Markov projection of semimartingales - application to comparison results

Benedikt Köpfer[1,1,*], Ludger Rüschendorf[1]

Abstract

In this paper we derive generalizations of comparison results for semimartingales. Our results are based on Markov projections and on known comparison results for Markov processes. The first part of the paper is concerned with an alternative method for the construction of Markov projections of semimartingales. In comparison to the construction in Bentata and Cont [1] which is based on the solution of a well-posed martingale problem, we make essential use of pseudo-differential operators as investigated in Böttcher [5] and of fundamental solutions of related evolution problems. This approach allows to dismiss with some boundedness assumptions on the differential characteristics in the martingale approach. As consequence of the construction of Markov projections, comparison results for path-independent functions (European options) of semimartingales can be reduced to the well investigated problem of comparison of Markovian semimartingales. The Markov projection approach to comparison results does not require one of the semimartingales to be Markovian, which is a common assumption in literature. An idea of Brunick and Shreve [7] to mimick updated processes leads to a related reduction result to the Markovian case and thus to thecomparison of related generators. As consequence, a general comparison result is also obtained for path functions of semimartingales.

Keywords: Markov projection, pseudo-differential operators, ordering of semimartingales.
2010 MSC: Primary 60G44; secondary 60E15, 60J25

1. Introduction

To a given stochastic process $X = (X_t)_{0\le t}$, a Markov projection is defined as a Markov process Y such that Y has the same marginal distributions as X, i.e. $X_t \stackrel{d}{=} Y_t, \forall t$. For diffusion processes Markov projections have been introduced and determined in Gyöngy [15] and Krylov [26]. Several constructions of this type considering also the martingale property were given in Madan and Yor [27] and Hirsch et al. [17] using the notion of peacocks. Gyöngy's results were rediscovered in mathematical finance in Dupire [10] who studied European option prices in a model in which the risk-neutral dynamics of the price process satisfy a time dependent diffusion equation. Dupire showed under smoothness assumptions that it is possible to construct such a "local volatility model"

*Corresponding author
Email addresses: `benedikt.koepfer@gmx.de` (Benedikt Köpfer), `ruschen@stochastik.uni-freiburg.de` (Ludger Rüschendorf)

[1] A LGFG grant of the state Baden-Württemberg is gratefully acknowledged

which is consistent with a given set of European option prices. In Dupire [11] this construction is extended to stochastic volatility price models, essentially corresponding to Gyöngy's result.

A basic extension of these results is given in Brunick and Shreve [7] who remove the conditions on non-degeneracy and boundedness on the volatility of the underlying Itô process to be mimicked. They also allow to match the joint distribution at each fixed time of a class of non-anticipating functions of the Itô process including the maximum-to-date or the running average. As consequence of these results a variety of path-dependent derivatives have the same price w.r.t. the underlying Itô process and w.r.t. the mimicking process. Forde [13] derives a Fourier based method for computing the coefficients of the Markov mimicking process extending Dupire's formula for the local volatility models and also gives mimicking results in some cases not covered by Brunick and Shreve [7], as for the case where the process to be mimicked is the local time or the quadratic variation. Hambly et al. [16] derive an extension of the Dupire PDE to the local volatility of barrier options. The introduced partial integro-differential equation (PIDE) is used as an efficient calibration routine including barrier option prices.

Our paper is based on Bentata and Cont [1, 2]. In these papers for general semimartingales with regular characteristics, a Markov projection is given as the solution to a martingale problem for a suitable integro-differential operator. For the flow of the one-dimensional distributions, the authors derive a characterization by a partial integro-differential equation.

In our paper we give an alternative construction method for Markov projections which is based on the construction of Markov processes as fundamental solution to evolution problems as developed in Böttcher [5]. The right or left generators of the transition operators of a Markov process are given by pseudo-differential operators when the test functions are contained in the domain. These pseudo-differential operators are characterized by the related symbols, see Jacob [19], Schnurr [33] and Rüschendorf et al. [31]. In consequence, the transition operators have a representation in terms of the characteristic functions of the conditional increments.

We describe this approach to Markov processes in brief form in Section 2. From this starting point we obtain the construction of a Markov projection for general semimartingales in Section 3. Besides pseudo-differential operators, the proof makes also essential use of the Itô formula and a characterization of the flow of marginal distributions by an integral equation due to Bentata and Cont [1]. In comparison to the martingale approach our procedure allows to relax some boundedness assumptions on the local characteristics.

The construction of Markov projections allows to reduce comparison results for functions of semimartingales to the well studied case of Markov processes. The application to comparison results for path-independent functions is the content of Section 4.1. For some general comparison results for Markov processes we refer to Rüschendorf et al. [31] and Köpfer and Rüschendorf [25]. The comparison between semimartingales as in Bergenthum and Rüschendorf [3, 4] has so far been restricted to the case that one semimartingale is Markovian. In Section 4.2 we make use of the idea of Brunick and Shreve to include additional path dependent functions Z_t of the underlying semimartingale process X_t. Based on the mimicking of the combined process we obtain as consequence of known comparison results for Markov processes corresponding comparison results for path dependent functions of a semimartingale.

2. Generators of Markov processes and evolution systems

For a Markov process $X = (X_t)_{0\le t\le T}$, $T < \infty$, on some metric space $(E, \mathfrak{B})$, $\mathfrak{B} = \mathfrak{B}(E)$ the Borel σ-algebra, we denote by $(P_{s,t})_{0\le s\le t\le T}$ the Markov transition functions and by $(T_{s,t})_{0\le s\le t\le T}$

the transition operators on $L_b(E)$, the space of bounded measurable functions. The family of transition operators forms an evolution system on the Banach space $(L_b(E), \|\cdot\|_\infty)$. Generally a family of bounded linear operators $(T_{s,t})_{0\le s\le t<T}$ on a Banach space $\mathbb{B}$ is called an **evolution system (ES)** if for all $0 \le s \le t \le u$ holds

1. $T_{s,s} = \mathrm{id}$,
2. $T_{s,u} = T_{s,t}T_{t,u}$ (evolution property).

An ES is called strongly continuous if for all $f \in \mathbb{B}$, $(s,t) \mapsto T_{s,t}f$ is continuous. The right and left generators of an ES are defined by

$$A_s^+ := \lim_{h\downarrow 0} \frac{T_{s,s+h}f - f}{h},\ \ s \ge 0$$

and

$$A_s^- := \lim_{h\downarrow 0} \frac{T_{s-h,s}f - f}{h},\ \ s > 0.$$

These operators are defined on their domains $\mathcal{D}(A_s^+)$ and $\mathcal{D}(A_s^-)$, i.e. for all $f \in \mathbb{B}$ for which the limit exists in the norm of the Banach space. If this is weakened to a pointwise limit we call the such defined operators the extended pointwise right and left generators.

Evolution systems arise naturally as solutions to homogeneous evolution problems of the form

$$\begin{aligned} \frac{\partial^+}{\partial s}u(s) &= \ - A_s^+ u(s), \\ \lim_{s\uparrow t} u(s) &= f, \end{aligned} \tag{2.1}$$

i.e. for a strongly continuous ES with

$$u(s) = T_{s,t}f,\ \ f \in \mathcal{D}_+(t) := \{f \in \mathbb{B}; s \mapsto T_{s,t}f \text{ is right differentiable on } (0,t)\}$$

equation (2.1) holds and similary for the left generator (see Gulisashvili and van Casteren [14]). If for $f \in \mathcal{D}^{A^+}(s,t) := \{f \in \mathbb{B}; f \in \mathcal{D}(A_u^+),\ \text{for all } s \le u \le t\}$ the right derivative $\frac{\partial^+}{\partial u}T_{s,u}f$ is integrable on $[s,t]$ we obtain the following representation

$$T_{s,t}f - f = \int_s^t T_{s,u}A_u^+ f du.$$

Assuming in the sequel $E = \mathbb{R}^d$, the left and right generators are pseudo-differential operators if the space of test functions $C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact support, is contained in the domains. A detailed treatment of this topic is given in Hoh [18], Jacob [20] and Schnurr [33]; we make in the sequel in particular use of Böttcher [5, 6].

Pseudo-differential operators are defined on the Schwartz space $S(\mathbb{R}^d)$ (also called the space of rapidly decreasing functions) which consists of all functions $f \in C^\infty(\mathbb{R}^d)$ such that for all $m_1, m_2 \in \mathbb{N}_0$

$$p_{m_1,m_2}(f) := \sup_{x\in\mathbb{R}^d} \left((1+|x|^2)^{\frac{m_1}{2}} \sum_{|\alpha|\le m_2} |\partial^\alpha f(x)| \right) < \infty$$

where the $\alpha \in \mathbb{N}_0^d$ are multi-indices and ∂^α is the corresponding multiple derivation.

The following definition of pseudo-differential operators is from Schnurr [33]. An operator A on the Schwartz space $S(\mathbb{R}^d)$ is called *pseudo-differential operator* with symbol ψ if it has for $f \in S(\mathbb{R}^d)$ the following representation

$$Af(x) = -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} \psi(x,\xi) \hat{f}(\xi) d\xi, \tag{2.2}$$

where ψ is locally bounded in both variables, $\psi(\cdot, \xi)$ is a measurable map for every ξ and $\psi(x, \cdot)$ is a continuous negative definite function for every x. Here $\hat{f}$ denotes the Fourier transform of f.

By a classical theorem from Courrège (see Jacob [20]), a linear operator $A : C_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is a pseudo-differential operator if and only if A satisfies the positive maximum principle, i.e. if $f \in \mathcal{D}(A)$ possesses a positive supremum at x_0, $f(x_0) = \sup_{x \in \mathbb{R}^d} f(x) \geq 0$, then $Af(x_0) \leq 0$.

Generators of an evolution system corresponding to a Markov process fulfill the positive maximum principle. This follows from the definition of the generators and the inequality

$$T_{s,t} f(x_0) = E[f(X_t) | X_s = x_0] \leq E[f^+(X_t) | X_s = x_0] \leq \|f\|_\infty = f(x_0),$$

where x_0 is a maximum of f. So under the condition that $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A_t^+)$ for all t, right generators of the evolution system of a Markov process are a family of pseudo-differential operators. The same holds for the left generators if $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A_t^-)$ for all t. The symbol $\psi(x, \xi)$ is a negative definite function in ξ and hence possesses a representation of the form

$$\psi(x,\xi) = c + i(b(x), \xi) + \xi' a(x) \xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i(y,\xi)} + i(y,\xi) \mathbb{1}_{|y| \leq 1}(y)) \mu(x, dy), \tag{2.3}$$

where $b \in \mathbb{R}^d$, $a \in M^{d \times d}(\mathbb{R})$ is a positive semidefinite matrix, $c \geq 0$ and μ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$, called the Lévy measure which fulfills $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2) \mu(dx) < \infty$. Here a, b, c and μ are uniquely determined. Conversely, each a, b, c and μ as above define a continuous negative definite function. Equation (2.3) is called Lévy–Khinchin formula. Note that for time-inhomogeneous Markov processes the symbol of the generators is time dependent, i.e. of the form $\psi(t, x, \xi)$.

There is a close connection of the characteristic function to the transition operators and generators. For a time-homogeneous Feller process, i.e. the corresponding transition operators are strongly continuous and map $C_0(\mathbb{R}^d)$ into itself, the transition operators are pseudo-differential operators themselves with symbol

$$\psi(t, x, \xi) = E\left[e^{-i((X_t - x), \xi)} \,\middle|\, X_0 = x \right],$$

see Jacob [19]. This means the characteristic function of the conditional increments are the symbols of the transition semigroup. Consequently, the generator of the semigroup has the symbol

$$\tilde{\psi}(x, \xi) = \frac{\partial}{\partial t}\Big|_{t=0} \psi(t, x, \xi).$$

This connection is used in Schnurr [33] to define the so-called probabilistic symbol of a Feller process directly as right derivative of the characteristic function. This definition is extended to time-inhomogeneous Markov processes in Rüschendorf et al. [31]. The probabilistic symbol of a time-inhomogeneous Feller process is defined by

$$\psi(t, x, \xi) = -\lim_{h \downarrow 0} \frac{E\left[e^{i((X_{t+h} - x), \xi)} \,\middle|\, X_t = x \right] - 1}{h}.$$

To describe the connection of evolution systems associated to Markov processes with their characteristic functions we shall make use of the connection to Fourier transforms and in particular to the Fourier inversion formula. Therefore we apply the evolution systems to integrable Fourier transforms and thus to the Wiener algebra.

The Wiener algebra, following the definition in Jacob [20], is the set of functions in $L^1(\mathbb{R}^d)$ such that the Fourier transform is again in $L^1(\mathbb{R}^d)$,

$$A(\mathbb{R}^d) := \left\{ f \in L^1(\mathbb{R}^d); \hat{f} \in L^1(\mathbb{R}^d) \right\}.$$

For all $f \in A(\mathbb{R}^d)$ we have for almost all $x \in \mathbb{R}^d$ the inversion formula:

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} \hat{f}(\xi) d\xi.$$

The Wiener algebra is dense in $C_0(\mathbb{R}^d)$ and in $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$. This is due to the fact that $S(\mathbb{R}^d) \subset A(\mathbb{R}^d)$.

For notational convenience we denote from now on the characteristic function of the conditional increments by

$$\varphi(s,t,x,u) := E\left[e^{i((X_t - x),u)} \middle| X_s = x \right].$$

In the following theorem λ denotes the Lebesgue measure.

Theorem 2.1 (representation of transition operators)**.** *Let X be a Markov process with transition operators $(T_{s,t})_{s \le t}$ defined on a Banach function space $\mathbb{B}$, such that it holds that $P^{X_t|X_s=x} \ll \lambda$ for all $s,t \in [0,T]$ and $x \in \mathbb{R}^d$. Then for all $f \in \mathbb{B} \cap A(\mathbb{R}^d)$ the following representation holds:*

$$T_{s,t} f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u) \varphi(s,t,x,u) du.$$

Proof. Since the conditional expectation is assumed to be absolutely continuous with respect to the Lebesgue measure the Fourier inversion formula can be applied. Then by Fubini's theorem

$$\begin{aligned}
T_{s,t} f(x) &= \int_{\mathbb{R}^d} f(y) P^{X_t|X_s=x}(dy) \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(y,u)} \hat{f}(u) du P^{X_t|X_s=x}(dy) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u) \int_{\mathbb{R}^d} e^{i((y-x),u)} P^{X_t|X_s=x}(dy) du \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u) \varphi(s,t,x,u) du.
\end{aligned}$$

Note that it is allowed to use Fubini's theorem since $\hat{f} \in L^1(\mathbb{R}^d)$. $\square$

This representation theorem can be stated for more general laws by demanding that the functions under consideration possess everywhere a Fourier inversion. This is for example the case for integrable functions which are bounded and continuous, see Deitmar [9]. In particular, this holds for the Schwartz space, the space of rapidly decreasing functions. Recall that the Schwartz space $S(\mathbb{R}^d)$ is a subset of $C_0(\mathbb{R}^d)$ and of $L^1(\mathbb{R}^d)$. In addition the Fourier transform maps $S(\mathbb{R}^d)$ into itself. Hence, $S(\mathbb{R}^d)$ is an example where the Fourier inversion holds everywhere. However, we state the next theorem in more general form. Therefore, we denote by $A_{bc}(\mathbb{R}^d) = C_b(\mathbb{R}^d) \cap A(\mathbb{R}^d)$ the functions in the Wiener algebra which are bounded and continuous.

We obtain on $A_{bc}(\mathbb{R}^d)$ the following representation.

Theorem 2.2. *Let X be a Markov process with transition operators $(T_{s,t})_{s\leq t}$ on a Banach space $\mathbb{B}$. Then for all $f \in \mathbb{B} \cap A_{bc}(\mathbb{R}^d)$ the following representation holds:*

$$T_{s,t}f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u)\varphi(s,t,x,u)du.$$

Proof. Since the Fourier transform maps $L^1(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$, the Fourier transform of $f \in A_{bc}(\mathbb{R}^d)$ is bounded, continuous and integrable. Thus, by Deitmar [9, Theorem 3.4.4], it follows that the Fourier inversion formula holds everywhere and the assertion follows as in the previous Theorem 2.1. □

From Theorem 2.2 we can compute the generators of the transition operators if differentiation under the integral sign is allowed. The following theorem is inspired by Jacob [20, Example 4.8.26]. The computation of the right generators is straightforward. We begin with the space $\mathbb{B} \cap A(\mathbb{R}^d)$.

Theorem 2.3 (Representation of generators)**.** *Let X be a Markov process with transition operators $(T_{s,t})_{s\leq t}$ on a Banach space $\mathbb{B}$. Assume that $P^{X_t|X_s=x} \ll \lambda$ for all $s,t \in [0,T]$ and $x \in \mathbb{R}^d$. Further assume that $\hat{f}(u)\frac{\partial^+}{\partial t}\Big|_{t=s}\varphi(s,t,x,u)$ is integrable in u. Then for all $f \in A(\mathbb{R}^d) \cap \mathcal{D}(A_s^+)$ the generator A_s^+ is given by*

$$A_s^+ f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u) \frac{\partial^+}{\partial t}\Big|_{t=s} \varphi(s,t,x,u)du. \tag{2.4}$$

Proof. By Theorem 2.1, the difference quotient has the form

$$\frac{T_{s,s+h}f(x) - f(x)}{h} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,u)} \hat{f}(u) \frac{1}{h}(\varphi(s,s+h,x,u) - 1)du.$$

We restrict the integral to $|u| \leq c$ thus considering

$$\Phi_c := \int_{|u|\leq c} e^{i(x,u)} \hat{f}(u) \frac{1}{h}(\varphi(s,s+h,x,u) - 1)du.$$

We see by a similar argument as in the proof for the inversion formula for the characteristic function via Dirichlet integrals (see e.g. Rüschendorf [30, Theorem 4.2.18]) that the integral term due to the characeristic function is bounded by a connstant b. Hence the total integrand is bounded by $b\hat{f}$. By bounded convergence the representation (2.4) is achieved. □

In the same way we obtain a representation for functions in $A_{bc}(\mathbb{R}^d)$. Since the representation of the evolution system has the same form on the space $A_{bc}(\mathbb{R}^d)$, the proof is identical to the proof of Theorem 2.3.

Theorem 2.4. *Let X be a Markov process with transition operators $(T_{s,t})_{s\leq t}$ on a Banach space $\mathbb{B}$. Assume that $\hat{f}(u)\frac{\partial^+}{\partial t}\Big|_{t=s}\varphi(s,t,x,u)$ is integrable in u. Then for all $f \in A_{bc}(\mathbb{R}^d)\cap\mathcal{D}(A_s^+)$ it holds that*

$$A_s^+ f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,u)}\hat{f}(u)\frac{\partial^+}{\partial t}\Big|_{t=s}\varphi(s,t,x,u)du.$$

The same representation can be achieved also for left generators.

Corollary 2.5. *Let X be a Markov process with transition operators $(T_{s,t})_{s\leq t}$ on a Banach space $\mathbb{B}$. Assume that $P^{X_t|X_s=x} \ll \lambda$ for all $x\in\mathbb{R}^d$ and all $s<t$. Further assume that $\hat{f}(u)\frac{\partial^-}{\partial s}\Big|_{s=t}\varphi(s,t,x,u)$ is integrable in u. Then for all $f\in A(\mathbb{R}^d)\cap\mathcal{D}(A_t^-)$ it holds that*

$$A_t^- f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,u)}\hat{f}(u)\frac{\partial^-}{\partial s}\Big|_{s=t}\varphi(s,t,x,u)du.$$

Corollary 2.6. *Let X be a Markov process with transition operators $(T_{s,t})_{s\leq t}$ on a Banach space $\mathbb{B}$. Assume that $\hat{f}(u)\frac{\partial^-}{\partial s}\Big|_{s=t}\varphi(s,t,x,u)$ is integrable in u. Then for all $f \in A_{bc}(\mathbb{R}^d)\cap\mathcal{D}(A_t^-)$ it holds that*

$$A_t^- f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,u)}\hat{f}(u)\frac{\partial^-}{\partial s}\Big|_{s=t}\varphi(s,t,x,u)du.$$

3. Markov projection of semimartingales by the evolution approach

Based on the representation results for transition operators and generators in Section 2 we develop in this section a method of constructing a Markov projection of a semimartingale based on the use of the fundamental solution of a suitable evolution problem. This evolution method for the construction of Markov projections is an alternative to the approach in Bentata and Cont [1] based on an associated martingale problem.

In the first part of this section we summarize the construction of Markov processes by solving an associated evolution problem as given in Hoh [18] and Jacob [21] in the time homogeneous case and in Böttcher [5] and Rüschendorf et al. [31] in the time inhomgeneous case.

In the second part we recollect several properties of (conditional) characteristic functions of semimartingales as used for the representation of generators and transition operators of Markov processes in Section 2.

Finally in the third part we give the construction of Markov projections by the evolution method.

3.1. Construction of Markov processes by the evolution method

For a family of operators $(A_t)_{t\in[0,T]}$ on a Banach space $\mathbb{B}$ and $f\in\mathbb{B}$, a differentiable function $u:[0,t]\to\mathbb{B}$ is a solution of the *homogeneous evolution problem* on $[0,t]$, $t\leq T$ if $u(s)\in\mathcal{D}(A_s), \forall s\leq t$

and

$$\begin{aligned}\frac{\partial}{\partial s}u(s) &= \; - A_s u(s),\\ u(t) &= f.\end{aligned} \tag{3.1}$$

An operator valued function $(T_{s,t})_{s\leq t\leq T}$ is called *fundamental solution* to the homogeneous evolution problem if $u(s) = T_{s,t}f$ solves (3.1) and $\lim_{s\uparrow t} T_{s,t}f = f$.

The existence and uniqueness of fundamental solutions is well studied (see Friedman [12]. The symbolic approach in Böttcher [5] provides the important positivity preserving property which allows to associate a family of measures to the fundamental solution. Some regularity notions of the symbolic calculus are needed in the sequel taken from Hoh [18]. Pseudo-differential operators will be defined on anisotropic Sobolev spaces for a nice mapping behaviour. To that purpose symbol classes are introduced.

Definition 3.1. *A continuous negative definite function $\eta : \mathbb{R}^d \to \mathbb{R}$ is of* class Λ *if for all multi indices $\alpha \in \mathbb{N}_0^d$ there exist constants $c_\alpha \geq 0$ such that*

$$|\partial_x^\alpha (1+\eta(x))| \leq c_\alpha (1+\eta(x))^{\frac{2-(|\alpha|\wedge 2)}{2}}. \tag{3.2}$$

Let $m \in \mathbb{R}$, $j \in \{0,1,2\}$ and $\eta \in \Lambda$.
A function $q : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is in the symbol class $S_j^{\eta,m}$ *if for all $\alpha, \beta \in \mathbb{N}_0^d$ there exist constants $c_{\alpha,\beta} \geq 0$ such that*

$$\left|\partial_x^\alpha \partial_y^\beta q(t,x,y)\right| \leq c_{\alpha,\beta}(1+\eta(x))^{\frac{m-(|\alpha|\wedge j)}{2}}$$

holds for all $t \in [0,T]$ and $x, y \in \mathbb{R}^d$. The number $m \in \mathbb{R}$ is called the order of the symbol.

The corresponding pseudo-differential operators (see (2.2)) are defined on an anisotropic Sobolev space. Let $r \in \mathbb{R}$ and $\eta : \mathbb{R}^d \to \mathbb{R}$ a continuous negative definite function. In Böttcher [5], the anisotropic Sobolev space $H^{\eta,r}(\mathbb{R}^d)$ is defined by

$$H^{\eta,r}(\mathbb{R}^d) := \{f \in S'(\mathbb{R}^d); \|f\|_{\eta,r} < \infty\}$$

with the norm

$$\|f\|_{\eta,r} := \left\|(1+\eta(D))^{\frac{r}{2}} f\right\|_{L^2},$$

where $\eta(D)$ is the pseudo-differential operator associated to η and $S'(\mathbb{R}^d)$ is the dual space of the Schwartz space, the space of tempered distributions.

The space $C_c^\infty(\mathbb{R}^d)$ is a dense subset of $H^{\eta,r}(\mathbb{R}^d)$ for all $r \in \mathbb{R}$ and all continuous negative definite η, see Jacob [20]. On the other hand if there exist constants $c_0 > 0$ and $r_0 > 0$ such that for η it holds that

$$\eta(x) \geq c_0 |x|^{r_0} \tag{3.3}$$

for large x and if $r > \frac{d}{r_0}$, then $H^{\eta,r}(\mathbb{R}^d)$ is contained in $C_0(\mathbb{R}^d)$. We assume this for the rest of this section. Further, (3.3) implies the existence of a constant $c > 0$ such that for all $f \in H^{\eta,r}$

$$\|f\|_\infty \leq c\|f\|_{\eta,r}.$$

In Böttcher [5] a Markov process corresponding to a family of pseudo-differential operators is constructed by finding a fundamental solution to the homogeneous evolution problem on $H^{\eta,r}(\mathbb{R}^d)$. This requires some assumptions on the symbols ψ of the pseudo-differential operators.

Assumption 1. *A family of functions* $(\psi_t)_{t\in[0,T]}$ *with* $\psi_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ *fulfills Assumption* **1** *if*

1. $\psi(\cdot, x, y)$ *is continuous for all* $x, y \in \mathbb{R}^d$,
2. $\psi(t, x, \cdot)$ *is continuous and negative definite for all* $t \in [0, T]$ *and all* $x \in \mathbb{R}^d$,
3. $\lim_{y\to 0} \sup_{x\in\mathbb{R}^d} |\psi(t, x, y)| = 0$ *holds uniformly in* t *on compact sets,*
4. $\psi \in S_j^{\eta,m}$ *is elliptic, i.e. uniformly in* t *on compact sets it holds*

$$\exists R, c > 0\ \forall x,\ |y| \geq R: \ Re\psi(t, x, y) \geq c(1 + \eta(y))^{\frac{m}{2}}.$$

Under Assumption **1** a unique fundamental solution on $H^{\eta,r}(\mathbb{R}^d)$ can be constructed by the method of Levi–Mizohata. This fundamental solution forms a strongly continuous evolution system on $H^{\eta,r}(\mathbb{R}^d)$ which can be extended uniquely to $C_0(\mathbb{R}^d)$ since $H^{\eta,r}(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. This holds because $C_c^\infty(\mathbb{R}^d)$ is on the one hand a subset of $H^{\eta,r}(\mathbb{R}^d)$ and on the other hand is dense in $C_0(\mathbb{R}^d)$. Note that we consider the space $C_0(\mathbb{R}^d)$ with the sup norm $\|\cdot\|_\infty$. Böttcher [5] establishes the following existence result.

Theorem 3.2. *Let* $\eta \in \Lambda$ *fulfill inequality* (3.3) *and let* $m \leq 2$. *For a family of pseudo-differential operators* $(A_t)_{t\in[0,T]}$ *with symbols in* $S_2^{\eta,m}$, *satisfying Assumption* **1**, *there exists a unique fundamental solution to the associated evolution problem. This defines a strongly continuous evolution system* $(T_{s,t})_{s\leq t\leq T}$ *on* $C_0(\mathbb{R}^d)$ *such that for all* $s \leq t$, $T_{s,t}$ *is a contraction and is positivity preserving. Further it holds that* $T_{s,t}1 = 1$.

As consequence one obtains the construction of a Markov process. By a variant of Riesz's representation theorem for each operator $T_{s,t}$ there exists a unique Borel measure $P_{s,t}(x, dy)$ on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$ such that for all $f \in C_0(\mathbb{R}^d)$ it holds that

$$T_{s,t}f(x) = \int_{\mathbb{R}^d} f(y)P_{s,t}(x, dy),$$

see Jacob [20, Section 4.8 and Theorem 2.3.4]. By Theorem 3.2 these measures are probability measures and define a projective system. Thus Kolmogorov's extension theorem provides the existence of a stochastic process corresponding to these Borel measures. This process is a Markov process and in fact a Feller process, $(T_{s,t})_{s\leq t\leq T}$ are its transition operators on $C_0(\mathbb{R}^d)$.

Remark 3.3. 1. *Note that by this construction the operators* $(A_t)_{t\in[0,T]}$ *are not necessarily the right or left generators of* $(T_{s,t})_{s\leq t\leq T}$. *This requires the additional assumption that the family* $(A_t)_{t\in[0,T]}$ *is strongly continuous and each operator* A_t *is bounded, see Pazy [29]. The strong continuity of* $(A_t)_{t\in[0,T]}$ *is implied by the continuity of the symbols, since they are the only part of the representation as pseudo-differential operator which depends on time.*
An alternative possibility is to suppose that $\frac{\partial}{\partial s}T_{s,t}$ *is integrable on* $[0, t]$ *for all* $f \in C_0(\mathbb{R}^d)$. *This condition implies as in Section 2 an integral representation*

$$T_{s,t}f - f = \int_s^t A_u T_{u,t} f du,$$

which is continuous as function in both time variables since the evolution system is strongly continuous. We can differentiate this in s *from the left and evaluate it at* t *to obtain that the* $(A_t)_{t\in(0,T]}$ *are left generators. Analogously we find that they are right generators.*

2. *Any fundamental strongly continuous and positivity preserving solution of the homogeneous evolution problem provides a Markov process. We choose the conditions from Böttcher [5] on the symbol, because of the positivity preservation. For other conditions on the existence of a strongly continuous fundamental solution, see Pazy [29] or Friedman [12]. The issue whether the fundamental solution is positivity preserving is not treated there, but this is the key point to apply the variant of Riez's representation theorem. In the time-homogeneous case the Hille–Yosida theorem (see Pazy [29]) gives exact conditions for the existence and uniqueness of a solution for the evolution problem.*

If the family $(A_t)_{t\in[0,T]}$ is a generator for the constructed evolution system, all results for generators and strongly continuous evolution systems in Section 2 are applicable. In particular, $(T_{s,t})_{s\leq t}$ is a fundamental solution to the evolution problem

$$\frac{\partial}{\partial t}T_{s,t} = T_{s,t}A_t. \tag{3.4}$$

Every strongly continuous evolution system whose right and left generator coincide has the property that it is a fundamental solution of the evolution problem in both variables.

By denseness of $C_c(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, the fundamental solution on $C_c(\mathbb{R}^d)$ determines the representing measures uniquely. As consequence of the integral representation of evolution systems (see Köpfer and Rüschendorf [25]) uniqueness of a fundamental solution implies the following uniqueness result which is just the integral form of the evolution problem.

Theorem 3.4. *Assume that for a family of operators $(A_t)_{t\in[0,T]}$ there is a unique fundamental solution $(T_{s,t})_{s\leq t}$ to the evolution problem (3.4) on the anisotropic Sobolev space $H^{\eta,r}(\mathbb{R}^d)$. Assume further that this fundamental solution is a positivity preserving Feller evolution. If the derivative in t of the fundamental solution is integrable on $[0,T]$, then there is a unique family of probability measures $(P_{s,t}(x,dy))_{s\leq t\leq T, x\in\mathbb{R}^d}$ such that for all $f \in C_c^\infty(\mathbb{R}^d)$ it holds*

$$\begin{gathered} T_{s,t}f(x) = \int_{\mathbb{R}^d} f(y)P_{s,t}(x,dy) = f(x) + \int_s^t \int_{\mathbb{R}^d} A_u f(y) P_{s,u}(x,dy)du, \\ P_{s,s}(x,\cdot) = \delta_x, \end{gathered} \tag{3.5}$$

where δ_x is the point mass at x.

Proof. This is just the integral form of the evolution problem. The uniqueness follows from the uniqueness of the fundamental solution. □

Remark 3.5. *Under the conditions of Theorem 3.2 the fundamental solution is a Feller evolution system. If additionally it is unique, the derivative in t is integrable and the right and left generator coincide, then the marginals of the constructed process fulfill equation (3.5).*

A corresponding uniqueness result for the marginals is given in Bentata and Cont [1, Section 2.2]. This result can be transfered to the present context.

Theorem 3.6 (Uniqueness of the marginal flow)**.** *Under the conditions of Theorem 3.4 there exists a unique family of probability measures $(P_t(x))_{t\in[0,t]}$, given by $(P_{0,t}(x,dy))_{t\in[0,T],x\in\mathbb{R}^d}$, such that*

$$\begin{gathered} \int_{\mathbb{R}^d} f(y)P_t(x,dy) = f(x) + \int_0^t \int_{\mathbb{R}^d} A_s f(y) P_s(x,dy)ds, \\ P_0(x,\cdot) = \delta_x, \end{gathered} \tag{3.6}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$.

The proof follows as in Bentata and Cont [1, Theorem 2.1]. There it is shown that equation (3.6) determines the one-dimensional marginals of a solution to a well-posed martingale problem uniquely. The martingale property yields the integral representation, the uniqueness follows from the Feller property for the semigroup of the space-time process. Under the conditions here, the integral representation follows from the integrability of the derivative of the fundamental solution and the Feller property is assumed. From Böttcher [6] we conclude that the semigroup for the corresponding space-time process is a Feller semigroup.

3.2. **Characteristic functions of semimartingales**

As shown in Section 2, generators of a Markov process are under weak assumptions pseudo-differential operators and allow an integral representation on the Schwartz space $S(\mathbb{R}^d)$ involving the right or the left derivative of the conditional characteristic function. In the frame of Section 3.1 $A^+ = A^-$ and derivatives instead of semidifferentials apply. For the following facts on characteristic functions of semimartingales we refer to Jacod and Shiryaev [22, Chapter II.2].

With respect to a truncation function h, let (B, C, ν) the triplet of characteristics of a semimartingale X related to the canonical decomposition, i.e.

$$X = X(h) + \tilde{X}(h),$$

with jump part

$$\tilde{X}_t = \sum_{s \le t} \Delta X_s - h(\Delta X_s)$$

and

$$X(h) = X_0 + M(h) + B(h),$$

where $M(h) \in M_{loc}$, $B(h)$ is predictable of finite variation and hence $X(h)$ is a special semimartingale. There exists a predictable process $A = (A_t)_{t\in[0,T]}$ such that the characteristics of X are given as Lebesgue–Stieltjes integrals with respect to A of the form:

$$B^i = b^i \cdot A, C^{ij} = c^{ij} \cdot A, \nu(\omega, dt, dx) = dA_t(\omega) K_{\omega,t}(dx), \tag{3.7}$$

This is called a good version of the semimartingale characteristics and the triplet (b, c, K) is called the triplet of differential characteristics of X with respect to A. Define

$$A(u)_t := i(u, B_t) - \frac{1}{2} u' C_t u + \int (e^{i(u,x)} - 1 - i(u, h(x)))\nu([0, t] \times dx); \tag{3.8}$$

then $A(u)$ is a complex-valued predictable process of finite variation. For a good version of the characteristics with respect to some process A, this process can be written as integral $A(u) = a(u) \cdot A$, where

$$a(u) := i(u, b) - \frac{1}{2} u' c u + \int (e^{i(u,x)} - 1 - i(u, h(x)))K(dx).$$

Then the following connection of the process $A(u)$ and the semimartingale X holds (see Jacod and Shiryaev [22]).

Proposition 3.7. *A stochastic process X is a semimartingale with characteristics (B, C, ν) if and only if for each $u \in \mathbb{R}^d$ the process $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ is a complex valued local martingale.*

In order to obtain the connection to characteristic functions, the process $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ has to be a proper martingale, equivalently to be of class (DL). For the definition of class (DL) see Revuz and Yor [34]. The martingale property then provides a direct connection of the characteristics of a semimartingale to the characteristic function. We restrict us to the case that X has a good version of the characteristics with respect to a deterministic predictable process $A \in \mathscr{A}_{loc}^{+}$. Here $\mathscr{A}_{loc}^{+}$ is the class of locally integrable adapted increasing processes as in Jacod and Shiryaev [22]. Then we can apply Fubini's formula. Note that processes with independent increments have deterministic semimartingale characteristics. Hence, processes with independent increments are candidates for the following corollary.

Corollary 3.8. *Let X be a semimartingale with differential characteristics (b, c, K) with respect to a deterministic, predictable process $A \in \mathscr{A}_{loc}^{+}$. Further, assume that $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ is of class (DL) for all $u \in \mathbb{R}^d$. Then the characteristic function φ_{X_t} of X_t is given by*

$$\begin{aligned} \varphi_{X_t}(u) = E\left[e^{i(u,X_0)}\right] &+ \int_0^t \left(i(u, E\left[e^{i(u,X_{s-})} b_s\right]) - \frac{1}{2}(u, E\left[e^{i(u,X_{s-})} c_s\right]) u \right) dA_s \\ &+ \int_0^t E\left[e^{i(u,X_{s-})} \int (e^{i(u,x)} - 1 - i(u, h(x))) K(dx)\right] dA_s. \end{aligned} \tag{3.9}$$

Proof. Since $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ is of class (DL), it is a proper martingale. So we obtain by the martingale property and Fubini

$$\begin{aligned} \varphi_{X_t}(u) &= E\left[e^{i(u,X_t)}\right] \\ &= E\left[e^{i(u,X_0)}\right] + E\left[\int_0^t e^{i(u,X_{s-})} a(u)_s dA_s\right] \\ &= E\left[e^{i(u,X_0)}\right] + \int_0^t E\left[e^{i(u,X_{s-})} a(u)_s\right] dA_s \\ &= E\left[e^{i(u,X_0)}\right] + \int_0^t \Bigg(i(u, E\left[e^{i(u,X_{s-})} b_s\right]) - \frac{1}{2} u' E\left[e^{i(u,X_{s-})} c_s\right] u \\ &\quad + E\left[e^{i(u,X_{s-})} \int (e^{i(u,x)} - 1 - i(u, h(x))) K(dx)\right] \Bigg) dA_s \end{aligned}$$

completing the proof. □

Remark 3.9.

1. *In Schnurr [33, Chapter 4] the proper martingale property of the characteristic function is obtained for semimartingales which possess a good version of the characteristics with respect to the identity. There it is assumed that the differential characteristics are finely continuous, i.e. continuous w.r.t. the fine topology, and bounded. For locally bounded differential characteristics a stopping procedure is used.*

2. *The restriction to a deterministic integrator A is necessary to use Fubini's theorem. Examples for classes of processes with deterministic A are Itô processes and extended Grigelionis processes. For the notion of extended Grigelionis processes see Kallsen [23].*

The following theorem gives a representation of the characteristic function of the conditional increments $\varphi(s,t,x,u)$ analogously to Corollary 3.8.

Theorem 3.10. *Let X be a semimartingale with differential characteristics (b,c,K) with respect to a deterministic predictable process $A \in \mathscr{A}^+_{loc}$ and assume that $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ is of class (DL) for all $u \in \mathbb{R}^d$. Then the characteristic function of the conditional increments φ has the following form*

$$\begin{aligned}\varphi(s,t,x,u) = 1 &+ \int_s^t i(u, E\left[e^{i((X_{r-}-x),u)}b_r \middle| X_s = x\right])dA_r - \frac{1}{2}\int_s^t u' E\left[e^{i((X_{r-}-x),u)}c_r \middle| X_s = x\right] u dA_r \\ &+ \int_s^t E\left[\int_{\mathbb{R}^d} (e^{i((X_{r-}-x+y),u)} - e^{i((X_{r-}-x),u)} - e^{i((X_{r-}-x),u)} i(u,h(y))K_r(dy) \middle| X_s = x\right] dA_r.\end{aligned}$$

Proof. Again $e^{i(u,X)} - \left(e^{i(u,X_-)}\right) \cdot A(u)$ is a martingale. Thus, we obtain

$$\begin{aligned}E\left[e^{i(u,X_t)} - \left(\left(e^{i(u,X_-)}\right)\cdot A(u)\right)_t \middle| X_s\right] &= E\left[E\left[e^{i(u,X_t)} - \left(\left(e^{i(u,X_-)}\right)\cdot A(u)\right)_t \middle| \mathcal{F}_s\right] | X_s\right] \\ &= E\left[e^{i(u,X_s)} - \left(\left(e^{i(u,X_-)}\right)\cdot A(u)\right)_s \middle| X_s\right].\end{aligned}$$

After factorization we get

$$E\left[e^{i(u,X_t)} | X_s = x\right] = e^{i(u,x)} + \int_s^t E[e^{i(u,X_{r-})}a(u)_r | X_s = x] dA_r.$$

Multiplication with $e^{-i(u,x)}$ then yields the desired representation. ∎

3.3. Markov projection of semimartingales

In this section we construct a Markov projection of a semimartingale by the evolution approach. Throughout we consider a semimartingale X with characteristics (B,C,ν). For the sake of simplicity we assume that the characteristics of the semimartingale X have a version which is absolutely continuous, i.e. the integrator A in (3.7) is the identity. Also we assume that $X_0 = x_0$ almost surely for $x_0 \in \mathbb{R}^d$.

Our aim is to specify a suitable family of pseudo-differential operators on $H^{\eta,r}(\mathbb{R}^d)$ by appropriate symbols. To motivate the choice of the symbols assume that the conditions of Theorem 3.10 hold. Then φ has the form

$$\begin{aligned}\varphi(s,t,x,u) = 1 &+ \sum_{j\le d}\int_s^t iu^j E\left[e^{i((X_{r-}-x),u)}b_r^j \middle| X_s = x\right] dr \\ &- \frac{1}{2}\sum_{j,k\le d}\int_s^t u^j u^k E\left[e^{i((X_{r-}-x),u)}c_r^{j,k} \middle| X_s = x\right] dr \\ &+ \int_s^t E\left[\int_{\mathbb{R}^d}\left(e^{i((X_{r-}-x+y),u)} - e^{i((X_{r-}-x),u)} \right.\right. \\ &\qquad \left.\left. - \sum_{j\le d} iu^j e^{i((X_{r-}-x),u)} h(y)^j\right) K_r(dy) \middle| X_s = x\right] dr.\end{aligned}$$

Differentiation w.r.t. t and evaluation at s yields

$$\frac{\partial}{\partial t}\Big|_{t=s}\varphi(s,t,x,u) = \sum_{j\le d} iu^j E\left[e^{i((X_{s-}-x),u)}b_s^j\Big| X_s=x\right] - \frac{1}{2}\sum_{j,k\le d} u^j u^k E\left[e^{i((X_{s-}-x),u)}c_s^{j,k}\Big| X_s=x\right]$$
$$+ E\left[\int_{\mathbb{R}^d}\left(e^{i((X_{s-}-x+y),u)} - e^{i((X_{s-}-x),u)} - \sum_{j\le d} iu^j e^{i((X_{s-}-x),u)}h(y)^j\right)K_s(dy)\Bigg| X_s=x\right].$$

Note that since X is càdlàg, we have Lebesgue almost surely $X_s = X_{s-}$. Hence, we obtain Lebesgue almost surely

$$\frac{\partial}{\partial t}\Big|_{t=s}\varphi(s,t,x,u)(s) = \sum_{j\le d} iu^j E\left[b_s^j\big| X_s=x\right] - \frac{1}{2}\sum_{j,k\le d} u^j u^k E\left[c_s^{j,k}\big| X_s=x\right]$$
$$+ E\left[\int_{\mathbb{R}^d}\left(e^{i(y,u)} - 1 - \sum_{j\le d} iu^j h(y)^j\right)K_s(dy)\Bigg| X_s=x\right].$$

This equation is basically the Lévy–Khinchin formula. Based on Theorem 2.3 we exploit this relation to define the family of pseudo-differential operators which generates the Markov projection of X. This motivates to use the conditioned differential characteristics to define a family of negative definite functions, which in turn define a family of pseudo-differential operators. For the following theorem, the conditions of Theorem 3.10 are not needed anymore, they were just used to motivate the choice of symbols.

We use the canonical representation of the semimartingale X as given in Jacod and Shiryaev [22]

$$X = x_0 + B + X^c + h*(\mu^X - \nu) + (x - h(x))*\mu^X. \tag{3.10}$$

Here $*$ denotes the integral with respect to random measures as introduced in Jacod and Shiryaev [22]. Further, we assume that B_t^i and C_t^{ij} have finite expectation for all $t\in[0,T]$ and $i,j\le d$. For notational convenience we use for at least once differentiable functions f the following functions on $\mathbb{R}^d\times\mathbb{R}^d$

$$H_f(x,y) := f(x+y) - f(x) - \sum_{i\le d}\frac{\partial}{\partial x^i}f(x)y^i,$$

and

$$H_{fh}(x,y) := f(x+y) - f(x) - \sum_{i\le d}\frac{\partial}{\partial x^i}f(x)h(y)^i. \tag{3.11}$$

For the proof we need that the integrals with respect to the continuous martingale part and the compensated jump measure in Itô's formula are proper martingales. Pleas note that in the following Assumption the $*$ integral is with respect to the jump, i.e. th second variable of H_f and H_{fh}.

Assumption 2. *A semimartingale X fulfills Assumption 2 if for all $f\in C_c^\infty(\mathbb{R}^d)$:*

1. *The processes $H_{fh}*(\mu^X-\nu)$, $\int_{[0,t]\times\mathbb{R}^d}\sum_{j\le d}\frac{\partial}{\partial x^j}f(X_{s-})h(z)^j[\mu^X(ds,dz) - K_s(dz)ds]$ and $\sum_{j\le d}\frac{\partial}{\partial x^j}f(X_-)\cdot X^c$ are of class (DL);*

2. $|H_{fh}| * \mu^X \in \mathscr{A}^+_{loc}$.

3. *There exist measurable functions* $\tilde{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\tilde{c} : [0,T] \times \mathbb{R}^d \to M^{d\times d}(\mathbb{R})$ *and a Markov kernel* $\tilde{K} : [0,T] \times \mathbb{R}^d \to \mathcal{R}(\mathbb{R}^d)$ *such that for all* $t \in [0,T]$ *and all* $x \in \mathbb{R}^d$ *the differential characteristics fulfill*

$$E[b_t|X_t = x] = \tilde{b}(t,x),\ E[c_t|X_t = x] = \tilde{c}(t,x),\ E[K_t(B)|X_t = x] = \tilde{K}(t,x,B).$$

This assumption makes sure that the processes in 1. are proper martingales and that by 2. $H_{fh} * \mu^X$ can be compensated. By Theorem 2.6 in Neufeld and Nutz [28], functions as in 3 exist for càdlàg semimartingales.

Theorem 3.11 (Markov projection of a semimartingale)**.** *Let X be a semimartingale fulfilling Assumption 2. Define a family of symbols by*

$$\begin{aligned}\psi(t,x,u) = &-\sum_{j\leq d} iu^j\tilde{b}^j(t,x) + \frac{1}{2}\sum_{j,k\leq d} u^j u^k \tilde{c}^{jk}(t,x)\\ &- \int_{\mathbb{R}^d}\left(e^{i(y,u)} - 1 - \sum_{j\leq d} iu^j h(y)^j\right)\tilde{K}(t,x,dy)\end{aligned} \tag{3.12}$$

and assume that ψ induces a family of pseudo-differential operators $(A_t)_{t\in[0,T]}$ on $H^{\eta,r}(\mathbb{R}^d)$. Further, assume that the evolution problem (3.4) for these operators possesses a unique fundamental solution which is a positivity preserving Feller evolution system whose derivative in t is integrable on $[0,T]$.
Then the operators $(A_t)_{t\in[0,T]}$ generate a Markov process $\tilde{X}$ which is a Markov projection of X.

Proof. Since the fundamental solution is assumed to be a positivity preserving Feller evolution system, these pseudo-differential operators generate a Markov process $\tilde{X}$. It remains to show that it mimicks X, i.e. it has the same marginals.

By Theorem 3.4 the transition operators of $\tilde{X}$ are uniquely determined by the behaviour of $(A_t)_{t\in[0,T]}$ on $C_c^\infty(\mathbb{R}^d)$. Consider these operators for $f \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}A_t f(x) = &-\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,u)}\psi(t,x,u)\hat{f}(u)du\\ = &-\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,u)}\hat{f}(u)\left(-\sum_{j\leq d} iu^j\tilde{b}^j(t,x) + \frac{1}{2}\sum_{j,k\leq d} u^j u^k\tilde{c}^{j,k}(t,x)\right.\\ &\left.-\int_{\mathbb{R}^d}\left(e^{i(y,u)} - 1 - \sum_{j\leq d} iu^j h(y)^j\right)\tilde{K}(t,x,dy)\right)du.\end{aligned}$$

We determine the integrals separately.

For the first integral we have by the Fourier inversion formula

$$
\begin{aligned}
\frac{1}{(2\pi)^{\frac{d}{2}}}\sum_{j\leq d}\int_{\mathbb{R}^d}\tilde{b}^j(t,x)e^{i(x,u)}iu^j\hat{f}(u)du &= \frac{1}{(2\pi)^{\frac{d}{2}}}\sum_{j\leq d}\int_{\mathbb{R}^d}\tilde{b}^j(t,x)e^{i(x,u)}\widehat{\frac{\partial}{\partial u^j}f}(u)du \\
&= \sum_{j\leq d}\tilde{b}^j(t,x)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}e^{i(x,u)}\widehat{\frac{\partial}{\partial u^j}f}(u)du \\
&= \sum_{j\leq d}\tilde{b}^j(t,x)\frac{\partial}{\partial x^j}f(x).
\end{aligned}
$$

Similarly the second integral is given by

$$
-\frac{1}{2}\sum_{j,k\leq d}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}e^{i(x,u)}\hat{f}(u)u^ju^k\tilde{c}^{j,k}(t,x)du = \frac{1}{2}\sum_{j,k\leq d}\tilde{c}^{j,k}(t,x)\frac{\partial^2}{\partial x^j\partial x^k}f(x).
$$

Analogously the third integral is given by

$$
\begin{aligned}
&\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}e^{i(x,u)}\hat{f}(u)\int_{\mathbb{R}^d}\left(e^{i(y,u)}-1-\sum_{j\leq d}iu^jh(y)^j\right)\tilde{K}(t,x,dy)du \\
&\qquad = \int_{\mathbb{R}^d}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}e^{i(x,u)}\hat{f}(u)\left(e^{i(y,u)}-1-\sum_{j\leq d}iu^jh(y)^j\right)du\tilde{K}(t,x,dy) \\
&\qquad = \int_{\mathbb{R}^d}H_{fh}(x,y)\tilde{K}(t,x,dy).
\end{aligned}
$$

Now, we have by Theorem 3.6 that the marginals of $\tilde{X}$ fulfill on $C_0^\infty(\mathbb{R}^d)$ uniquely

$$
\begin{aligned}
\int_{\mathbb{R}^d}f(x)P_{0,t}(x_0,dx) &= f(x_0)+\int_0^t\int_{\mathbb{R}^d}A_uf(x)P_{0,u}(x_0,dx)du \\
&= f(x_0)+\int_0^t\int_{\mathbb{R}^d}\sum_{j\leq d}\tilde{b}^j(u,x)\frac{\partial}{\partial x^j}f(x)P_{0,u}(x_0,dx)du \\
&\quad +\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\sum_{j,k\leq d}\tilde{c}^{j,k}(u,x)\frac{\partial^2}{\partial x^j\partial x^k}f(x)P_{0,u}(x_0,dx)du \\
&\quad +\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}H_{fh}(x,y)\tilde{K}(u,x,dy)P_{0,u}(x_0,dx)du.
\end{aligned}
$$

On the other hand we obtain by Itô's formula and the canonical decomposition (3.10) of X

$$\begin{aligned} f(X_t) &= f(x_0) + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})dX_s^j + \frac{1}{2}\sum_{j,k\le d}\int_0^t \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}ds \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} H_f(X_{s-},y)\mu^X(ds,dy) \\ &= f(x_0) + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})b_s^j ds + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})dX_s^c \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \sum_{j\le d}\frac{\partial}{\partial x^j} f(X_{s-})h(y)^j[\mu^X(ds,dy) - K_s(dy)ds] \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \sum_{j\le d}\frac{\partial}{\partial x^j} f(X_{s-})(y^j - h(y)^j)\mu^X(ds,dy) \\ &\quad + \frac{1}{2}\sum_{j,k\le d}\int_0^t \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}ds + \int_{[0,t]\times\mathbb{R}^d} H_f(X_{s-},y)\mu^X(ds,dy). \end{aligned}$$

Combining the integrals with respect to the jump measure we get

$$\begin{aligned} f(X_t) &= f(x_0) + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})b_s^j ds + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})dX_s^c \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \sum_{j\le d}\frac{\partial}{\partial x^j} f(X_{s-})h(y)^j[\mu^X(ds,dy) - K_s(dy)ds] \\ &\quad + \frac{1}{2}\sum_{j,k\le d}\int_0^t \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}ds + \int_{[0,t]\times\mathbb{R}^d} H_{fh}(X_{s-},y)\mu^X(ds,dy). \end{aligned}$$

Taking the expectation which yields together with the assumptions above

$$\begin{aligned} E[f(X_t)] &= f(x_0) + \sum_{j\le d} E\left[\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})b_s^j ds\right] + \frac{1}{2}\sum_{j,k\le d} E\left[\int_0^t \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}ds\right] \\ &\quad + E\left[\int_{[0,t]\times\mathbb{R}^d} H_{fh}(X_{s-},y)K_s(dy)ds\right]. \end{aligned}$$

Since f and its derivatives are bounded and B and C are integrable, we can apply Fubini's theorem on the first two integrals. By assumption $H_{fh}*(\mu-\nu)$ is of class (DL) and thus $H_{fh}*\nu$ is integrable. We obtain by Fubini's theorem

$$\begin{aligned} E[f(X_t)] &= f(x_0) + \sum_{j\le d}\int_0^t E\left[\frac{\partial}{\partial x^j} f(X_{s-})b_s^j\right] ds + \frac{1}{2}\sum_{j,k\le d}\int_0^t E\left[\frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}\right] ds \\ &\quad + \int_0^t E\left[\int_{\mathbb{R}^d} H_{fh}(X_{s-},y)K_s(dy)\right] ds. \end{aligned}$$

Conditioning on X_{s-} yields

$$\begin{aligned}
E[f(X_t)] &= f(x_0) + \sum_{j\le d}\int_0^t E\left[\frac{\partial}{\partial x^j} f(X_{s-}) E[b_s^j | X_{s-}]\right] ds \\
&\quad + \frac{1}{2}\sum_{j,k\le d}\int_0^t E\left[\frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-}) E[c_s^{jk} | X_{s-}]\right] ds \\
&\quad + \int_0^t E\left[E\left[\int_{\mathbb{R}^d} H_{fh}(X_{s-}, y) K_s(dy) | X_{s-}\right]\right] ds \\
&= f(x_0) + \sum_{j\le d}\int_0^t E\left[\frac{\partial}{\partial x^j} f(X_{s-}) \tilde{b}^j(s, X_{s-})\right] ds \\
&\quad + \frac{1}{2}\sum_{j,k\le d}\int_0^t E\left[\frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-}) \tilde{c}^{jk}(s, X_{s-})\right] ds \\
&\quad + \int_0^t E\left[\int_{\mathbb{R}^d} H_{fh}(X_{s-}, y) \tilde{K}(s, X_{s-}, dy)\right] ds.
\end{aligned}$$

Since $X_- = X$ Lebesgue almost surely we obtain for the law P_t^X of X_t

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) p_t^X(dx) = f(x_0) &+ \int_0^t \int_{\mathbb{R}^d} \left(\sum_{j\le d} \frac{\partial}{\partial x^j} f(x) \tilde{b}^j(s,x) \right. \\
&\left. + \frac{1}{2}\sum_{j,k\le d} \frac{\partial^2}{\partial x^j \partial x^k} f(x) \tilde{c}^{ik}(s,x) \right) p_s^X(dx) ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H_{fh}(x,y) \tilde{K}(s,x,dy) p_s^X(dx) ds.
\end{aligned} \tag{3.13}$$

So the family $(p_t^X)_{t\in[0,T]}$ fulfills the same integral equation as the marginals of $\tilde{X}$. Since by Theorem 3.6 the probability measures are uniquely determined by this equation, the marginals of X and $\tilde{X}$ are the same. $\square$

We next construct a Markov projection based on an martingale problem in a similar way as in Bentata and Cont [1]. In comparison we replace the boundedness assumption on the coefficients in Bentata and Cont [1] by the weaker Assumption **2** as used in Theorem 3.11 in the evolution approach. For a family of linear operators $(A_t)_{t\in[0,T]}$ on $C_c^\infty(\mathbb{R}^d)$ a probability measure P_x on $D[0,T]$ is a solution to the martingale problem for (A_t) if $P_x(X_0 = x) = 1$ and

$$f(X_t) - f(x) - \int_0^t A_s(f(X_s)ds \tag{3.14}$$

is a martingale w.r.t. P_x on $D[0,T]$.

In the construction by a pseudo-differential operator we see by Itô's formula in (3.13) that for all $f \in C_c^\infty(\mathbb{R}^d)$ we have the representation

$$E[f(X_t)] = f(x_0) + E\left[\int_0^t \sum_{j\le d} \frac{\partial}{\partial x^j} f(X_{s-})\tilde{b}^j(s, X_{s-}) + \frac{1}{2}\sum_{j,k\le d} \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})\tilde{c}^{jk}(s, X_{s-})ds\right] + E\left[\int_0^t \int_{\mathbb{R}^d} H_{fh}(X_{s-}, y)\tilde{K}(s, X_{s-}, dy)ds\right].$$

This suggests to choose the operators for the martingale problem as

$$A_t f(x) := \sum_{j\le d} \frac{\partial}{\partial x^j} f(x)\tilde{b}^j(t,x) + \frac{1}{2}\sum_{j,k\le d} \frac{\partial^2}{\partial x^j \partial x^k} f(x)\tilde{c}^{jk}(t,x) + \int_{\mathbb{R}^d} H_{fh}(x,z)\tilde{K}(t,x,dz). \tag{3.15}$$

This particular choice provides a Markov projection.

Theorem 3.12. *Let X be a semimartingale fulfilling Assumption 2. Define a family of operators $(A_t)_{t\in[0,T]}$ on $C_c^\infty(\mathbb{R}^d)$ by equation (3.15) and assume that the martingale problem (3.14) for the family $(A_t)_{t\in[0,T]}$ is well-posed and that the associated evolution system is a Feller evolution system. Then the solution $\tilde{X}$ to the martingale problem is a Markov projection of X.*

Proof. The well-posedness provides the Markov property. By Bentata and Cont [1] we have that the one-dimensional marginals $(P_t)_{t\in[0,T]}$ fulfill the following equation on $C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)P_t(dx) &= f(x_0) + \int_0^t \int_{\mathbb{R}^d} A_s f(x)P_s(dx)ds \\ &= f(x_0) + \int_0^t \int_{\mathbb{R}^d} \left(\sum_{j\le d} \tilde{b}^j(s,x)\frac{\partial}{\partial x^j} f(x) + \frac{1}{2}\sum_{j,k\le d} \tilde{c}^{j,k}(s,x)\frac{\partial^2}{\partial x^j \partial x^k} f(x)\right) P_s(dx)ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_{fh}(x,y)\tilde{K}(s,x,dy)P_s(dx)ds.\end{aligned}$$

On the other hand we obtain by Itô's formula as in the proof of Theorem 3.11

$$\begin{aligned}f(X_t) &= f(x_0) + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})b_s^j ds + \sum_{j\le d}\int_0^t \frac{\partial}{\partial x^j} f(X_{s-})dX_s^c \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \sum_{j\le d} \frac{\partial}{\partial x^j} f(X_{s-})h(y)^j[\mu^X(ds,dy) - K_s(dy)ds] \\ &\quad + \frac{1}{2}\sum_{j,k\le d}\int_0^t \frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}ds + \int_{[0,t]\times\mathbb{R}^d} H_{fh}(X_{s-},y)\mu^X(ds,dy).\end{aligned}$$

We take expectation and apply Fubini which yields

$$\begin{aligned}E[f(X_t)] &= f(x_0) + \sum_{j\le d}\int_0^t E\left[\frac{\partial}{\partial x^j} f(X_{s-})b_s^j\right]ds + \frac{1}{2}\sum_{j,k\le d}\int_0^t E\left[\frac{\partial^2}{\partial x^j \partial x^k} f(X_{s-})c_s^{jk}\right]ds \\ &\quad + \int_0^t E\left[\int_{\mathbb{R}^d} H_{fh}(X_{s-},y)K_s(dy)\right]ds.\end{aligned}$$

Analog to Theorem 3.11 we obtain, after conditioning, for the law P_t^X of X_t

$$\int_{\mathbb{R}^d} f(x)p_t^X(dx) = f(x_0) + \int_0^t \int_{\mathbb{R}^d} \left(\sum_{j\le d} \frac{\partial}{\partial x^j} f(x)\tilde{b}^j(s,x) + \frac{1}{2}\sum_{j,k\le d} \frac{\partial^2}{\partial x^j \partial x^k} f(x)\tilde{c}^{ik}(s,x) \right) p_s^X(dx)ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H_{fh}(x,y)\tilde{K}(s,x,dy)p_s^X(dx)ds.$$

So the family $(p_t^X)_{t\in[0,T]}$ fulfills the same integral equation as the marginals of the constructed Markov process. By the uniqueness result in Bentata and Cont [1] for the solution of the integral equation, X and $\tilde{X}$ have the same marginals. □

4. Comparison of semimartingales

As consequence of the existence and construction of Markov projections it is possible to reduce the comparison of path-independent functions and of path-dependent functions of two semimartingales X, Y to the comparison of their Markov projections which is essentially based on a comparison of their generators. We call our approach to comparison results for semimartingales the "Markov projection comparison method". An important advantage of this approach, compared to the martingale comparison method used in literature, is that the Markov projection comparison method allows to dismiss with the assumption that one of the processes is Markovian. This assumption has been used in most of the literature on comparison theorems for semimartingales so far. We describe this method by a sequence of applications.

We make use of the following two basic comparison results for Markov processes from Köpfer and Rüschendorf [25].

Theorem 4.1. *Assume that $(T_{s,t}^X)_{s\le t}$ and $(T_{s,t}^Y)_{s\le t}$ are strongly continuous evolution systems on $\mathbb{B}$ and let $f \in \mathbb{B}$. If for fixed $t \in \mathbb{R}_+$ it holds that for all $s \le t$*

1. $T_{s,t}^X f \in \mathcal{D}(A_s^{Y+})$;
2. $r \mapsto T_{s,r}^Y (A_r^{X+} - A_r^{Y+}) T_{s,r}^X f$ *is integrable on* $[s,t]$;
3. $A_s^{X+} T_{s,t}^X f \le A_s^{Y+} T_{s,t}^X f$ *a.s.*.

Then it holds that

$$T_{s,t}^X f \le T_{s,t}^Y f \quad \text{a.s. for all } s \le t.$$

Theorem 4.2. *Let $(T_{s,t}^X)_{s\le t}$ and $(T_{s,t}^Y)_{s\le t}$ be strongly continuous on a Banach space $\mathbb{B}$. For $f \in \mathbb{B}$ and fixed $t \in \mathbb{R}_+$ assume that for all $s \le t$ the following holds*

1. $T_{\cdot,t}^X f \in \mathcal{D}_+(A^{X+}) \cap \mathcal{D}_+(A^{Y+})$;
2. $\frac{\partial^+}{\partial u} E[T_{u,t}^X f(X_u)|X_s]$ *and* $\frac{\partial^+}{\partial u} E[T_{u,t}^X f(Y_u)|Y_s]$ *are integrable on* $[0,t]$;
3. $\operatorname{supp}(P^{Y_s}) \subset \operatorname{supp}(P^{X_s})$;
4. $A_s^{X+} T_{s,t}^X f \ge A_s^{Y+} T_{s,t}^X f$ *a.s.*

Then it holds that

$$E[f(Y_t)] \le E[f(X_t)].$$

4.1. **Comparison of path-independent functions of semimartingales**

For the comparison of path-independent functions $Z_t = f(X_t)$ and $Z'_t = f(Y_t)$ of two semimartingales X, Y, we make use of the Markov projections $\tilde{X}$, $\tilde{Y}$ of these processes as given in Theorem 3.11.

Let (b, c, K) denote the semimartingale characteristics of X and (a, σ, L) the characteristics of Y. Furthermore let $\tilde{b}, \tilde{c}, \tilde{K}$ resp. $\tilde{a}, \tilde{\sigma}, \tilde{L}$ denote measurable versions of the conditional expectations $\tilde{b}(t, x) = E[b_t|X_t = x]$, $\tilde{c}(t, x) = E[c_t|X_t = x]$, $\tilde{K}(t, x, \cdot) = E[K_t(\cdot)|X_t = x]$, resp. $\tilde{a}(t, x) = E[a_t|Y_t = x]$, $\tilde{\sigma}(t, x) = E[\sigma|Y_t = x]$, $\tilde{L}(t, x, \cdot) = E[L_t(\cdot)|Y_t = x]$. Assuming that the corresponding families of symbols $\psi(t, x, u)$, $\tau(t, x, u)$ as defined in (3.12) induce families of pseudo-differential operators $(A_t)_{t\in[0,T]}$, $(B_t)_{t\in[0,T]}$ on $H^{\eta,r}(\mathbb{R}^d)$ such that the evolution problems (3.4) for these operators possess unique fundamental solutions which are positivity preserving Feller evolution systems and whose derivatives in t are integrable on $[0, T]$. Then by Theorem 3.11 the operators $(A_t)_{t\in[0,T]}$, $(B_t)_{t\in[0,T]}$ generate Markov projections $\tilde{X}$, $\tilde{Y}$ of X, Y. Assume that the transition operators $T^{\tilde{X}}$, $T^{\tilde{Y}}$ of $\tilde{X}$, $\tilde{Y}$ and the function f satisfy Assunptions 1., 2. of Theorem 4.1, then we get as immediate consequence of the Markov comparison result in Theorem 4.1 and of the existence theorem for Markov projections in Theorem 3.11 the following comparison result for path-independent functions of semimartingales.

Corollary 4.3 (Comparison of path-independent functions of semimartingales)**.** *Under the assumptions on $\tilde{X}$, $\tilde{Y}$ and f as stated above, the comparison condition on the generators*

$$A_s T^{\tilde{X}}_{s,t} f \le B_s T^{\tilde{X}}_{s,t} f \text{ for all } s \le t \le T$$

implies the comparison result

$$T^{\tilde{X}}_{s,t} f \le T^{\tilde{Y}}_{s,t} f \text{ a.s. for all } s \le t \le T,$$

in particular

$$E[f(X_t)|X_0 = x] \le E[f(Y_t)|Y_0 = x] \text{ a.s.}$$

Applications

Throughout we assume as in Section 3.3 that the integrator process A of the good version of the characteristic triplet is the idendity id.

1. In the first application, we pose some continuity and boundedness conditions on the characteristics. Let X be a semimartingale with differential characteristics (b, c, K). Assume that $\tilde{b}^X$ is bounded and $\tilde{c}^X$ is bounded, continuous on $[0, T] \times \mathbb{R}^d$ and everywhere invertible. Further, assume that for all $B \in \mathscr{B}(\mathbb{R}^d)$ the function

$$(t, y) \mapsto \int_B (\|z\|^2 \wedge 1) \tilde{K}_t(y, dz)$$

is bounded and continuous. Then by Jacod and Shiryaev [22, Corollary III.2.41] the martingale problem is well-posed. Also X^c is a square-integrable martingale, see Revuz and Yor [34, Proposition IV.1.23]. Since $f \in C^\infty_c(\mathbb{R}^d)$ is bounded with bounded derivatives, we obtain by Jacod and Shiryaev [22, Theorem I.4.40] that the stochastic integrals $\frac{\partial}{\partial x^i} f(X-) \cdot X^c$ are square-integrable martingales as well. In Bentata and Cont [1] the Feller property of the semigroup of the space-time process is shown using only continuity and boundedness. Thus, if we assume further that $\tilde{b}$ is continuous in time and space, we can apply Theorem 3.12 and obtain as result a Markov projection for X.

If Y is another semimartingale which fulfills the conditions above, it possesses a Markov projection as well. So under the conditions of Theorem 4.2, an ordering of generators leads to

$$E[f(Y_t)] \leq E[f(X_t)].$$

2. The setting from Bentata and Cont [1] allows to use the comparison theorems above. Consider a semimartingale X with differential characteristics (b, c, K). Denote by δ an adapted process with values in $M^{d\times n}(\mathbb{R})$ which has the property that $\delta_t'\delta_t = c_t$. Assume that b and δ are bounded almost surely and continuous on $[0,T]\times\mathbb{R}^d$. Further, assume that K is such that

$$\int_{\mathbb{R}^d} (\|y\|^2 \wedge 1) K_t(\cdot, dy)$$

is bounded for all $t \in [0,T]$ and continuous on $[0,T]\times\mathbb{R}^d$ and

$$\lim_{R\to\infty} \int_0^T K_t(\cdot, \{\|y\| \geq R\}) = 0 \quad a.s.$$

In addition local non-degeneracy is assumed. This means that either

$$\exists \varepsilon > 0, \forall t \in [0,T) \; : c_t \geq \varepsilon I \quad a.s.$$

or $\delta = 0$ and there exist constants $\beta \in (0,2), c_1, c_2 > 0$ and a family of positive measures $m^\beta(t,dy)$ such that for all $t\in[0,T]$

$$K_t(dy) = m^\beta(t,dy) + \mathbb{1}_{\{\|y\|\leq 1\}} \frac{c_1}{\|y\|^{d+\beta}} \quad a.s.,$$

$$\int_{\mathbb{R}^d} (\|y\|^\beta \wedge 1) m^\beta(t,dy) \leq c_2 \text{ and}$$

$$\lim_{\varepsilon\to 0} \int_{\|y\|\leq\varepsilon} \|y\|^\beta m^\beta(t,dy) = 0 \quad a.s.$$

Then the martingale problem is well posed (see Bentata and Cont [1]). Further, it is shown there that the corresponding semigroup for the homogenized process is a Feller semigroup. So the conditions for Theorem 3.12 are fulfilled and a Markov projection exists and is given by Theorem 3.12.

Let X and Y be semimartingales satisfying the assumptions above. Then there exist Markov projections $\tilde{X}$ and $\tilde{Y}$. Further, $(T^{\tilde{X}}_{s,t})_{s\leq t}$ and $(T^{\tilde{Y}}_{s,t})_{s\leq t}$ are strongly continuous evolution systems on $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$. Note that in the sequel we consider the family of generators as an operator on $C_0([0,T]\times\mathbb{R}^d)$. We fix t and a choose a function $f\in C_0(\mathbb{R}^d)$ such that $T^{\tilde{X}}_{\cdot,t} f \in \mathcal{D}_+(A^{\tilde{Y}+})$. If for $s \leq t$ the right derivatives $\frac{\partial^+}{\partial u} E[T^{\tilde{X}}_{u,t} f(\tilde{X}_u)|\tilde{X}_s]$ and $\frac{\partial^+}{\partial u} E[T^{\tilde{X}}_{u,t} f(\tilde{Y}_u)|\tilde{Y}_s]$ are integrable on $[0,t]$ and $\operatorname{supp}(P^{\tilde{Y}_s}) \subset \operatorname{supp}(P^{\tilde{X}_s})$, then an ordering of the generators

$$A^{\tilde{Y}+}_s T^{\tilde{X}}_{s,t} f \leq A^{\tilde{X}+}_s T^{\tilde{X}}_{s,t} f.$$

leads by Theorem 4.2 to

$$E[f(Y_t)] \leq E[f(X_t)].$$

A possible scenario where the generators are ordered is when $T^{\tilde{X}}_{s,t} f$ is increasing and directionally convex in s and the coefficients $\tilde{b}$ and $\tilde{c}$ are ordered. The differentiability of $T^{\tilde{X}}_{s,t} f$ holds for example if $\tilde{X}$ is a Lévy process (i.e. $\tilde{b}^X$ and $\tilde{c}^X$ are deterministic) and possesses a smooth Lebesgue density, see Cont and Tankov [8]. In addition sometimes the supports of Lévy processes are known, see Sato [32, Section 24]. So the condition $\operatorname{supp}(P^{\tilde{Y}_s}) \subset \operatorname{supp}(P^{\tilde{X}_s})$ can easily be obtained if we impose conditions such that the support of the Lévy process $\tilde{X}$ is the whole space. For example type C Lévy processes on $\mathbb{R}$ have the property that $\operatorname{supp}(P^{X_s}) = \mathbb{R}$ for all s. Type C means that $\tilde{c} \neq 0$ or $\int_{|x|\leq 1} |x| \tilde{K}(dx) = \infty$. As consequence our comparison result allows to compare general semimartingales to Lévy processes.

3. We give an application based on the specification of a symbol as in Theorem 3.11. Assume that X is a semimartingale with differential characteristics (b, c, K). We define a symbol as in (3.12) by

$$\begin{aligned}\psi(t,x,u) = & -\sum_{j\leq d} iu^j \tilde{b}^j(t,x) + \frac{1}{2}\sum_{j,k\leq d} u^j u^k \tilde{c}^{jk}(t,x) \\ & - \int_{\mathbb{R}^d} (e^{i(y,u)} - 1 - \sum_{j\leq d} iu^j h(y)^j) \tilde{K}(t,x,dy).\end{aligned}$$

   By Theorem 3.2 a fundamental solution to the evolution problem (3.1) exists if it is of class $S^{\eta,m}_2$ and satisfies Assumption **1**. Therefore, it needs to be infinitely often differentiable in x and u. The differentiability in x is a question of differentiability of $\tilde{b}, \tilde{c}$ and $\tilde{K}$. It holds for example if the conditional law is smooth in x and the characteristics (b, c, K) are bounded. The smoothness in u for the first two summands is clear. For the smoothness in the last integrand it suffices that if interchange of differentiation and integration is admitted. Further, if we assume that the bounding function in (3.2) is $x \mapsto |x|^2$, the left and right generator coincide (see Böttcher [6]).

   The continuity of ψ in t posed in Assumption **1** depends only on the continuity of the conditional laws. That ψ is continuous negative definite follows from the construction. Part 3. and 4. of Assumption **1** are assumed. Further, assume that the derivative in t of the evolution system is integrable. Then X possesses a Markov projection by Theorem 3.12.

4. Continuous semimartingales. Let X and Y be continuous semimartingales such that its differential semimartingale characteristics $(b^X, c^X, 0)$ and $(b^Y, c^Y, 0)$ are bounded. Then $\tilde{b}^X$, $\tilde{c}^X$, $\tilde{b}^Y$ and $\tilde{c}^Y$ are bounded as well. Under the additional condition that they are continuous on $[0,T] \times \mathbb{R}^d$, it follows by Example 1 that both possess a Markov projection $\tilde{X}$ and $\tilde{Y}$. The generator of $\tilde{X}$ is given by

$$A^{\tilde{X}}_t f(x) := \sum_{j\leq d} \tilde{b}^{Xj}(t,x) \frac{\partial}{\partial x^j} f(x) + \frac{1}{2} \sum_{j,k\leq d} \tilde{c}^{Xjk}(t,x) \frac{\partial^2}{\partial x^j \partial x^k} f(x).$$

   Note that by the boundedness and continuity of $\tilde{b}^X$ and $\tilde{c}^X$, $(A^{\tilde{X}}_t f)_{t\in[0,T]}$ is bounded and continuous as well. The generator of $\tilde{Y}$ has the same form.

Now fix $t \in [0,T]$ and assume that for all $s \leq t$ and for some $f \in C_0(\mathbb{R}^d)$, $T^{\tilde{X}}_{s,t} f \in C_0^2(\mathbb{R}^d)$. Then for all $s \leq t$,

$$T^{\tilde{X}}_{s,t} f \in \mathcal{D}(A^{\tilde{X}}_s) \cap \mathcal{D}(A^{\tilde{Y}}_s).$$

Further, by the continuity of the operators involved, we have that $r \mapsto T^{\tilde{Y}}_{s,r}(A^{\tilde{X}+}_r - A^{\tilde{Y}+}_r) T^{\tilde{X}}_{s,r} f$ is continuous and hence integrable on $[s,t]$. In consequence under the condition that

$$A^{\tilde{X}+}_s T^{\tilde{X}}_{s,t} f \leq A^{\tilde{Y}+}_s T^{\tilde{X}}_{s,t} f$$

we obtain by the comparison theorem for Markov processes in Theorem 4.1 that

$$E[f(X_t)] = E[f(\tilde{X}_t)] = T^{\tilde{X}}_{0,t} f \leq T^{\tilde{Y}}_{0,t} f = E[f(\tilde{Y}_t)] = E[f(Y_t)].$$

4.2. **Comparison of path-dependent functions of semimartingales**

For the comparison of path-dependent real-valued functions Z_t, Z'_t of underlying semimartingale processes X, Y we make use of the idea of Brunick and Shreve [7] and consider pairs of processes (X_t, Z_t) resp. (Y_t, Z'_t), where Z and Z' are from some class of non-anticipating functionals of X and Y. Typical examples considered in the diffusion case in the literature mentioned above are the integrated process $Z_t = Z_0 + \int_0^t X_s ds$, the maximum (or minimum) process $Z_t = \max(0, \max_{0 \leq s \leq t} X_s)$, the path to date process $Z_t = E[W_t | X^t]$, $X^t = (X_s)_{s \leq t}$ and W, X measurable w.r.t. a common filtration $(\mathcal{F}_t)$, the local time or the quadratic variation. We consider more generally real functions $f(X_t, Z_t)$ resp. $f(Y_t, Z'_t)$ to be compared.

Assume that (X, Z) and (Y, Z') are $\mathbb{R}^d$-valued semimartingales with characteristics (b, c, K) resp. (a, σ, L). Let $(\tilde{b}, \tilde{c}, \tilde{K})$ resp. $(\tilde{a}, \tilde{\sigma}, \tilde{L})$ denote measurable versions of $\tilde{b}(t,x,z) = E[b_t | X_t = x, Z_t = z]$, $\tilde{c}(t,x,z) = E[c_t | X_t = x, Z_t = z]$ and $\tilde{K}(t,x,z,\cdot) = E[K_t(\cdot) | X_t = x, Z_t = z]$ resp. of $\tilde{a}(t,x,z) = E[a_t | Y_t = x, Z'_t = z]$, $\tilde{\sigma}(t,x,z) = E[\sigma_t | Y_t = x, Z'_t = z]$ and $E\tilde{L}(t,x,z,\cdot) = [L_t(\cdot) | Y_t = x, Z'_t = z]$. Now as in Section 4.1 we assume that the corresponding families of symbols $\psi(t,x,z,u)$ and $\tau(t,x,z,u)$ as defined in (3.12) induce families of pseudo-differential operators $(A_t)_{t \in [0,T]}$ and $(B_t)_{t \in [0,T]}$ on $H^{\eta,r}(\mathbb{R}^d)$ such that the evolution problems (3.4) for those operators posses unique fundamental solutions which are positivity preserving Feller evolution systems with integrable derivatives on $[0,T]$. Then by Theorem 3.11 the families of operators $(A_t)_{t \in [0,T]}$ and $(B_t)_{t \in [0,T]}$ generate Markov projections $(\tilde{X}, \tilde{Z})$, $(\tilde{Y}, \tilde{Z}')$ of (X, Z), (Y, Z'). Assuming that the transition operators $T^{(\tilde{X},\tilde{Z})}_{s,t}$, $T^{(\tilde{Y},\tilde{Z}')}_{s,t}$ and f satisfy the assumptions of Theorem 4.1 we get as corollary the following general comparison result for path-dependent functions of semimartingales

Corollary 4.4 (Comparison of path-dependent functions)**.** *Under the assumptions on* $(\tilde{X}, \tilde{Z})$*,* $(\tilde{Y}, \tilde{Z}')$ *and* f *as stated above, the comparison condition on the generators*

$$A_s T^{(\tilde{X},\tilde{Z})}_{s,t} f \leq B_s T^{(\tilde{X},\tilde{Z})}_{s,t} f$$

for all $0 \leq s \leq t \leq T$ *implies the comparison result*

$$T^{(\tilde{X},\tilde{Z})}_{s,t} f \leq T^{(\tilde{Y},\tilde{Z}')}_{s,t} f \ \text{a.s. for all } s \leq t \leq T;$$

in particular

$$E[f(X_t, Z_t) | X_0 = x, Z_0 = z] \leq E[f(Y_t, Z'_t) | Y_0 = x, Z'_0 = z].$$

As mentioned before in particular for the case of underlying diffusion processes and for several path-dependent functions stochastic functional differential equations (SFDE) for the mimicking processes and for the computation of the conditional characteristics $\tilde{b}, \tilde{c}, \tilde{K}$ are given in the above mentioned literature. The case of comparison of path-dependent multivariate functions of semimartingales of the form $E[f(X_{t_1}, \ldots, X_{t_n})] \leq E[f(Y_{t_1}, \ldots, Y_{t_n}]$ has been dealt with in Köpfer [24] based on the method of filtration enlargement. Alternatively, this case can be treated by the results in Corollary 4.4 using the process $Z_t = (X_{t\wedge t_1}, \ldots, X_{t\wedge t_n})$.

References

[1] Bentata A. and Cont R. (2009). Mimicking the marginal distributions of a semimartingale. arXiv:0910.3992 [math.PR].

[2] Bentata A. and Cont R. (2015). Forward equations for option prices in semimartingale models. *Finance and Stochastics Vol. 19, No. 3)*, 617–651.

[3] Bergenthum J. and Rüschendorf L. (2006). Comparison of option prices in semimartingale models. *Finance and Stochastics 10*, 222–249.

[4] Bergenthum J., Rüschendorf L. (2007). Comparison of semimartingales and Lévy processes. *The Annals of Probability, 35*, 228–254.

[5] Böttcher B. (2008). Construction of time inhomogeneous Markov processes via evolution equations using pseudo-differential operators. *Journal of the London Mathematical society (2) 78*, 605–621.

[6] Böttcher B. (2014). Feller evolution systems: Generators and approximation. *Stochastics and Dynamics 14*, 15p.

[7] Brunick G., Shreve S. (2013). Mimicking an Itô process by a solution of a stochastic differential equation. *The Annals of Applied Probability Vol. 23, No 4.*, 1584–1628.

[8] Cont R., Tankov P. (2004). Financial Modelling with Jump Processes. *Chapman and Hall/CRC*.

[9] Deitmar, A. (2005). A First Course in Harmonic Analysis. *Springer*.

[10] Dupire, B. (1994). Pricing with a smile. *Risk* **7**, 18–20.

[11] Dupire, B. (1996). A unified theory of volatility. In: *Derivative Pricing: The Classic Collection (P. Carr ed.)* Risk Books, New York, 18–20.

[12] Friedman A. (2008). Partial Differential Equations. *Dover*.

[13] Forde M. (2014). On the Markovian pprojection in the Brunick–Shreve mimicking result. *Statistics and Probability Letters 85*, 98–105.

[14] Gulisashvili A., van Casteren J.A. (2006). Non-autonomous Kato Classes and Feynman–Kac Propagators. *World Scientific*.

[15] Gyöngy, I. (1986). Mimicking the one-dimensional marginal distributions of processes having an Itô differential. *Probability Theory and Related Fields 71*, 501–516.

[16] Hambly B., Mariapragassam M., Reisinger C. (2016). A forward equation for barrier options under the Brunick & Shreve Markovian projection. *Quantitative Finance Vo. 16, No.6*, 827–838.

[17] Hirsch F., Profeta C., Roynette B., Yor M. (2011). Peacocks and Associated Martingales, with Explicit Constructions. *Springer*.

[18] Hoh W. (1998). Pseudo differential operators generating Markov processes. *Habilitationsschrift, Universität Bielefeld*.

[19] Jacob N. (1998). Characteristic Functions and Symbols in the Theory of Feller Processes. *Potential Analysis 8*, 61–68.

[20] Jacob N. (2001). Pseudo Differential Operators and Markov Processes I. Fourier Analysis and Semigroups. *Imperial College Press*.

[21] Jacob N. (2005). Pseudo Differential Operators and Markov Processes III. Markov Processes and Applications. *Imperial College Press*.

[22] Jacod J., Shiryaev A. N. (2003). Limit Theorems for Stochastic Processes. *Springer*.

[23] Kallsen J. (1998). Semimartingale modelling in finance. *Phd-thesis, Albert-Ludwigs-Universität Freiburg*.

[24] Köpfer B. (2019). Comparison of stochastic processes by Markov projection and functional Itô calculus. *Phd-thesis, Albert-Ludwigs-Universität Freiburg*.

[25] Köpfer B., Rüschendorf L. (2021). Comparison of Markov processes by the martingale comparison method. *Journal of Applied Probability* , **58**, 164–176.

[26] Krylov N.V. (1985). Once more about connection between elliptic operators and Itô's stochastic equations. In: *Statistics and Control of Stoch. Proc., Optimiz. Software* , New York, 214–229.

[27] Madan D., Yor M. (2002). Making Markov martingales meet marginals: with explicit construction. *Bernoulli 8 (4)*, 509–536.

[28] Neufeld A., Nutz M. (2014). Measurability of semimartingale characteristics with respect to the probability law. *Stochastic Processes and their Applications Vol. 124, No. 11*, 3819–3845.

[29] Pazy A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equations. *Springer*.

[30] Rüschendorf L. (2016). Wahrscheinlichkeitstheorie. *Springer*.

[31] Rüschendorf L., Schnurr A., Wolf V. (2016). Comparison of time-inhomogeneous Markov processes. *Advances in Applied Probability 48*, 1015–1044.

[32] Sato K. (1999). Lévy Processes and Infinitely Divisible Distributions. *Cambridge University press*.

[33] Schnurr A. (2009). The symbol of a Markov semimartingale. *Phd Thesis*.

[34] Revuz D., Yor M. (1999). Continuous Martingales and Brownian Motion. *Springer* 3rd edition.