Approximative solutions of optimal stopping and selection problems

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Abstract

In this paper we review a series of developments over the last 15 years in which a general method for the approximative solution of finite discrete time optimal stopping and selection problems has been developed. This method also enables us to deal with multiple stopping and selection problems and to deal with stopping or selection problems for some classes of dependent sequences.

The basic assumption of this approach is that the sequence of normalized observations when embedded in the plane converges in distribution to a Poisson or to a cluster process. For various classes of examples the method leads to explicit or numerically accessible solutions.

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1 Introduction

Optimal stopping and best choice problems are a classical subject in probability theory whose theory is expounded in a series of by now classical text books, such as Chow, Robbins, and Siegmund (1971), Neveu (1975), Shiryaev (1978) or Ferguson (2007). The optimal stopping problem for a discrete time sequence $X_1, X_2, \ldots, X_n$, $n \leq \infty$ is to determine

$$V^n = \sup_{\tau \in \gamma^n} EX_\tau,$$

the optimal stopping value as well as optimal stopping times $\tau \in \gamma^n$. Classical examples of this problem are e. g., the optimal choice problem, the parking problem, the house selling problem or the (S_n/n)-problem with applications to deriving prophet (and other) inequalities.

Some basic methods for finding solutions are the martingale approach based on Snell's envelope, the Markovian approach based on super-harmonic dominating functions and in continuous time relations to free boundary problems resp. approximations using Brownian motions or extremal processes.

In particular, the method of backward induction for $n < \infty$ allows in principle to determine optimal stopping times and the optimal stopping value. Defining by backward induction the sequence,

$$W_n := X_n, \quad W_i := X_i \vee E(W_{i+1} \mid \mathcal{F}_i), \quad i = n-1, \ldots, 0,$$

in reverse time, then the optimal stopping time is given by

$$\tau^* = \min\{1 \le i \le n;\ \ X_i \ge W_i\}.$$

A problem however arises with the calculation of (W_i). In many cases, this prevents the use of backward induction in order to get explicit solutions.

In a series of papers it has been noticed in the literature that optimal stopping problems are easier to solve in a Poisson context. This was observed for the house selling problem by Karlin (1962), Elfving (1967) and Siegmund (1967). Finite intensity of the Poisson process enables us to treat such a problem as a stationary Markov case and to derive a differential equation for the optimal boundary. Flatau and Irle (1984) used embedding of max sequences in extremal processes to derive asymptotics and Bruss and Rogers (1991) as well as Saario and Sakaguchi (1992), Gnedin and Sakaguchi (1992) and Gnedin (1996) embedded optimal selection problems for the i. i. d. case in Poisson processes in the plane (with stationary intensity), and identified the asymptotic value of solutions with the value of optimal selection (stopping) in homogeneous Poisson processes. Also, approximations for optimal stopping for the S_n/n problem have been established by stopping Brownian motion.

In this paper we describe a general approximation method for the optimal stopping of sequences by the optimal stopping of a continuous time point process. The idea of developing the method of approximation using optimal stopping resp. selection in limiting point processes in the plane is founded on a basic paradigm used in asymptotic statistics. After a normalization $(X_{n,i})_{1 \le i \le n}$ of the underlying sequence as e. g. of the form $X_{n,i} = (X_i - b_n)/a_n$, the analysis is based on the following scheme:

$$\begin{array}{ccc} \text{statistical experiment} & \rightarrow & \text{limiting experiment} \\ \uparrow & & \downarrow \\ \text{asymptotic optimal procedure} & \leftarrow & \text{optimal statistical procedure} \end{array}$$

In the case of optimal stopping resp. selection problems this program amounts to the following 4 steps:

1) Convergence of the embedded point processes

$$N_n = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N = \begin{cases} \mathcal{P}(\mu), \\ \text{cluster process.} \end{cases}$$

This step determines the limiting experiment.

2) Solving the optimal stopping problem in the limit model.

3) Approximation of the stopping problem.
This step amounts to the convergence of the statistical experiment.

4) Construction of asymptotically optimal stopping times.

While in a series of papers in the literature point process convergence has been used as a tool to derive asymptotics of stopping time distributions and values, the aim of our method is to systematically use the analysis of the limit process to derive approximately optimal stopping times. Some examples in the literature in which this has been done by embedding techniques were mentioned before. This approach has been described in a series of papers in Kühne and Rüschendorf (1998–2003) and in Faller and Rüschendorf (2011–2013).

Particularly influential for developing this approach were some papers of Kennedy and Kertz (1990–1992) considering the stopping of an i. i. d. sequence $X_1, \dots, X_n$. The backward induction then amounts to

$$v_{n-1} = EX_n, \quad v_{n-k} = E(X_{n-k+1} \vee v_{n-k+1}), \quad k = 2, \dots, n-1,$$

with optimal stopping times

$$\tau_n^* = \begin{cases} \min\{k < n;\ \ X_k \geq v_k\} & \text{if } \neq \emptyset, \\ n & \text{else.} \end{cases}$$

Kennedy and Kertz obtained the asymptotics of v_n by approximating the corresponding functional of $X_1, \dots, X_n$ by the functional of the limiting Poisson process using point process convergence. However, they did not base their asymptotics on an analysis of the limiting Poisson process model. The challenge and innovation involved in our approach comes from the fact that the limit Poisson process or cluster process model typically has infinite intensity on a lower boundary curve, which was not the case considered in previous approaches.

In the following, we go through the four steps of our general scheme and then discuss various applications and extensions, such as e. g. to multiple stopping resp. best choice problems. A fascinating consequence of this approach is the possibility to obtain, under quite general conditions, explicit or numerically tractable approximations for optimal stopping/choice problems. Details for the following results, along with several further extensions and examples, are given in the papers Kühne and Rüschendorf resp. Faller and Rüschendorf as cited above, as well as in the dissertations of Kühne (1997) and Faller (2009).

2 Approximation of point processes

The basic assumption of the approach in this paper is convergence of the embedded point processes

$$N_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N = \begin{cases} \mathcal{P}(\mu), \\ \text{cluster process.} \end{cases} \tag{2.1}$$

There are several relevant constraints and classes of examples ensuring condition (2.1).

2.1 Extreme value distributions for i. i. d. sequences

The classical maximum central limit theorem of Fisher, Tippett and Gnedenko for i. i. d. sequences states that only three types of limiting distributions – the extreme value distributions – arise as limits of normalized maxima $M_n := \max_{i \leq n} X_i$, i. e. for some sequences

(a_n) and (b_n) one has

$$\frac{M_n - b_n}{a_n} \xrightarrow{\mathcal{D}} G, \quad G \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}, \tag{2.2}$$

where $F \in D(G)$, i. e., F is in the domain of G, and

$$\begin{aligned} \Lambda(x) &= e^{-e^{-x}}, \quad x \in \mathbb{R}, \\ \Phi_\alpha(x) &= \exp(-x^{-\alpha}), \quad x \geq 0, \alpha > 0, \text{ and} \\ \Psi_\alpha(x) &= \exp((-x)^\alpha), \quad x < 0, \alpha > 0 \end{aligned}$$

are the Gumbel, Fréchet, and Weibull distributions respectively. This convergence result is closely connected with convergence of the embedded point processes in (2.1) for $X_{n,i} = (X_i - b_n)/a_n$ to a Poisson point process $\mathcal{P}(\mu)$, where the normalizations a_n, b_n are from the max CLT in (2.2). More precisely (see Resnick (1987)), the following holds:

$$N_n \xrightarrow{\mathcal{D}} N \overset{d}{=} \mathcal{P}(\mu) \text{ with intensity measure } \mu = \lambda_{[0,1]} \otimes \nu, \tag{2.3}$$

where for

$$\begin{aligned} G &= \Lambda, & \nu(x, \infty] &= e^{-x}, & x &\in (-\infty, \infty], \\ G &= \Phi_\alpha, & \nu(x, \infty] &= x^{-\alpha}, & x &\in (0, \infty], \\ G &= \Psi_\alpha, & \nu[x, \infty] &= (-x)^\alpha, & x &\in (-\infty, 0]. \end{aligned}$$

As a result, the limiting Poisson process has infinite intensity at the lower boundary (see Figure 2.1).

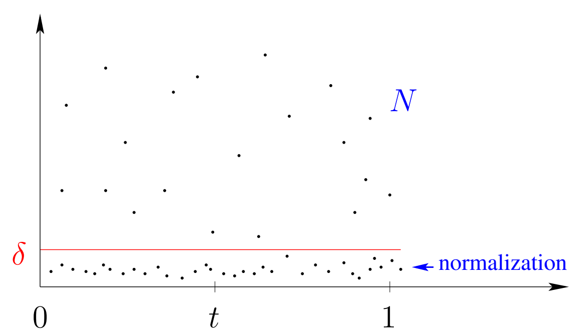

Figure 2.1 Infinite intensity at lower boundary

The limit result in (2.1) can be extended to several inhomogeneous cases as, such to the case $X_i = c_i Y_i + d_i$ with discounting and observation costs. Under some stabilization conditions on the costs c_i and d_i, such as e. g. $(c_{[nt]})/c_n \to \gamma_t$ and $d_n/(a_n c_n) \to b$ one obtains point process convergence as in (2.3) to a transformed Poisson process with modified intensity.

2.2 Moving average processes

There is a rich variety of convergence results for finite or infinite moving average processes (and thus also for autoregressive sequences)

$$X_n = \sum_{j=1}^{\infty} c_j Y_{n-j+1} \quad \text{resp.} \quad \sum_{j=1}^{k} c_j Y_{n-j+1}$$

for some i. i. d. sequence (Y_i) see Rootzén (1978), Davis and Resnick (1985, 1988, 1991)). We just state two special cases.

2.2.1) If, for example, $\sum |c_j|^\delta < \infty$ for some $0 < \delta < \alpha < 1$ and $c_j \geq 0$, and if $F \in D(\Phi_\alpha)$, then for some normalizations a_n

$$N_n = \sum_i \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}\right)} \xrightarrow{\mathcal{D}} N = \sum_k \sum_{i: c_i \neq 0} \varepsilon_{\tau_k, c_i \widetilde{Y}_k}. \tag{2.4}$$

N is a cluster process with deterministic clusters and underlying basic Poisson process $\widetilde{N} = \sum \varepsilon_{(\tau_k, \widetilde{Y}_k)}$.

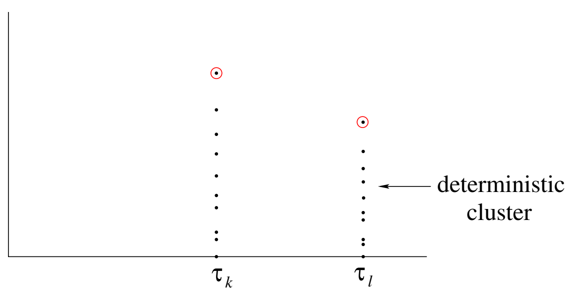

Figure 2.2

2.2.2) If $X_i = Y_i + Y_{i-1}$ and $F \in D(\Lambda) \cap S_\gamma(1)$, i. e. $(1 - F(x-y))/(1 - F(x)) \to e^{\gamma y}$, then

$$N_n \xrightarrow{\mathcal{D}} N = \sum_k \sum_{i=1}^{2} \varepsilon_{(\tau_k, \widetilde{Y}_k + Z_{k,i})}, \tag{2.5}$$

where $Z_{k,i} \sim F$ are independent. N is a cluster process with random clusters.

2.3 Stationary Markov chains

A point process convergence result of the form (2.3) for the case of stationary Markov chains is given in Perfekt (1994).

2.4 General dependent sequences

Under the assumption of asymptotic independence on the compensator, a general convergence result for embedded point processes to a Poisson process was given in Durrett and Resnick (1978) and Liptser and Shiryaev (2001).

Theorem 2.1 *If* $(X_{n,i}, \mathcal{A}_{n,i})_{1\le i\le n}$ *is a triangular array and* $\mu \in M([0,1]\times\mathbb{R})$ *satisfies* $\mu([0,1]\times\{x\}) = 0,\ \forall x$*, then:*

$$\sum_{i=1}^{\lceil nt\rceil} P(X_{n,i} > x \mid \mathcal{A}_{n,i-1}) \xrightarrow{P} \mu([0,t]\times[x,\infty)) \quad \textit{and}$$

$$\sup_n P(X_{n,i} > x \mid \mathcal{A}_{n,i-1}) \xrightarrow{P} 0, \quad \textit{imply:}$$

$$N_n = \sum \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N \sim \mathcal{P}(\mu).$$

3 Optimal stopping of Poisson processes

In this section we consider optimal stopping problems for Poisson processes which arise as the limits of embedded point processes. The main result is that this problem can be reduced to a differential equation of first order which can be solved in several classes of interesting examples. The solution in the case of Poisson processes can be extended to the class of cluster processes (see the corresponding literature as mentioned above).

We consider a Poisson point process $N = \sum_k \varepsilon_{(\tau_k, y_k)} \sim \mathcal{P}(\mu)$ on $M_f = \{(t,y) \in [0,1]\times\mathbb{R};\ f(t) < y\}$, where f is a decreasing function on $[0,1]$ and μ is a Radon measure on M_f. In the max stable case we have $f \equiv 0$ or $f \equiv -\infty$. So the Poisson process may have infinite intensity on the lower boundary as in the max stable case (2.3), see Figure 3.1.

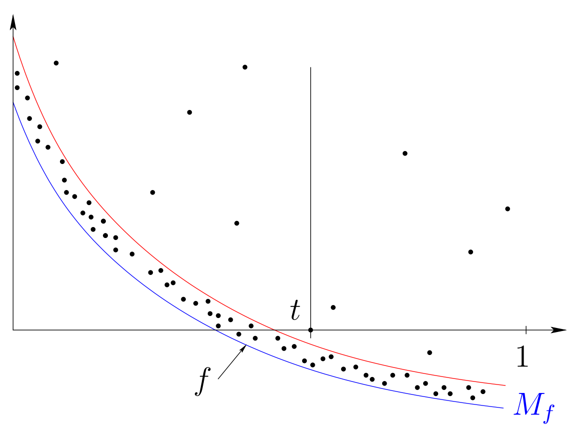

Figure 3.1 Point process with accumulation points at the lower boundary f

A stopping time τ for N is assumed to satisfy

— $\{\tau \le t\} \in \mathcal{A}_t = \sigma(N_s, s\le t) = \sigma(\{(\tau_i, y_i);\ \tau_i \le t\}), \quad \forall t \le 1,$

— $\bigcup_i \{\tau = \tau_i\} \cup \{\tau = 1\} = \Omega.$

We denote by $\overline{y}_\tau := \sup\{y_k;\ \tau_k = \tau\}$ the maximal payoff at τ and by K^τ the stopping index of τ. Let

$$u(t) = \sup\{E\overline{y}_{K^\tau};\ \tau \text{ a stopping time } \geq t\} \tag{3.1}$$

denote the optimal stopping curve. For $\tau = 1$ we define the guaranteed value $y_{K^\tau} = c \geq f(1)$ and without loss of generality assume $c = f(1)$. More generally, we define

$$u(t,x) = \sup\{E(\overline{y}_{K^\tau} \vee x);\ \tau \text{ a stopping time } \geq t\} \tag{3.2}$$

be the the optimal stopping value after time t with guaranteed value x.

We state three types of assumptions:

(S) Separation Condition
A curve v satisfies the separation condition if $v - f \geq c_t > 0$ on $[0,t]$ for all $t < 1$.

(D) Differentiability Condition
μ has a density on M_f, i. e. $\mu/M_f = h_f\, \lambda^2/M_f$ and

$$(t,z) \longrightarrow \int_z^\infty \int_x^\infty h_f(t,y)dy\, dx = \int_z^\infty G(t,x)dx$$

is continuous on M_f. The function G is called the intensity function.

(B) Boundedness condition $\quad E \sup_k \overline{y}_k < \infty$

The following convergence theorem for threshold stopping times is a basic tool in many of the following results.

Theorem 3.1 (Convergence of threshold stopping times)
Assume that

a) $N_n = \sum \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N = \sum \varepsilon_{(\tau_i, y_i)} \sim \mathcal{P}(\mu)$

b) $v_n \downarrow v$*, where* v *satisfies the separation condition (S)*

c) Condition (D) for the intensity measure μ

and define the threshold stopping times

$$T_n = \tau_n^{v_n} = \inf\{i \leq n;\ X_{n,i} \geq v_n(\tfrac{i}{n})\}, \quad T = \tau^v = \inf\{\tau_i;\ y_i \geq v(\tau_i)\}.$$

Then we have joint convergence of the stopping times and stopping values, i. e.

$$\left(\frac{T_n}{n}, X_{n,T_n}\right) \xrightarrow{\mathcal{D}} (T, y_{K^T}).$$

The proof of this theorem makes ample use of the Skorohod Theorem and the continuous mapping theorem. The following result characterizes optimal stopping values and optimal stopping times using a first order differential equation.

Theorem 3.2 (Optimal stopping of a Poisson process)
Let $N \sim \mathcal{P}(\mu)$, and assume that (B) and (D) hold and $c = f(1)$.

a) *If u satisfies condition (S), then $T = \tau^u$ is an optimal stopping time. If $\mu(\{(t,y); y = u(t)\}) = 0$, then uniqueness holds.*

b) *If u satisfies condition (S), then u is a solution of the following first order differential equation*
$$\begin{cases} u'(t) &= -\int_{u(t)}^{\infty}\int_x^{\infty} h_f(t,y)dy\,dx, \quad t<1, \\ u(1) &= c = f(1). \end{cases} \tag{3.3}$$

c) *If $c > -\infty$, if v satisfies condition (S) and is a solution of (3.3), then v is an optimal stopping curve.*

d) *$c = -\infty$. If (3.3) has a unique solution v, $v(t) > -\infty$, $\forall t < 1$, then v is a solution of the optimal stopping problem.*

e) *If $c = -\infty$ and v solves (3.3), then $v(t) \leq E\overline{y}_{\tau^u(t)}$, i. e. the optimal stopping curve is the largest solution of (3.3).*

Proof: The idea of the proof is as follows:

a) For a dyadic decomposition, define $Z_{n,i} := M_{\frac{i-1}{2^n},\frac{i}{2^n}}$, to be the maximum value in the dyadic interval $((i-1)/(2^n), i/(2^n)]$, then

1) The stopping problem for $(Z_{n,i})$ majorizes the stopping problem for N, $\forall n$.
2) Let w_n denote the optimal stopping boundary for $(Z_{n,i})$ extended as a curve on $[0,1]$. Then $w_n \downarrow w \geq u$.
3) We establish point process convergence of the point process $N_n := \sum \varepsilon_{(\frac{i}{n}, Z_{n,i})}$ to N, i. e.
$$N_n := \sum \varepsilon_{(\frac{i}{n}, Z_{n,i})} \xrightarrow{\mathcal{D}} N.$$
With $\widetilde{T}_n := \tau^{w_n}$, $\widetilde{T} = \tau^w$ we, therefore, obtain from the threshold convergence theorem
$$\Big(\frac{\widetilde{T}_n}{n}, Z_{n,\widetilde{T}_n}\Big) \xrightarrow{\mathcal{D}} (\widetilde{T}, y_{K^{\widetilde{T}}}).$$

This implies with the help of Fatou's theorem.
$$\limsup EZ_{n,\widetilde{T}_n} \leq Ey_{K^{\widetilde{T}}} \leq u(0).$$
But on the other hand from the majorization property we get:
$$EZ_{n,\widetilde{T}_n} = w_n(0) \geq u(0) \text{ and thus } w(0) = u(0).$$
This argument can be extended to show that $w(t) = u(t)$ for all t and thus $\widetilde{T} = T$ is an optimal stopping time.

If T is an optimal stopping time for N, then we find by an improvement argument that T is identical to the threshold stopping time τ^u with the *optimal stopping curve* u as its threshold function. This step also justifies calling u the optimal stopping curve.

b) From the definition of τ^u, the following holds

$$P(T \geq t) = P(\tau^u \geq t) = e^{-\mu_u([0,t]\times\mathbb{R})}.$$

This implies using partial integration and the intensity of the Poisson process

$$\begin{aligned} u(s) &= \int y_{K^{T/T\geq s}} dP = \int_s^1 E(y_{K^T} \mid T = t) dP^T(t) \\ &= \int_s^1 (u(t) + \int_{u(t)}^\infty K([y,\infty), t) dy) \frac{d\mu^1}{d\lambda}(t) e^{-\mu^1([s,t])} dt, \text{where} \end{aligned}$$

$K([x,\infty), t) := \dfrac{\int_{x\vee u(t)}^\infty \frac{d\mu}{d\lambda^2}(t,y)dy}{\int_{u(t)}^\infty \frac{d\mu}{d\lambda^2}(t,y)dy}$. This implies the differential equation in (3.3).

c), d) One gets uniqueness of the solution of (3.3) for a modification of μ satisfying the separation condition and then one uses a comparison argument.

e) This point needs a more involved argument. For the details, we refer the reader to Faller and Rüschendorf (2011a). □

Solution of the differential equation (3.3) In various cases one obtains explicit or numerically tractable solutions of equation (3.3) for the optimal boundary curve.

a) If the intensity function $G(t, y) = a(t)H(y)$ is a function in separate variables then equation (3.3) is of the form

$$\varphi'(t) = f(t)g(\varphi(t)), \quad \text{where } \varphi(1) = y_0.$$

This is a differential equation in separate variables. From classical theory of differential equations, for $g(y) \neq 0$, this equation has a unique solution φ, which is characterized as the unique solution of

$$G(\varphi(t)) = F(t), \tag{3.4}$$

where $G(y) := \int_{y_0}^y 1/(g(s))ds$ and $F(x) := \int_1^x f(t)dt$. In many cases of interest equation (3.4) can be solved explicitly or numerically.

b) In several cases of optimal stopping with discounting and observation costs, when the limit process is a transformed Poisson process, this leads to equations of the form

$$\begin{aligned} &G(t,y) = H\Big(\frac{y}{v(t)}\Big)\frac{v'(t)}{v(t)} \\ \text{and} \quad &G(t,y) = H(y - v(t))v'(t). \end{aligned} \tag{3.5}$$

A detailed solution theory for these classes of equations has been developed in Faller and Rüschendorf (2011a).

4 Approximation of optimal stopping problems

In this section we show that the basic assumption

$$N_n = \sum_i \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N \tag{4.1}$$

together with some further integrability assumptions implies convergence of the stopping problems. We first consider the case of independent observations, then dependent observations and conclude with some examples.

4.1 Approximation for independent sequences

The following example shows that convergence according to (4.1) does in general not imply convergence of the stopping problems.

Example 4.1 *Let (X'_i) be i. i. d., exponentially $\mathcal{E}(1)$ distributed and let (X_i) be independent random variables with $P(X_i \geq x) = e^{-x}$, $x \geq e^{-i}$. Further, define a sequence (a_i) such that $P(X_i = a_i) = 1 - e^{-e^{-i}}$ where (a_i) is choosen such that $EX_1 = 0$, $EX_2 = a_1$, $EX_3 = a_2, \ldots$ As consequence one finds $a_n \to -\infty$.*

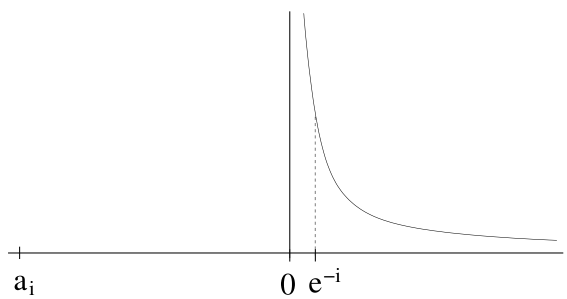

Figure 4.1

Then $N'_n = \sum \varepsilon_{(\frac{i}{n}, X'_i - \log n)} \xrightarrow{\mathcal{D}} N \sim \mathcal{P}(\mu)$, $\mu = \lambda_{[0,1]} \otimes \nu$, $\nu([x, \infty]) = e^{-x}$; also, we have

$$N_n = \sum \varepsilon_{(\frac{i}{n}, X_i - \log n)} \xrightarrow{\mathcal{D}} N \sim \mathcal{P}(\mu).$$

But these problems have different stopping behavior. $T_n = 1$ is an optimal stopping time for (X_n) with $EX_{T_n} = 0$, while the optimal stopping time (T'_n) for (X'_n) satisfies the following:

$$EX'_{T'_n} - \log n \to c, \quad \frac{T'_n}{n} \xrightarrow{\mathcal{D}} T',$$

where T' is an optimal stopping time for the limiting Poisson process (see Kennedy and Kertz (1990, 1991) and Kühne and Rüschendorf (2000b)).

We introduce the following additional integrability assumptions:

(G) **Uniform integrability**
$\{(M_n)_+\}$ is uniformly integrable.

(L) **Lower curve condition** (in the case $f(1) = -\infty$)
For the optimal stopping curve $u_n(s) = u_{n,[ns+1]\wedge n}$ assume:

$$\underline{\lim}\, u_n(1-\varepsilon) > -\infty, \quad \forall \varepsilon > 0.$$

Then the following approximation result holds:

Theorem 4.2 (Approximation of optimal stopping, independent observations)

Let $(X_{n,i})$ be independent and assume condition (G), as well as basic point process convergence

$$N_n = \sum \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N = \mathcal{P}(\mu) \text{ on } M_f.$$

Let u denote the optimal stopping curve for N, $T = \tau^u$ the corresponding threshold stopping time, and assume that u satisfies conditions (S) and (D). Then the following holds:

a) If $u_n(1) \to c = f(1) \in \mathbb{R}$, then $u_n(s) \to u(s)$, uniformly on $[o,t]$, $\quad \forall t < 1$. Furthermore,

$$\left(\tfrac{T_n}{n}, X_{n,T_n}, M_{n,1,T_{n-1}}, M_{n,T_{n+1},n}\right) \xrightarrow{\mathcal{D}} (T, y_{K^T}, M_{0,T-}, M_{T+,1}),$$

i. e. the stopping problem converges, and u is a solution of the differential equation

$$\begin{cases} u'(t) = -\displaystyle\int_{u(t)}^{\infty}\int_x^{\infty} h_f(t,y)\,dy\,dx \\ u(1) = c = f(1) \end{cases} \tag{4.2}$$

b) If $u_n(1) \to -\infty$, and assuming conditions (L), (D), and (S), then for all subsequences $(n') \subset \mathbb{N}$ with $u_{n'} \to \widehat{u}$ the following holds: $\widehat{u}$ solves (4.2) with $\widehat{u}(1) = -\infty$.

If (4.2) has a unique solution, then $u_n \to u$.

c) Under the lower uniform integrability condition (L′) in Theorem 4.4, it also holds in the case $u_n(1) \to -\infty$ that $u_n \to u$.

The proof uses compactness and subsequence arguments, followed by the threshold convergence theorem and the construction of a comparison stopping time. The lower uniform integrability condition (L′) will be detailed in the following subsection.

4.2 Approximation for dependent sequences

In such stopping problems the following prediction effect may happen.

Example 4.3 *Let (T_n) be a sequence of optimal stopping times for the case of an i. i. d. sequence (Y_i) where $Y_i \sim \mathcal{E}(1)$. For $n_k = 10^{2^k}$ let $m_k \in \{n_{k-1}+2, \dots, n_k\}$ be such that $Y_{m_k} = \max\{Y_{n_{k-1}+2}, \dots, Y_{n_k}\}$ and define the dependent sequence*

$$X_i = \begin{cases} Y_i, & i \neq m_k - 1, \\ -1, & i = m_k - 1. \end{cases}$$

Thus observing -1 in the sequence (X_i) allows us to predict that the next observation is a block maximum. Define

$$T_k^0 = \inf\{i \in (n_{k-1}+1,\dots,n_k); \quad X_i = -1\} + 1,$$

to be a stopping time for $(X_i)_{i\le n_k}$. Then

$$\lim EX_{T_k^0} - \log n_k = \lim(EM_{n_k} - \log n_k) = \gamma > 0,$$

but

$$\lim E(Y_{T_{n_k}} - \log n_k) = 0.$$

So both sequences (X_i) and (Y_i) have completely different stopping behavior, which is caused by the prediction effect, even if the induced point processes have the same limit behavior.

Consider a sequence $(X_{n,i}, \mathcal{A}_{n,i})_{1\le i\le n}$ and a Poisson process $N \stackrel{d}{=} \mathcal{P}(\mu)$ with optimal stopping curve u. We state the following assumptions:

Asymptotic conditional independence

(A) $\quad P^{N_n(\cdot\cap([t,1]\times\mathbb{R}))|\mathcal{A}_{n,[nt]-1}} \xrightarrow{\mathcal{D}} P^{N(\cdot\cap([t,1]\times\mathbb{R}))}$

(A′) $\quad \max\limits_{s<i/n\le t} X_i^n \vee f(s) \xrightarrow{\mathcal{D}} \sup\limits_{s<\tau_k\le t} y_k \vee f(s), \quad 0\le s<t\le 1.$

(L′) **Integrability assumption**
Assume there exists a sequence $(u_{n,i}) \downarrow i$ such that $u_{n,[ns]} \to u(s)$, $s<1$. Furthermore, for $T_n' := \inf\{i : X_{n,i} \ge u_{n,i}\}$ the following holds:

$$\lim_{t\to 1} \overline{\lim}_n E \mid X_{n,T_n'^{\ge[ns]}} \mid 1_{\left\{T_n'^{\ge[ns]} \ge nt\right\}} = 0.$$

(L) **Lower curve condition**
For the Snell hull $\gamma_{n,i} := \max\{X_{n,i}, E(\gamma_{n,i+1} \mid \mathcal{A}_{ni})\}$, the following holds

$$\underline{\lim}\, E\gamma_{n,[nt]} > -\infty, \quad 0\le t<1.$$

(A) and (A′) are asymptotic independence assumptions, which prevent the prediction effect and typically are met in our applications. (L′) and (L) are integrability conditions on the stopping problem.

Then the following approximation holds for dependent sequences.

Theorem 4.4 (Approximation of optimal stopping, dependent observations) *Let $N_n = \sum \varepsilon_{(\frac{i}{n},X_{n,i})} \xrightarrow{\mathcal{D}} N = \mathcal{P}(\mu)$ on M_f and assume conditions (D), (A′), (G), and (L′). Furthermore, assume that the optimal stopping curve u is the unique solution of the differential equation* (3.3). *Then:*

$$u_n(t,x) \xrightarrow{P} u(t,x), \quad \forall (t,x)\in[0,1)\times[c,\infty)$$

and $\left(\frac{T_n}{n}, X_{n,T_n}\right) \xrightarrow{\mathcal{D}} (T, \overline{y}_{K^T})$, $EX_{n,T_n} \to u(0) = E\overline{y}_{K^T}$.

Furthermore, (T_n') is an asymptotically optimal sequence of stopping times.

Versions of this result also hold true for cluster processes and without the uniqueness condition. Also, some examples where conditions on the asymptotic independence do not hold are dealt with for special types of dependent sequences.

4.3 Examples

We illustrate the results of this section by a sequence of examples. In the case where (X_i) are i. i. d. with distribution function $F \in D(G)$ for a max-stable G, the optimal reward sequence (v_i) satisfies the backward induction (Bellman) equation

$$v_n = EX_n, \quad v_{i-1} = E\max\{X_{i-1}, v_i\}, \quad i = n, \dots, 2.$$

The asymptotics of $(v_i) = (v_i^n)$ in this case have been given by Kennedy and Kertz (1990, 1991) based on a point process approximation.

a) Independent sequences with discounting or observation costs

Let $X_i = c_i Y_i + d_i$ with $F \in D(G)$ and assume that

$$N_n = \sum \varepsilon_{\left(\frac{i}{n}, \frac{X_i - \widehat{b}_n}{\widehat{a}_n}\right)} \xrightarrow{\mathcal{D}} N \stackrel{d}{=} \mathcal{P}(\mu),$$

where $\widehat{b}_n = c_n b_n + d_n$, $\widehat{a}_n = a_n c_n$ and thus

$$\frac{X_i - \widehat{b}_n}{\widehat{a}_n} = \frac{c_i}{c_n}\frac{Y_i - b_n}{a_n} - \frac{d_n}{a_n c_n} + \frac{c_i - c_n}{c_n}\frac{b_n}{a_n}.$$

Assume that uniformly in $t : (c_{[nt]})/c_n \to \gamma_t$, $d_n/(a_n c_n) \to d$, then $\gamma_t = t^c$. All these cases have been dealt with and lead to explicit results.

We explain these type of results in the particular case where only observation costs are present, i. e. $F \in D(\Lambda)$, $X_i = Y_i + d_i$, $0 < d_i$.

If $(d_n - d_{[nt]})/a_n \to \gamma_t$, then one finds that $\gamma_t = -c\log\ t$, $c \in \mathbb{R}$ and $\widehat{b}_n = b_n + d_n$, $\widehat{a}_n = a_n$. In consequence, we obtain point process convergence

$$N_n \xrightarrow{\mathcal{D}} N = \mathcal{P}(\mu), \quad \text{with the intensity given by } \frac{d\mu(\cdot \times (y,\infty))}{d\lambda} = e^{-y}\, t^c.$$

Let $c > -1$, and w_n be a $(1 - 1/n)$-quantile, i. e. $n(1 - F(w_n)) \to 1$ and let (T_n) be a sequence of optimal stopping times, then we get the following asymptotics of the optimal stopping values and stopping distribution.

1) **Optimal stopping values and distribution:**

$$\frac{EX_{T_n} - \widehat{b}_n}{a_n} \to -\log(1+c) \text{ and}$$

$$P\Big(\frac{X_{T_n} - \widehat{b}_n}{a_n} \le x\Big) \to \begin{cases} 1 - \frac{1}{2}\frac{e^{-x}}{1+c}, & x \ge \log(1+c), \\ \frac{1}{2}e^x(1+c), & x > \log(1+c). \end{cases}$$

The case $c = 0$ corresponds to the i. i. d. case. In consequence, we obtain the following:

2) **Prophet inequality**:

$\frac{EM_n - EX_{T_n}}{a_n} \to \gamma = 0.577\dots$, the Euler constant; the limit being independent of c and $\frac{EM_n - b_n}{a_n} \to EM = \gamma + \log(1+c)$.

3) **Asymptotic independence and distribution** of the optimal stopping time, stopping value and the pre- and the post-maxima. The following convergence result holds:

$$\Big(\frac{T_n}{n}, \frac{M_{1,T_{n-1}} - u_{n,T_n}}{a_n}, \frac{X_{T_n} - u_{n,T_n}}{a_n}, \frac{M_{T_{n+1,n}} - u_{n.T_n}}{a_n}\Big)$$
$$\xrightarrow{\mathcal{D}} (T, M_{0,T-} - u(T), y_{K^T} - u(T), M_{T+,1} - u(T)).$$

Furthermore, the limiting variables are independent.

4) An **asymptotically optimal stopping time sequence** is given by

$$T_n' = \inf\Big\{ i \le n;\ \Big(i \ge n - [n\varepsilon], \frac{X_i - \widehat{b}_n}{a_n} \ge u_c\Big(\frac{i}{n}\Big) - u_0\Big(\frac{i}{n}\Big) + \frac{w_{n-i} - b_n}{a_n}\Big)$$
$$\text{or } \Big(i < n - [n\varepsilon], \frac{X_i - \widehat{b}_n}{a_n} \ge u_c\Big(\frac{i}{n}\Big)\Big)\Big\}.$$

5) **Differential equation for optimal stopping curve:**
The optimal stopping curve u_c is a solution of the differential equation

$$u'(t) = -\int_{u(t)}^{\infty} e^{-x} t^c dx = -t^c e^{-u(t)}, \quad t < 1,$$
$$u(1) = -\infty.$$

This is an equation in separate variables and has the solution

$$u_c(t) = \log \frac{1 - t^{1+c}}{1 + c}.$$

As a result, one obtains a complete description of the asymptotic optimal stopping times, distributions, and stopping values. Similar explicit results have also been obtained for other cases depending on the domain of attraction and the conditions on the coefficients.

b) Moving averages

As a second class of applications, we consider some examples of moving average processes.

Let $F \in D(\Phi_\alpha)$, $\alpha > 1$, let (Y_i) be i. i. d., and consider the moving average sequence $X_i = \sum_{j=1}^{\infty} c_j Y_{i-j+1}$, with $c_j \ge 0$, $\sum_{j=1}^{\infty} c_j^\delta < \infty$ for some $0 < \delta < 1$. W. l. o. g. assume that $\sup c_i = 1$. Then the embedded point processes N_n converge,

$$N_n = \sum \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}\right)} \xrightarrow{\mathcal{D}} N = \sum_k \sum_{i: c_i \neq 0} \varepsilon_{(\tau_k, c_i y_k)}.$$

The limit N is a cluster process based on the Poisson process $N' = \sum \varepsilon_{(\tau_k, y_k)}$ with intensity measure $\mu = \lambda \otimes \nu$, $\quad \nu([x, \infty)) = x^{-\alpha}, x > 0$.

For this process we can verify the independence condition (A′) and obtain as a consequence of Theorem 4.4 the following optimal stopping result:

Theorem 4.5 (Optimal stopping of moving averages) *Under the conditions above, the following hold:*

a) $\frac{EX_{T_n}}{a_n} \to \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha}$

b) $P\left(\frac{X_{T_n}}{a_n} \le x\right) \to \begin{cases} 1 - x^{-\alpha}\frac{1}{2-1/\alpha}, & x \ge \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} \\ \frac{\alpha}{2\alpha-1}(\frac{\alpha-1}{\alpha})^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < (\frac{\alpha}{\alpha-1}), \\ 0, & x \le 0 \end{cases}$

c) $$T_n' := \inf\Big\{ i \ge w+1; \quad X_i \ge a_n \tfrac{c_1}{c_m} u_\alpha\left(\tfrac{i}{n}\right) \\ X_{i-1} \vee \dots \vee X_{i-w} < \tfrac{1}{2} a_n \tfrac{c_1}{c_m} u_\alpha\left(\tfrac{i}{n}\right) \Big\} + m - 1$$

is an asymptotically optimal sequence of stopping times, where $m =$ *index of* $\sup c_j$ *and* $w = \max\{i : c_i \ge c_1\}$.

The idea of the construction in c) is to wait until the first crossing of the boundary and then until a point in the cluster with the greatest coefficient appears. The optimal stopping curve of the Poisson process N' has the following explicit form

$$u_\alpha(t) = \left(\frac{\alpha}{\alpha^2-1}\right)^{1/2} (1 - t^{1+\alpha})^{1/\alpha}.$$

As a result we thus have an explicit description of the approximative optimal stopping behavior. Similar explicit results have also been obtained in the other domains under conditions on the coefficients which lead to a deterministic cluster.

In a limit case with a **random cluster** (as in Section 2.2) a modification of this idea works. One compares the observed point in a cluster with the expected value of future cluster points and with the stopping curve. This leads to an additional estimation problem to be solved for the expected values of future cluster points (for more details see Kühne and Rüschendorf (2003a,b)).

c) Chain dependent sequences (hidden Markov chain)

Let (J_n) be an irreducible Markov chain on $\{1,\dots,m\}$ with transition probabilities $(p_{i,j})$ and stationary distribution (π_i). Furthermore, let $F_1,\dots,F_m$ be distribution functions.

A sequence (X_n) is called chain dependent (hidden Markov chain) w. r. t. (J_n) if

$$\begin{aligned} P(J_n = j, &X_n \le x \mid J_0,\dots,J_{n-1}, X_1,\dots,X_{n-1}) \\ &= P(J_n = j, X_n \le x \mid J_{n-1}) = p_{J_{n-1},j}\, F_{J_{n-1}}(x). \end{aligned}$$

For chain dependent sequences, we obtain as a consequence of our approach the following optimal stopping result:

Theorem 4.6 (Optimal stopping of hidden Markov chains)
Let (X_i) *be a chain dependent sequence and assume that there exist* a_n *and* b_n *such that* $n\sum_{i=1}^m \pi_i \overline{F}_i(a_n x + b_n) \to \nu(x,\infty)$. *Then:*

a) $N_n \xrightarrow{\mathcal{D}} N = \mathcal{P}(\lambda \otimes \nu)$*, where* $e^{-\nu(x,\infty)}$ *is an extreme value distribution function*

b) If $\nu([x,\infty)) = e^{-x^\alpha}$, $\alpha > 1$*, then*

b1) $\dfrac{EX_{T_n}}{a_n} \to \left(\dfrac{\alpha}{\alpha-1}\right)^{1/\alpha}$

b2) $P\left(\dfrac{X_{T_n}}{a_n} \le x\right) \to \begin{cases} 1 - \frac{1}{2-1/\alpha} x^{-\alpha}, & x \ge \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} \\ \frac{\alpha}{2\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < (\frac{\alpha}{\alpha-1})^{1/\alpha} \\ 0, & x \le 0 \end{cases}$

b3) $T_n' := \inf\left\{ i \le n : \quad X_i \ \ge \ \left(\dfrac{\alpha}{\alpha-1}\right)^{1/\alpha}\left(1-\dfrac{i}{n}\right)^{1/\alpha}\right\}$

is an asymptotically optimal sequence of stopping times.

The proof of a) follows from the general limit theorem for point process convergence in Theorem 2.1. The optimal stopping curve for a limit Poisson point process is given by: $u(t) = (\alpha/(\alpha-1))^{1/\alpha}(1-t)^{1/\alpha}$.

For this example, the independence condition (A) can be verified and the approximation theorem for optimal stopping in the case of a dependent sequence, Theorem 4.4 applies and gives the explicit results in b).

d) max AR(1)-sequence

For an integrable i. i. d. sequence (Y_i) with distribution function $F \in D(\Phi_\alpha)$, $\alpha > 1$ and an independent scaling sequence (Z_i), $0 \le Z_i \le 1$, define a max AR(1)-sequence (X_i) by

$$X_i = Z_i(Y_i \vee X_{i-1}),$$

where X_0 is a real random variable with $EX_o^+ < \infty$. The max in the definition leads to an increase which however is compensated by the downscaling factors (Z_i). Then we have the following optimal stopping result.

Theorem 4.7 (Optimal stopping of max AR(1)-sequence)
Assume that $\frac{1}{n}\sum_{i=1}^n EZ_i^\alpha \to d$ *and* $EZ_i^\alpha \le \beta < 1$*. Then*

$$\frac{EX_{T_n}}{a_n} \to d\left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha}$$

and $T_n' := \inf\left\{i \le n; X_i \ge (\alpha/(\alpha-1))^{1/\alpha}(1-i/n)^{1/\alpha}\right\}$ *is an asymptotically optimal sequence of stopping times.*

The proof of Theorem 4.7 is based on a modification of the approach to deriving Theorem 4.4. In this example, the point process convergence of N_n is not obvious. Instead, we use a comparison with the stopping of a related sequence defined by $\widetilde{X}_i = Z_i Y_i$. Then the corresponding embedded point processes $\widetilde{N}_n$ can be shown to converge as follows:

$$\widetilde{N}_n = \sum \varepsilon_{\left(\frac{i}{n}, \frac{\widetilde{X}_i}{a_n}\right)} \xrightarrow{\mathcal{D}} N = \mathcal{P}(\mu).$$

The optimal stopping curve of N is given by $u(t) = d(\alpha/(\alpha-1))^{1/\alpha}(1-t)^{1/\alpha}$. From the approximation result in Theorem 4.2 we obtain that the associated sequence

$$\widetilde{T}'_n = \inf\left\{i; \widetilde{X}_i \geq a_n u\left(\frac{i}{n}\right)\right\}$$

of stopping times is asymptotically optimal for the stopping problem w. r. t. $(\widetilde{X}_i)$ and thus $E(X_{\widetilde{T}'_n})/a_n \to u(0) = d(\alpha/(\alpha-1))^{1/\alpha}$.

In the next step, we obtain using uniform integrability that

$$\overline{\lim}\, \frac{E\gamma_{n,1}}{a_n} \leq u(0).$$

This implies that $u(0)$ is identical to the asymptotically optimal stopping value for the majorizing sequence $(\widetilde{X}_i)$. In the final step, we prove the asymptotic optimality of the sequence (T'_n) using an appropriate comparison.

5 Approximation of best choice problems

An interesting class of optimal stopping problems are best choice problems. We consider in this section the best choice problem for independent sequences (X_i), i. e. in the full information case. The probability of choosing the best observation is given by

$$v_n = P(X_{T_n} = M_n) = \sup_{\tau \in \gamma^n} P(X_\tau = M_n), \tag{5.1}$$

where $M_n = \max\{X_1, \dots, X_n\}$ and $\gamma^n = \{\tau \text{ stopping time}; \tau \leq n\}$. Let T_n be optimal stopping times, $n \in \mathbb{N}$. Using the normalization $X_{n,i} = (X_i - b_n)/a_n$ and point process convergence

$$N_n = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, X_{n,i})} \xrightarrow{\mathcal{D}} N \stackrel{d}{=} \mathcal{P}(\mu), \tag{5.2}$$

where N is a Poisson process on $M_c = [0,1] \times (c, \infty)$ with intensity measure μ, we approximate the best choice problem for the sequence $X_1, \dots, X_n$ by the best choice problem for the Poisson point process N on M_c.

In the i. i. d. case, a classical result of Gilbert and Mosteller (1966) states that $v_n \downarrow 0.58016\dots$, see also Bruss and Rogers (1991) and Gnedin (1996). Using a simple splitting strategy, one gets $v_2 = 3/4$. For the stopping problem without information (the secretary problem), the classical result states that $v_n \to 1/e$.

Before stating a general optimal best choice result, we consider a special case the best choice problem with discounting, since in this case the basic arguments are simple to explain. Let (Y_i) be an i. i. d. sequence with $F \in D(\Psi_\alpha)$, $\alpha > 0$, $\omega_F = 0$, and let (c_n) be a sequence with $c_{[nt]}/c_n \to t^{-c}$, $c > -1/\alpha$ (see Section 4). Define

$$X_i = c_i Y_i, \quad M_n = \max\{X_1, \dots, X_n\} \quad \text{and}$$
$$N_n = \sum \varepsilon_{(\frac{i}{n}, \frac{X_i}{\overline{a}_n})}, \quad \overline{a}_n = c_n a_n.$$

Then we establish the following explicit best choice result.

Theorem 5.1 (Optimal best choice with discounts)
Under the above conditions the following hold:

a) $v_n \to 0.58016\ldots$

b) $T_n' = \inf\left\{i \le n; X_i \ge \overline{a_n}\left(\frac{b}{1-(\frac{i}{n})^{1+c\alpha}}\right)^{1/\alpha}, \;\; X_i = M_i\right\}$
is an asymptotically optimal sequence of stopping times. Here $b = 0.804352\ldots$ *is the solution of* $\sum_{i=1}^{\infty} b^i/(i!\,i) = 1$.

Idea of the proof: For the proof we use the following steps:

1) $N_n \xrightarrow{\mathcal{D}} N = \mathcal{P}(\mu)$, where the intensity measure μ is given by

$$\mu([0,t] \times [x,\infty)) = (-x)^\alpha \quad \frac{t^{1+c\alpha}}{1+c\alpha}, \quad x \le 0.$$

2) $p = P(\text{optimal choice in Poisson PP}) = 0.5801$

   The proof is obtained by a transformation to the stationary case

3) $\overline{\lim}\, v_n \le p$

   For the proof of this inequality, we obtain from the Bellman equation that the optimal stopping time is a threshold stopping time. Now we use the threshold convergence theorem and choose an appropriate subsequence to obtain the stated inequality.

4) We finally check that $P(X_{T_n'} = M_n) \to p$ using point process convergence.

   1)–4) imply the result stated in Theorem 5.1.

The following theorem determines approximations to best choice probabilities and gives approximative optimal stopping times for the case of general independent sequences with infinite intensity

Theorem 5.2 (Independent sequences, best choice, infinite intensity)
Let $N_n \xrightarrow{\mathcal{D}} N \stackrel{d}{=} \mathcal{P}(\mu)$ *and assume*

(I) ***Intensity condition***
$\forall t \in [0,1]$ *holds:* $\mu((t,1] \times (c,\infty)) = \infty$

Then for the best choice problem for independent sequences $X_1, \ldots, X_n$ *the following hold:*

a) The optimal best choice stopping time T *for the Poisson process* N *is given by*

$$T = \inf\left\{\tau_k; y_k = \sup_{\tau_j \in [0,\tau_k]} y_j, \;\; y_k > v(\tau_k)\right\},$$

where $v : [0,1] \to [c, \infty)$ *is a solution of the equation*

$$\int_t^1 \int_{v(t)}^{\infty} e^{\mu((r,1]\times(v(t),y])} \mu(dr, dy) = 1, \quad v(1) = c.$$

The probability of making the best choice is given by

$$s_1 := P\Big(y_T = \sup_k y_k\Big) = \int_0^1 e^{-\mu([0,r]\times(c,\infty])} \int_{v(r)}^{\infty} e^{-\mu((r,1]\times(y,\infty])} \mu(dr, dy)$$
$$+ \int_0^1 \int_c^{v(r')} \int_{r'}^1 e^{-\mu([0,r]\times(y',\infty])} \int_{y'\vee v(r)}^{\infty} e^{-\mu((r,1]\times(y,\infty])} \mu(dr, dy)\mu(dr', dy').$$

b) ***Approximation:*** $\lim_{n\to\infty} v_n = \lim_{n\to\infty} P(X_{T_n} = M_n) = s_1$

c) $\widehat{T}_n := \min\left\{1 \le i \le n : X_i = M_i,\ X_i > a_n v\big(\frac{i}{n}\big) + b_n\right\}$ *is an asymptotically optimal sequence of stopping times.*

There also exists a modified version of this result for the case of finite intensity. In particular, these theorems have been applied to describe, in detail and in explicit form, the probabilities of making the best choice for i. i. d. sequences (Y_i) and for the related sequences $X_i = c_i Y_i + b_i$ with discounting and observation costs for (Y_i) in the domain of attraction of an extreme value distribution under various conditions on the coefficients.

6 Approximative solutions of multi-stopping and multi-choice problems

The approximation approach has also been applied to approximatively solve several types of multi-stopping and multi-choice problems. We first consider optimal two-stopping problems of the form

$$V^{(2)}(X_1, \dots, X_n) = \sup\{EX_{T_1} \vee X_{T_2};\ T_1 < T_2 \le n \text{ stopping times}\}, \tag{6.1}$$

where two choices are possible. This case was solved in Kühne and Rüschendorf (2002). Similar results were also obtained in Assaf, Goldstein, and Samuel-Cahn (2004, 2006) and Goldstein and Samuel-Cahn (2006).

The solution is based on the following structural result. This result implies that optimal two-stopping problems can be reduced to an optimal one-stopping problem, but for a more complicated sequence.

Proposition 6.1 *Let* $u^x_{n,i} := V(X_i, \dots, X_n \vee x)$ *and define the following two stopping times*

$$T_1 = \inf\{i \le n-1;\ u^{X_i}_{n,i+1} \ge V^{(2)}(X_{i+1}, \dots, X_n)\},$$
$$T_2 = \inf\Big\{i > T_1;\ X_i \ge u^{X_{T_1}}_{n,i+1}\Big\}.$$

Then T_1 and T_2 are optimal stopping times, i. e. solutions of (6.1) *and*

$$V^{(2)}(X_1,\dots,X_n) = V\left(u_{n,2}^{X_1},\dots,u_{n,n}^{X_{n-1}}\right). \tag{6.2}$$

Based on (6.2), the optimal two-stopping problem for (X_i) is reduced to an optimal one-stopping problem for the sequence $(u_{n,i}^{X_{i-1}})$. In consequence we are able to derive a solution of the optimal two-stopping problem. As an example, we consider the case where (X_i) are in the domain of the Gumbel distribution.

Theorem 6.2 (Optimal solution of the two-stopping problem) *Let (X_i) be an i. i. d. sequence with $P^{X_1} \in D(\Lambda)$. Let w_n be a sequence such that $n(1-F(w_n)) \to e-1$. Then*

$$T_n^1 = \inf\{i : X_i \geq w_{n-i}\},$$
$$T_n^2 = \inf\left\{i > T_n^1 : X_i \geq a_n \log\left(1 + e^{\frac{X_{T_n^1} - b_n}{a_n}} - \frac{i}{n}\right) + b_n\right\}$$

are asymptotically optimal stopping times and

$$\frac{EX_{T_n^1} \vee X_{T_n^2} - b_n}{a_n} \to 0.7649\ldots \tag{6.3}$$

Remark 6.3 *In comparison, as shown in Section 4, to the one-stopping case the limiting stopping value is zero.*

Proof: From the result on the structure of solutions in Proposition 6.1, we have to analyse the stopping problem for $N_n \sim (Y_{n,2},\dots,Y_{n,n}) := (u_{n,2}^{X_1},\dots,u_{n,n}^{X_{n-1}})$.

Since $\sum \varepsilon_{\left(\frac{i}{n},\frac{X_i-b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N' = \mathcal{P}(\mu)$, we obtain from the approximation result for optimal stopping for an independent sequence in Theorem 4.2, that

$$V\left(\frac{X_{[nt]\vee 2} - b_n}{a_n},\dots,\frac{X_n - b_n}{a_n} \vee x\right) \to u^x(t),$$

where $u^x(t)$ is a solution of the differential equation $\frac{d}{dt}u^x(t) = -e^{-u^x(t)}$, $u^x(1) = x$.

This equation has the solution

$$u^x(t) = \log(e^x + 1 - t).$$

From the convergence of the stopping boundaries, as a consequence of the point process convergence defined in (2.3), we obtain

$$N_n \xrightarrow{\mathcal{D}} N^* = \mathcal{P}(\mu^*), \quad \text{where } \frac{d\mu^*(\cdot \times [x,\infty])}{d\lambda^1}(t) = \frac{1}{e^x + t - 1}.$$

Next we solve the stopping problem for the point process N^* and verify the conditions for the approximation theorem, Theorem 4.2. □

Similar solutions for optimal stopping problems have also been obtained in the other domains of max stable distributions. The method of solution for two-stopping problems has been extended to several types of multi-stopping and multiple best choice problems in Faller and Rüschendorf (2011–2013), where solutions are obtained in explicit or numerically tractable form.

The **multi-stopping problem**

$$E \max_{1\le i\le m} X_{T_i} = \sup_{1\le\tau_1<\tau_2<\ldots<\tau_m\le n} E \max_{1\le i\le m} X_{\tau_i}$$

which allows m choices has been reduced in the case of dependent sequences to the solution of a system of m recursive differential equations of first order. In particular, explicit approximative solutions are given in the i. i. d. case with discounting and observation costs.

Also, multiple stopping problem with total payoff

$$E \sum_{i=1}^{m} X_{T_i} = \sup_{1\le\tau_1<\tau_2<\ldots<\tau_m\le n} E \sum_{i=1}^{m} X_{\tau_i}$$

has been solved in a similar way. Furthermore, **multiple best choice problems** of the form

$$P\Big(\max_{i\le m} X_{T_i} = M_n\Big) = \sup_{1\le\tau_1<\tau_2<\ldots<\tau_m\le n} P\Big(\max_{i\le m} X_{\tau_i} = M_n\Big)$$

for independent sequences have been reduced to recursive systems of first order differential equations, which lead to explicit or numerical solutions in certain cases. For details of these results we refer the reader to the above mentioned literature.

7 Conclusion

In this review we have described a general approach to approximatively solving some general classes of optimal stopping and best choice problems for discrete time sequences, as well as extensions to several versions of multi-stopping and multiple choice problems. The basic assumption of this approach is the convergence of the embedded point process to some Poisson or cluster process.

This assumption, together with some kind of integrability conditions, implies that the stopping problem can be approximated by the stopping problem for the limit process, which can be reduced to the solution of a differential equation (or a system of differential equations) of first order. This approach can be applied to several classes of independent sequences and also to some classes of dependent sequences. In several cases, it leads to exact or to numerically tractable solutions. For some classes of multi-choice problems, so far we have results only for independent sequences, due to the complexity of more general cases. It would be of interest to see if this approach can also be applied to further interesting classes of stopping problems with alternative criteria, such as expected rank, or competitive rank selection as considered in Bruss and Ferguson (1993, 1996, 1997), Bruss and Louchard (1998) or Bruss and Swan (2009).

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