\,,\\
&\Delta^2_n D_{j-1,n}(\tfrac{r}{n}, \cdot) = \Delta^2_n D_{j,n}(\tfrac{r}{n}, \cdot), \quad r \in \{1,\dots,n\} \setminus \{k\}\,.
\end{aligned}
$$

This implies (D.4). To show (D.6), the above equations yield

$$
D^1_{j-1,n}(u,t) = D^1_{j-1,n}(\tfrac{\lfloor nu\rfloor}{n}, \tfrac{\lfloor nt\rfloor}{n}) = \sum_{\ell=1}^{\lfloor nt\rfloor} \Delta^2_n D^1_{j-1,n}(\tfrac{\lfloor nu\rfloor}{n}, \tfrac{\ell}{n}) \leq \sum_{\ell=1}^{\lfloor nt\rfloor} \Delta^2_n D^1_{j,n}(\tfrac{\lfloor nu\rfloor}{n}, \tfrac{\ell}{n}) = D^1_{j,n}(\tfrac{\lfloor nu\rfloor}{n}, \tfrac{\lfloor nt\rfloor}{n}) = D^1_{j,n}(u,t)\,,
$$

for $u,t \in [0,1]$, using that $t_* < t^*$.

To show (D.5), we obtain for $\Delta^2_n D^1_{j,n}$ that

$$
\begin{aligned}
&0 < \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t_*) + \eta = \Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t_*) \leq \Delta^2_n E^1_n(\tfrac{k}{n}, t_*) \leq 1\,,\\
&0 \leq \Delta^2_n E^1_n(\tfrac{k}{n}, t^*) \leq \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t^*) - \eta = \Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t^*) < \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t^*) \leq 1\,,\\
&0 \leq \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t) = \Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t) \leq 1, \quad t \in \mathbb{G}^1_n \setminus \{t_*, t^*\}\,,
\end{aligned}
$$

using that $\eta > 0$, that $0 \leq \Delta^2_n E^1_n(k/n,t) \leq 1$ for all t and using the induction hypothesis $0 \leq \Delta^2_n D^2_{j-1,n}(k/n,t) \leq 1$, $t \in \mathbb{G}^1_n$. Then the statement follows with $\Delta^2_n D^1_{j-1,n}(r/n,\cdot) = \Delta^2_n D^1_{j,n}(r/n,\cdot)$ for all $r \in \{1,\dots,n\} \setminus \{k\}$.

To show (D.7), we need to prove that $\Delta^2_n D^1_{j,n}(u,t)$ is decreasing in t for all $u \in [0,1]$. Since $D^1_{j-1,n}$ and $D^1_{j,n}$ are signed n-grid copulas, it is sufficient to show this statement for $(u,t) \in \mathbb{G}^2_n$. Due to the induction hypothesis and the definition of $D^1_{j,n}$, it follows that $\Delta^2_n D^1_{j,n}(r/n,\cdot)$ is decreasing for all $r \in \{1,\dots,n\} \setminus \{k\}$. For $\Delta^2_n D_{j,n}(k/n,\cdot)$, we obtain that

$$
\begin{aligned}
\Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t_1) &= \Delta^2_n E^1_n(\tfrac{k}{n}, t_1) \geq \Delta^2_n E^1_n(\tfrac{k}{n}, t_*) \geq \Delta^2_n D^2_{j,n}(\tfrac{k}{n}, t_*)\\
&= \Delta^2_n D^2_{j-1,n}(\tfrac{k}{n}, t_*) + \eta > \Delta^2_n D^2_{j-1,n}(\tfrac{k}{n}, t_*) \geq \Delta^2_n D_{j-1,n}(\tfrac{k}{n}, t_2) = \Delta^2_n D_{j,n}(\tfrac{k}{n}, t_2)\,,
\end{aligned}
$$

$t_1 \in \{1/n, 2/n, \dots, t_* - 1/n\}$ and $t_2 \in \{t_* + 1/n, t_* + 2/n, \dots, t^* - 1/n\}$, where each equality holds by definition of $D^1_{j,n}$. The first inequality holds true because E^1 and thus E^1_n is CIS. The second inequality follows with the definition of $\eta > 0$. For the last inequality, we use the induction hypothesis that $D^1_{j-1,n}$ is CIS. Similarly, we obtain

$$
\begin{aligned}
\Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t_2) &= \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t_2) \geq \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t^*) > \Delta^2_n D^1_{j-1,n}(\tfrac{k}{n}, t^*) - \eta\\
&= \Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t^*) \geq \Delta^2_n E^1_n(\tfrac{k}{n}, t^*) \geq \Delta^2_n E^1_n(\tfrac{k}{n}, t_3) = \Delta^2_n D^1_{j,n}(\tfrac{k}{n}, t_3)\,,
\end{aligned}
$$

$t_2 \in \{t_* + 1/n, t_* + 2/n, \ldots, t^* - 1/n\}$ and $t_3 \in \{t^* + 1/n, t^* + 2/n, \ldots, 1\}$. This proves that $D^1_{j,n}$ is CIS and thus (D.7) holds true.

Next, we prove for the discretized $*$-products that

$$*_{\mathbf{B},G_n} D^i_{j-1,n} \leq_{lo} *_{\mathbf{B},G_n} D^i_{j,n}, \quad j \in \{1, \ldots, m_n\}. \tag{D.8}$$

Note that $D^i_{j-1,n} = D^i_{j,n}$ for all $i \in \{2, \ldots, d\}$. Since for all i and j the signed grid copula $D^i_{j,n}$ is CIS, it follows that $\partial_2^{G_n} D^i_{j,n}(u,t)$ is decreasing in t f.a. $u \in [0,1]$. Define for $i \in \{1, \ldots, d\}$ and $u_i \in [0,1]$ the functions $f_i, g_i \colon [0,1] \to [0,1]$ by $f_i(t) := \partial_2^{G_n} D^i_{j-1,n}(u_i, t)$ and $g_i(t) := \partial_2^{G_n} D^i_{j,n}(u_i, t)$. Similar to the proof of [4, Theorem 3.7], we obtain from $D^i_{j-1,n} \leq_{lo} D^i_{j,n}$ that

$$\int_0^v f_i(t)\,\mathrm{d}t = D^i_{j-1,n}(u_i, \iota^-_{G_n}(v)) + (v - \iota^-_{G_n}(v))\,\partial_2^{G_n} D^i_{j-1,n}(u_i, v) \leq D^i_{j,n}(u_i, \iota^-_{G_n}(v)) + (v - \iota^-_{G_n}(v))\,\partial_2^{G_n} D^i_{j,n}(u_i, v) = \int_0^v g_i(t)\,\mathrm{d}t$$

with equality if $v = 1$. This implies with the decreasingness of f_i and g_i that $f_i \prec_S g_i$. Since $(B^{G_n}_t)_{t\in[0,1]}$ defined by (A.5) is a family of componentwise convex copulas satisfying the submodularity Assumption 1 as $\mathbf{B}$ does, see [4, Lemma 3.6], it follows from the Ky Fan–Lorentz theorem (see [22, Theorem 1]) that

$$*_{\mathbf{B},G_n} D^i_{j-1,n}(u) = \int_0^1 B^{G_n}_t(f_1(t), \ldots, f_d(t))\,\mathrm{d}t \leq \int_0^1 B^{G_n}_t(g_1(t), \ldots, g_d(t))\,\mathrm{d}t = *_{\mathbf{B},G_n} D^i_{j,n}(u)$$

for $u = (u_1, \ldots, u_d)$, where the continuity assumption in the Ky Fan–Lorentz Theorem can be relaxed to piecewise continuity in t. Since the above inequality holds true for all $u \in [0,1]^d$ and for all $j \in \{1, \ldots, m_n\}$, we obtain (D.8). Since (D.3) - (D.6) and (D.8) are valid for all $n \in \mathbb{N}$, we obtain from Theorem 5 that $*_{\mathbf{B},G} D^i \leq_{sm} *_{\mathbf{B},G} E^i$. □

For the proof of Theorem 2, we need the following lemma which is a version of Skorohod's theorem; see, e.g., [10, Theorem 25.6].

Lemma 5. *Let $f_n, f \colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, be integrable functions such that $f_n(t) \to f(t)$ for Lebesgue-almost all t. Then also the decreasing rearrangements converge, i.e., $f_n^*(t) \to f^*(t)$ for Lebesgue-almost all t.*

Proof of Theorem 2: First, we consider the case where G is continuous. We show that

$$*_{\mathbf{B}} D^i \leq_{sm} *_{\mathbf{B}} D^i_{\uparrow}, \tag{D.9}$$

where $D^i_{\uparrow}$ is the uniquely determined CIS copula such that $D^i_{\uparrow} =_{\partial_2 S} D^i$, see [4, Proposition 3.17]. Since $D^i_{\uparrow}$ and E^i are CIS, we obtain from $D^i_{\uparrow} \leq_{\partial_2 S} E^i$ that $D^i_{\uparrow} \leq_{lo} E^i$; see [4, Lemma 3.16(ii)]. Then, Theorem 1 yields

$$*_{\mathbf{B}} D^i_{\uparrow} \leq_{sm} *_{\mathbf{B}} E^i. \tag{D.10}$$

Hence, (15) follows from (D.9) and (D.10) with the transitivity of the supermodular order.

To show (D.9), we may assume without loss of generality that G has compact support; see [4, Proposition 2.14]. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of distribution functions defined by $G_n(x) := \lceil nG(x) \rceil / n$ for $x \in \mathbb{R}$. Since G is continuous and has compact support, it holds that $Ran(G_n) = \{0, 1/n, 2/n, \ldots, 1\}$ and $\iota_{G_n}(t) \to \iota_G(t) = t$ for all $t \in [0,1]$.

For $n \in \mathbb{N}$, denote by $j = j(k,\ell) = 1, \ldots, n(n-1)/2$ a counting of all pairwise combinations of columns $(k,\ell)_{1\leq k<\ell\leq n}$ of an $(n \times n)$-matrix. Then, define for $i = 1, \ldots, d$ the n-grid copula $D^i_{0,n}$ as well as the n-grid copula sequences $(D^i_{j,n})_{1\leq j\leq m_n} = (D_{j(k,\ell),n})_{1\leq k<\ell\leq n}$, $m_n = n(n-1)/2$, by

$$D^i_{0,n} := \mathbb{G}_n(D^i), \qquad D^i_{j,n}(u,t) := \sum_{r=0}^{\lfloor nt \rfloor} \Delta_n^2 D^i_{j,n}(u, \tfrac{r}{n}), \quad (u,t) \in [0,1]^2,\ j = j(k,\ell) \in \{1, \ldots, m_n\},\ k < \ell.$$

$$\Delta_n^2 D^i_{j,n}(u,t) := \begin{cases} \Delta_n^2 D^i_{j-1,n}(u,t) & \text{for } t \in \mathbb{G}_n^1 \setminus \{\frac{k}{n}, \frac{\ell}{n}\}, \\ \max\{\Delta_n^2 D^i_{j-1,n}(u, \frac{k}{n}), \Delta_n^2 D^i_{j-1,n}(u, \frac{\ell}{n})\} & \text{for } t = \frac{k}{n}, \\ \min\{\Delta_n^2 D^i_{j-1,n}(u, \frac{k}{n}), \Delta_n^2 D^i_{j-1,n}(u, \frac{\ell}{n})\} & \text{for } t = \frac{\ell}{n}, \end{cases} \tag{D.11}$$

Note that $\Delta_n^2 D^i_{j,n}(u,0)=0$ by definition of the difference operator Δ_n^2 in (B.1). We show for $i\in\{1,\dots,d\}$ that

$$
\begin{aligned}
&\mathfrak{Ch}_n(D^i_{0,n}) \xrightarrow{\partial_2} D^i\,,\quad \mathfrak{Ch}_n(D^i_{m_n,n}) \xrightarrow{\partial_2} E^i\,, && n\to\infty\,, && \text{(D.12)}\\
&\Delta_n^2 D^i_{j,n}(u,t) = \Delta_n^2 D^i_{j-1,n}(u,t), && t\in\mathbb{G}_n^1\setminus\{\tfrac{k}{n},\tfrac{\ell}{n}\},\ u\in[0,1]\,,\ j=j(k,\ell),\ k<\ell\,, && \text{(D.13)}\\
&0\le \Delta_n^2 D^i_{j,n}(u,t)\le 1, && u,t\in[0,1],\ j\in\{0,\dots,m_n\}\,, && \text{(D.14)}\\
&D^i_{j-1,n}\le_{lo} D^i_{j,n}, && j\in\{1,\dots,m_n\} && \text{(D.15)}\\
&\mathfrak{Ch}_n(D^i_{j-1,n}) =_{\partial_2 S} \mathfrak{Ch}_n(D^i_j), && j\in\{1,\dots,m_n\} && \text{(D.16)}\\
&\mathfrak{Ch}_n(D^i_{m_n,n}) \text{ is CIS.} && && \text{(D.17)}
\end{aligned}
$$

To show (D.16), observe that, by construction of the grid copula $D^i_{j,n}$, for all rows of the associated mass matrix, the entries of the columns k and ℓ (where $k<\ell$) are rearranged into a decreasing order. Since rearrangements do not affect the Schur-order, (D.16) follows. After all such rearrangement with respect to all pairwise combinations of columns, the mass matrix of the resulting grid copula $D^i_{m_n,n}$ has decreasing entries in each row. Thus, the associated checkerboard copula is CIS, i.e., (D.17) follows.

To show (D.12), define for $u\in[0,1]$ and $n\in\mathbb{N}$ the functions $f_{n,i},g_{n,i},f_i,g_i\colon[0,1]\to[0,1]$ by

$$
\begin{aligned}
f_{n,i}(t) &:= \partial_2^{G_n}\mathfrak{Ch}_n(D^i_{0,n})(u,t) = \partial_2^{G_n} D^i_{0,n}(u,t)\,, && f_i(t):=\partial_2 D^i(u,t)\,,\\
g_{n,i}(t) &:= \partial_2^{G_n}\mathfrak{Ch}_n(D^i_{m_n,n})(u,t) = \partial_2^{G_n} D^i_{m_n,n}(u,t)\,, && g_i(t):=\partial_2 D^i_{\uparrow}(u,t)\,,
\end{aligned}
$$

whenever the derivative exists, and by 0 elsewise. From (D.16) and (D.17), we obtain that $g_{n,i}$ is the decreasing rearrangement of $f_{n,i}$, i.e., $g_{n,i}=f^*_{n,i}$ Lebesgue-almost surely. Since $D^i_{\uparrow}$ is the uniquely determined CIS copula such that $D^i_{\uparrow} =_{\partial_2 S} D^i$, it follows that g_i is the decreasing rearrangement of f_i, i.e., $g_i=f_i^*$. Then, Lemma 5 implies that $g_{n,i}\to g_i$ almost surely using that $f_{n,i}\to f_i$ almost surely. Since $g_{n,i}$, $n\in\mathbb{N}$, and g_i are bounded, it follows that $g_{n,i}\to g_i$ in L^1, which implies $\mathfrak{Ch}_n(D^i_{m_n,n}) \xrightarrow{\partial_2} D^i_{\uparrow}$. Note that by definition of $D^i_{0,n}$ it holds that $\mathfrak{Ch}_n(D^i_{0,n}) \xrightarrow{\partial_2} D^i$, see (C.5).

Properties (D.13), (D.14), and (D.15) follow immediately by construction of $\Delta_n^2 D^i_{j,n}$ in (D.11).

Now, let $u=(u_1,\dots,u_d)\in[0,1]^d$. Consider for $j=j(k,\ell)$ the union $J=((k-1)/n,k/n)\cup((\ell-1)/n,\ell/n)$ of intervals on which, for all $i\in\{1,\dots,d\}$, the values of $\partial_2^{G_n}D^i_{j-1,n}(u_i,t)$ are rearranged with respect to $t\in J$ in decreasing order for $j-1\mapsto j$ due to the construction in (D.11). Then we obtain from a version of the Lorentz theorem, see [37], that

$$
\int_J B_t^G\left((\partial_2^{G_n}D^i_{j-1,n}(u_i,t))_{1\le i\le d}\right)\mathrm{d}t \le \int_J B_t^G\left((\partial_2^{G_n}D^i_{j,n}(u_i,t))_{1\le i\le d}\right)\,, \tag{D.18}
$$

because, restricted on the interval J, it holds that $\partial_2^{G_n}D^i_{j,n}(u_i,\cdot)|_J$ is the decreasing rearrangement of $\partial_2^{G_n}D^i_{j-1,n}(u_i,\cdot)|_J$. Since for $J^c=[0,1]\setminus J$, the values of $\partial_2^{G_n}D^i_{j-1,n}(u_i,\cdot)|_{J^c}$ remain for $j-1\mapsto j$ essentially (with respect to the Lebesgue measure) unchanged, (D.18) implies for the $*$-product of grid copulas that

$$
*_{\mathbf{B},G_n}D^i_{j-1,n}(u) = \int_0^1 B_t^G\left((\partial_2^{G_n}D^i_{j-1,n}(u_i,t))_{1\le i\le d}\right)\mathrm{d}t \le \int_0^1 B_t^G\left((\partial_2^{G_n}D^i_{j,n}(u_i,t))_{1\le i\le d}\right)\mathrm{d}t = *_{\mathbf{B},G_n}D^i_{j,n}(u)\,.
$$

This applies for all $u\in[0,1]^d$ and $j=j(k,\ell)\in\{1,\dots,m_n\}$, which yields

$$
*_{\mathbf{B},G_n}D^i_{j-1,n}\le_{lo} *_{\mathbf{B},G_n}D^i_{j,n},\quad j\in\{1,\dots,m_n\}\,. \tag{D.19}
$$

Now, we obtain, for the case that G is continuous, from (D.12) - (D.15) and (D.19) by an application of Theorem 5 the supermodular comparison in (D.9).

In the case that G is discontinuous, consider the bivariate copulas $\tilde{D}^i$ and $\tilde{E}^i$ defined by

$$
\tilde{D}^i(u,t) := \begin{cases} D(u,t) & \text{if } \iota_G^-(t)=\iota_G(t)\,,\\ t\cdot\partial_2^G D(u,t) & \text{if } \iota_G^-(t)\ne\iota_G(t)\,,\end{cases}\,,\qquad \tilde{E}^i(u,t) := \begin{cases} E(u,t) & \text{if } \iota_G^-(t)=\iota_G(t)\,,\\ t\cdot\partial_2^G E(u,t) & \text{if } \iota_G^-(t)\ne\iota_G(t)\,,\end{cases}
$$

for $(u,t)\in[0,1]^2$. By an application of the chain rule, we obtain

$$\partial_2\tilde{D}^i(u,t)=\partial_2^G D^i(u,t) \text{ and } \partial_2\tilde{E}^i(u,t)=\partial_2^G E^i(u,t) \quad \text{for all } u\in[0,1] \text{ and for Lebesgue-almost all } t\in[0,1]\,. \tag{D.20}$$

Note that $\partial_2^G D(u,t)$ and $\partial_2^G E(u,t)$ are constant in t if $t\in(\iota_G^-(t),\iota_G(t))$; see [4, Lemma (xi) and (xii)]. Due to the assumption that $D^i\leq_{\partial_2 S,G}E^i$, we have that

$$\partial_2^G D^i(u_i,\cdot)\prec_S \partial_2^G E^i(u,\cdot),\quad u\in[0,1]\,. \tag{D.21}$$

Hence, (D.20) and (D.21) imply $\tilde{D}^i\leq_{\partial_2 S}\tilde{E}^i$. Since $\mathbf{B}=(B_t)_{t\in[0,1]}$ is a family of supermodular functions satisfying the submodularity assumption 1, also $\mathbf{B^G}:=(B_t^G)_{t\in[0,1]}$ does, see [4, Lemma 3.6], where B_t^G is defined by (A.5). Note that B_t^G is not continuous in t . However, applying an approximation argument to the conditional copulas due to [4, Theorem 2.23], we obtain from the first part of the proof that

$$\ast_{\mathbf{B}^G}\tilde{D}^i=\int_0^1 B_t^G\left(\partial_2\tilde{D}^1(\cdot,t),\dots,\partial_2\tilde{D}^d(\cdot,t)\right)\mathrm{d}t\leq_{sm}\int_0^1 B_t^G\left(\partial_2\tilde{E}^1(\cdot,t),\dots,\partial_2\tilde{E}^d(\cdot,t)\right)\mathrm{d}t=\ast_{\mathbf{B}^G}\tilde{E}^i\,.$$

This implies with (D.20) that

$$\ast_{\mathbf{B},G}D^i=\int_0^1 B_t^G\left(\partial_2^G D^1(\cdot,t),\dots,\partial_2^G D^d(\cdot,t)\right)\mathrm{d}t\leq_{sm}\int_0^1 B_t^G\left(\partial_2^G E^1(\cdot,t),\dots,\partial_2^G E^d(\cdot,t)\right)\mathrm{d}t=\ast_{\mathbf{B},G}E^i\,,$$

which concludes the proof. □

References

[1] J. Ansari, L. Rüschendorf, Ordering risk bounds in factor models, Dependence Modeling 6 (2018) 259–287.
[2] J. Ansari, L. Rüschendorf, Upper risk bounds in internal factor models with constrained specification sets, Prob. Uncert. Quant. Risk 5 (2020) 1–30.
[3] J. Ansari, L. Rüschendorf, Ordering results for elliptical distributions with applications to risk bounds, J. Multivariate Anal. 182 (2021) 14.
[4] J. Ansari, L. Rüschendorf, Sklar's theorem, copula products, and ordering results in factor models, Dependence Modeling 9 (2021) 267–306.
[5] O. Barndorff-Nielsen, J. Kent, M. Sorensen, Normal variance-mean mixtures and z distributions, Int. Stat. Rev. 50 (1982) 145–159.
[6] N. Bäuerle, Inequalities for stochastic models via supermodular orderings, Commun. Stat., Stochastic Models 13 (1997) 181–201.
[7] N. Bäuerle, A. Müller, Stochastic orders and risk measures: consistency and bounds., Insurance Math. Econom. 38 (2006) 132–148.
[8] T. Bedford, R. M. Cooke, Vines – a new graphical model for dependent random variables., Ann. Statist. 30 (2002) 1031–1068.
[9] C. Bernard, L. Rüschendorf, S. Vanduffel, R. Wang, Risk bounds for factor models., Finance Stoch. 21 (2017) 631–659.
[10] P. Billingsley, Probability and Measure, John Wiley & Sons Ltd., Chichester, 1995.
[11] C. Burgert, L. Rüschendorf, Consistent risk measures for portfolio vectors, Insurance Math. Econom. 38 (2006) 289–297.
[12] M. M. Carhart, On persistence in mutual fund performance, The Journal of finance 52 (1997) 57–82.
[13] G. Connor, R. A. Korajczyk, A test for the number of factors in an approximate factor model, The Journal of Finance 48 (1993) 1263–1291.
[14] C. Czado, Analyzing Dependent Data with Vine Copulas. A Practical Guide with R, volume 222 of Lecture Notes in Statist., Springer, Cham, 2019.
[15] W. F. Darsow, B. Nguyen, E. T. Olsen, Copulas and Markov processes., Illinois J. Math. 36 (1992) 600–642.
[16] F. Durante, E. P. Klement, J. J. Quesada-Molina, P. Sarkoci, Remarks on two product-like constructions for copulas., Kybernetika 43 (2007) 235–244.
[17] F. Durante, C. Sempi, Principles of Copula Theory, CRC Press, Boca Raton, FL, 2016.
[18] M. L. Eaton, A review of selected topics in multivariate probability inequalities, Ann. Statist. 10 (1982) 11–43.
[19] R. F. Engle, V. K. Ng, M. Rothschild, Asset pricing with a factor-arch covariance structure: Empirical estimates for treasury bills, J. Econometrics 45 (1990) 213–237.
[20] S. Fallat, S. Lauritzen, K. Sadeghi, C. Uhler, N. Wermuth, P. Zwiernik, Total positivity in Markov structures., Ann. Statist. 45 (2017) 1152–1184.
[21] E. F. Fama, K. R. French, Common risk factors in the returns on stocks and bonds, J. Financ. Econom. 33 (1993) 3–56.
[22] K. Fan, G. G. Lorentz, An integral inequality., Amer. Math. Monthly 61 (1954) 626–631.
[23] H. Föllmer, A. Schied, Stochastic Finance. An Introduction in Discrete Time., De Gruyter Textb., de Gruyter, Berlin, 4th revised edition edition, 2016.
[24] J. Glaz, B. M. Johnson, Probability inequalities for multivariate distributions with dependence structures, J. Amer. Statist. Assoc. 79 (1984) 436–440.
[25] M. B. Gordy, A comparative anatomy of credit risk models, Journal of Banking & Finance 24 (2000) 119–149.
[26] M. B. Gordy, A risk-factor model foundation for ratings-based bank capital rules, Journal of financial intermediation 12 (2003) 199–232.
[27] H. Joe, Multivariate Models and Dependence Concepts, volume 73 of Monogr. Stat. Appl. Probab., Chapman and Hall, London, 1997.

[28] E. Jouini, W. Schachermayer, N. Touzi, Law invariant risk measures have the Fatou property, in: Advances in Mathematical Economics. Vol. 9, Springer, Tokyo, 2006, pp. 49–71.

[29] S. Karlin, Y. Rinott, Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions, J. Multivariate Anal. 10 (1980) 467–498.

[30] S. Karlin, Y. Rinott, Classes of orderings of measures and related correlation inequalities. II. Multivariate reverse rule distributions, J. Multivariate Anal. 10 (1980) 499–516.

[31] E. P. Klement, A. Kolesárová, R. Mesiar, S. Saminger-Platz, Copula constructions using ultramodularity, in: Copulas and Dependence Models with Applications, Springer, 2017, pp. 135–156.

[32] E. P. Klement, M. Manzi, R. Mesiar, Ultramodularity and copulas, Rocky Mountain J. Math. 44 (2014) 189–202.

[33] P. Krupskii, H. Joe, Factor copula models for multivariate data., J. Multivariate Anal. 120 (2013) 85–101.

[34] D. Kurowicka, R. Cooke, Uncertainty Analysis with High Dimensional Dependence Modelling., Wiley Ser. Probab. Stat., John Wiley & Sons, Chichester, 2006.

[35] D. Lee, H. Joe, Multivariate extreme value copulas with factor and tree dependence structures, Extremes 21 (2018) 147–176.

[36] A. Lewbel, The rank of demand systems: theory and nonparametric estimation, Econometrica: Journal of the Econometric Society (1991) 711–730.

[37] G. G. Lorentz, An inequality for rearrangements., Amer. Math. Monthly 60 (1953) 176–179.

[38] A. W. Marshall, I. Olkin, B. C. Arnold, Inequalities: Theory of Majorization and Its Applications, Springer Ser. Stat., Springer, New York, 2nd edition edition, 2011.

[39] A. J. McNeil, R. Frey, P. Embrechts, Quantitative Risk Management. Concepts, Techniques and Tools, Princeton University Press, Princeton, NJ, revised edition edition, 2015.

[40] R. C. Merton, On the pricing of corporate debt: The risk structure of interest rates, The Journal of finance 29 (1974) 449–470.

[41] M. A. Meyer, B. H. Strulovici, The supermodular stochastic ordering, available at SSRN: https://ssrn.com/abstract=2271937 (2013).

[42] P. Mikusiński, M. D. Taylor, Some approximations of n-copulas., Metrika 72 (2010) 385–414.

[43] A. Müller, Duality theory and transfers for stochastic order relations., in: Stochastic Orders in Reliability and Risk., Springer, New York, 2013, pp. 41–57.

[44] A. Müller, M. Scarsini, Some remarks on the supermodular order, J. Multivariate Anal. 73 (2000) 107–119.

[45] A. Müller, M. Scarsini, Stochastic comparison of random vectors with a common copula., Math. Oper. Res. 26 (2001) 723–740.

[46] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks., Wiley, Chichester, 2002.

[47] R. B. Nelsen, An Introduction to Copulas. 2nd ed., Springer, New York, 2006.

[48] R. B. Nelsen, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, M. Úbeda-Flores, Quasi-copulas and signed measures, Fuzzy Sets and Systems 161 (2010) 2328–2336.

[49] L. Rüschendorf, Inequalities for the expectation of Δ-monotone functions, Z. Wahrscheinlichkeitstheor. Verw. Geb. 54 (1980) 341–349.

[50] L. Rüschendorf, Stochastically ordered distributions and monotonicity of the OC-function of sequential probability ratio tests., Math. Operationsforsch. Stat., Ser. Stat. 12 (1981) 327–338.

[51] L. Rüschendorf, Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios, Springer Ser. Oper. Res. Financ. Eng., Springer, Berlin, 2013.

[52] S. Saminger-Platz, A. Kolesárová, R. Mesiar, E. P. Klement, The key role of convexity in some copula constructions, Eur. J. Math. 6 (2020) 533–560.

[53] A. A. Santos, F. J. Nogales, E. Ruiz, Comparing univariate and multivariate models to forecast portfolio value-at-risk, J. Financ. Econom. 11 (2013) 400–441.

[54] M. Shaked, A concept of positive dependence for exchangeable random variables, Ann. Statist. 5 (1977) 505–515.

[55] M. Shaked, J. G. Shantikumar, Stochastic Orders, Springer Ser. Stat., Springer, New York, 2007.

[56] W. F. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk, The journal of finance 19 (1964) 425–442.

[57] C. Strothmann, H. Dette, K. F. Siburg, Rearranged dependence measures, arXiv preprint arXiv:2201.03329 (2022).

[58] A. H. Tchen, Inequalities for distributions with given marginals, Ann. Probab. 8 (1980) 814–827.

[59] Y. L. Tong, An ordering theorem for conditionally independent and identically distributed random variables, Ann. Statist. 5 (1977) 274–277.

[60] Y. L. Tong, Inequalities for a class of positively dependent random variables with a common marginal, Ann. Statist. 17 (1989) 429–435.

[61] Y. L. Tong, Probability Inequalities in Multivariate Distributions, Academic Press, 2014.

[62] W. Trutschnig, On a strong metric on the space of copulas and its induced dependence measure, J. Math. Anal. Appl. 384 (2011) 690–705.

[63] W. Trutschnig, J. F. Sánchez, Some results on shuffles of two-dimensional copulas, J. Statist. Plann. Inference 143 (2013) 251–260.