On approximative solutions of optimal stopping problems

Andreas Faller and Ludger Rüschendorf
University of Freiburg

Abstract

In this paper an extension is established of the method to approximate optimal discrete time stopping problems by related limiting stopping problems for Poisson type processes. The extension allows to apply this method to a larger class of examples as arising f.e. from point process convergence results in extreme value theory. The second main point in this paper is the development of new classes of solutions of the differential equations which characterize optimal threshold functions. As particular application we give a fairly complete discussion of the approximative optimal stopping behavior of i.i.d. sequences with discount and observation costs.

Keywords: optimal stopping, max stable distributions, Poisson processes

Mathematics Subject Classification 2000: 60 G40, 62 L15

1 Introduction

The theory of optimal stopping of independent and dependent sequences $X_1, \dots, X_n$ is a classical subject of probability theory which still has a lot of open problems and new applications as for instance in the area of financial mathematics. In a series of papers an approximation method has been developed in order to solve approximatively optimal stopping problems for $X_1, \dots, X_n$ by some limiting stopping problems for Poisson and related point processes (see [KR][1] (2000a; 2000b; 2003; 2004). The basic assumption in this approach is convergence of the imbedded planar point process

$$N_n = \sum_{i=1} \delta_{(\frac{i}{n}, X_i^n)} \xrightarrow{d} N \tag{1.1}$$

[1]Kühne and Rüschendorf is abbreviated within this paper with [KR], Faller with [F], Faller and Rüschendorf with [FR].

to some Poisson (or related) point process N. Here $X_i^n = \frac{X_i - b_n}{a_n}$ is a normalization of X_i induced typically from the central limit theorem for maxima. For the limiting Poisson type process N, which has accumulation points along a lower boundary curve, an optimal stopping problem in continuous time can be formulated.

The optimal solution for this limiting stopping problem is given by a threshold stopping time. The threshold function is determined by a differential equation of first order. This is in analogy to stopping problems for diffusion processes which typically lead to free boundary value problems with differential equations of second order for the stopping curve (Stefan free boundary problem). The differential equation for the optimal threshold function in the Poisson case can be solved in several cases explicitly or numerically. Under some uniform integrability and separation conditions, a differentiability condition for the intensity measure of N as well as an asymptotic independence condition in the dependent case the optimal stopping problem for $X_1, \dots, X_n$ can be approximated by the optimal stopping problem for the limiting Poisson type process.

In this paper we establish some relevant extensions of interest for this approximation method. In Section 2 we give a new and simplified derivation of the optimal stopping curve u for the optimal stopping problem for continuous time Poisson processes as above. These curves solve a differential equation of the form

$$u'(t) = -\int_{u(t)}^{\infty} G(t,y)\,dy, \quad t \in [0,1), \qquad\qquad u(1) = c \tag{1.2}$$

with some guarantee value $c \in \mathbb{R} \cup \{-\infty\}$. Here G is defined explicitly via the intensity measure of N and is called 'intensity function'. For $c \in \mathbb{R}$ (1.2) has a unique solution and thus characterizes the optimal stopping curve. For $c = -\infty$ there may exist several solutions of (1.2). While the finite case in $c \in \mathbb{R}^1$ has been dealt with in [KR] (2000a) the case where $c = -\infty$ has been left mostly open in the previous literature and has been dealt with only under an uniqueness assumption on equation (1.2). Based on our new derivation of the approximation result we characterize the optimal stopping curve for $c = -\infty$ as the maximal solution of (1.2). We also establish some uniqueness criteria for (1.2) in the case $c = -\infty$. There are several interesting applications with $c = -\infty$ (see e.g. the examples in Section 5 of this paper) which can be solved with our new extension of the approximation method.

The second main contribution of this paper concerns the differential equation (1.2), which characterizes optimal stopping boundaries. The classical results from differential equations for equation (1.2) concern the so-called separable case where $G(t,y) = a(t)b(y)$. But even in this case the known characterization results for solutions are typically not explicit but need numerical tools. In this paper we introduce two interesting new classes of intensity functions G – not leading to the case of separate variables in (1.2) – which allow us to solve the differential equation (1.2) in 'explicit' form. For these classes of intensity functions the so-called 'separation condition', which is needed in our approximation approach to optimal stopping problems, can be verified. In Section 4 we state an extension of the approximation theorem in [KR] (2004, Theorem 2.1) for the optimal stopping of dependent sequences. This is the second main ingredient of the approximation method. Our version gives a precise uniform integrability condition which allows us to treat also the case $c = -\infty$ (which was not included in previous results) and allows for more general filtrations.

As application of our extensions we are able in Section 5 to discuss in fairly complete form the optimal stopping of i.i.d. sequences (Z_i) with discount factors (c_i) and observation costs (d_i), i.e. of

$$X_i = c_i Z_i + d_i. \tag{1.3}$$

Here c_i, d_i fulfill some criteria to ensure point process convergence, (Z_i) is an i.i.d. sequence in the max-domain of an extreme value distribution Γ, Φ_α, Ψ_α. The new results on the solution of the optimality equation in (1.2) and the inclusion of the case $c = -\infty$ allow us to complete some partial results on this problem in [KR] (2000b). This kind of stopping problem was first considered in the i.i.d. case without discount and observation cost in Kennedy and Kertz (1990, 1991).

It has been observed in several papers in the literature that optimal stopping problems may have an easier solution in a related form for Poisson numbers of points as for instance in the classical house selling problem (see Chow et al. (1971); Bruss and Rogers (1991), and Gnedin and Sakaguchi (1992)). The approach extended in this paper makes this method applicable to a wider class of examples. Based on the new results in this paper an interesting extension to multistopping problems has been given recently in [FR] (2010). Several details and proofs in this paper have been omitted for the reason of space and concentration. For them we refer to the dissertation of Faller (2009) on which this paper is based.

2 Optimal stopping of Poisson processes

We consider optimal stopping of a Poisson process $N = \sum_k \delta_{(\tau_k, Y_k)}$ in the plane restricted to some set

$$M_f = \{(t,x) \in [0,1] \times \overline{\mathbb{R}} :\ x > f(t)\}$$

where $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ is a continuous function describing the lower boundary of N. We consider Poisson processes restricted to M_f which may have infinite intensity along the lower boundary f. We assume that the intensity measure μ of N is a Radon measure on M_f with the topology on M_f induced by the usual topology on $[0,1] \times \overline{\mathbb{R}}$. Thus any compact set $A \subset M_f$ has only finitely many points. By convergence in distribution '$N_n \xrightarrow{d} N$ on M_f' we mean convergence in distribution of the restricted point processes.

We generally assume the *boundedness condition*

$$\text{(B)} \qquad E(\sup_k Y_k)^+ < \infty. \tag{2.1}$$

Let $\mathcal{A}_t = \sigma(N(\cdot \cap [0,t] \times \overline{\mathbb{R}} \cap M_f))$, $t \in [0,1]$, denote the relevant filtration of the point process N. A stopping time $T : \Omega \to [0,1]$ for N is a stopping time w.r.t. filtration $(\mathcal{A}_t)$, i.e. $\{T \le t\} \in \mathcal{A}_t$, $t \in [0,1]$. In general N may have multiple points. Denote by $\overline{Y}_T := \sup\{Y_k : T = \tau_k\}$, $\sup \emptyset := -\infty$, the reward w.r.t. stopping time T. For any *guarantee value* $x \in [c, \infty]$, $c := f(1)$, define the *optimal stopping curve* w.r.t. x by

$$\begin{aligned} u(t,x) &:= \sup\{E[\overline{Y}_T \vee x] : \ T > t \text{ a stopping time}\}, \quad t \in [0,1), \\ u(1,x) &:= E[\overline{Y}_1 \vee x]. \end{aligned} \tag{2.2}$$

In comparison to [KR] (2000a) we consider stopping times $T > t$ in this paper and introduce a guarantee value. This has some technical advantages w.r.t. continuity properties and leads to some changes in the optimal stopping time formulas. For notational convenience we write

$$u(t) := u(t,c), \quad t \in [0,1].$$

Every instance of u without arguments is to be understood as $u(\cdot, c)$. The *critical point* of N for x is given by

$$t_0(x) := \inf\{t \in [0,1] : \ N((t,1] \times (x,\infty] \cap M_f) = 0 \quad P\text{-a.s.}\}, \quad \inf \emptyset := 1. \tag{2.3}$$

The following lemma gives some basic properties of the optimal stopping curve u.

Lemma 2.1 *(a) For any $x \in [c,\infty]$ the optimal stopping curve $u(\cdot, x)$ is right continuous on the interval $\{t \in [0,1] : u(t,x) > -\infty\}$.*

(b) For x, $y \in \mathbb{R}$, $x \le y$ holds

$$0 \le u(t,y) - u(t,x) \le y - x, \quad t \in [0,1].$$

(c) For $x \in [c,\infty]$, $x > -\infty$ holds

$$t_0(x) = \inf\{t \in [0,1] : \ u(t,x) = x\}, \quad \inf \emptyset := 1.$$

Proof: For the proof see [F] (2009). □

In order to identify the optimal stopping curve u for the stopping of the Poisson process N we *generally assume* in the following: $t_0(c) = 1$ and that the intensity measure μ of N is a Radon measure on M_f and Lebesgue-continuous with density g.

An important condition is the *separation condition*:

(S) Assume that $u > f$ on $[0, 1)$.

If $c = -\infty$, then (S) implies right-continuity of u on $[0, 1]$ and the validity of Lemma 2.1c) also for $x = -\infty$. By our general assumption $t_0(c) = 1$, the separation condition (S) is fulfilled generally for $c \in \mathbb{R}$ if $f \leq c$.

With the additional guarantee parameter x, Theorem 2.5 a) in [KR] (2000a) in our slightly modified form can be formulated as

Theorem 2.2 (Existence and uniqueness of an optimal stopping time) *Let N fulfill (B) and (S). Then for any $x \geq c$ the optimal stopping curve $u(\cdot, x) : [0, 1] \to [x, \infty)$ is continuous on $[0, 1]$. Furthermore,*

$$u(t, x) = E[\overline{Y}_{T(t,x)} \vee x], \quad t \in [0, 1], \tag{2.4}$$

where the optimal stopping time $T(t, x)$ at time t is given by

$$T(t, x) = \inf\{\tau_k > t : \ Y_k > u(\tau_k, x)\}, \quad \inf \emptyset := 1. \tag{2.5}$$

Thus $T(0, x)$ is an optimal stopping time for N, x at time 0. It is P-a.s. unique.

Note that by condition (S) in case $c = -\infty$ we have $P(T(t, c) < 1) = 1$ or equivalently $\mu((t, 1] \times \mathbb{R} \cap M_u) = \infty$ for all $t \in [0, 1)$.

In the following we want to characterize the optimal stopping curve u by a differential equation. The following lemma will be needed in the case $c = -\infty$.

Lemma 2.3 *Let N fulfill the boundedness condition (B) and define*

$$v(t) := \lim_{x \downarrow c} u(t, x), \quad t \in [0, 1].$$

If for some continuous function $w : [0, 1) \to \mathbb{R}$ holds $v \geq w > f$ on $[0, 1)$, then $v = u$. In particular the separation condition (S) is fulfilled.

Proof: For $t \in [0, 1]$ let $T_v(t) := \inf\{\tau_k > t : \ Y_k > v(\tau_k)\}$, $\inf \emptyset := 1$ denote the threshold stopping time of v. This is a stopping time for N since $v \geq w > f$ on $[0, 1)$. Then $u \leq v \leq u(\cdot, x)$ and thus $T_v(t) \leq T(t, x)$. Further $\overline{Y}_{T(t,x)} \vee x \to \overline{Y}_{T_v(t)} \vee c$ P-a.s. for $x \downarrow c$. This follows from our modified definition of $T(t, x) = \inf\{\tau_k > t : Y_k > u(\tau_k, x)\}$ (in comparison to the usual '$\geq$' definition) and using that the thresholds $u(\cdot, x)$ converge to v. Thus by Fatou's Lemma it follows

$$u(t) \leq v(t) = \lim_{x \downarrow c} u(t, x) = \lim_{x \downarrow c} E[\overline{Y}_{T(t,x)} \vee x] \leq E[\overline{Y}_{T_v(t)} \vee c] \leq u(t). \qquad \square$$

For any continuous function $v : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ with $v > f$ on $[0, 1)$ and its threshold stopping times

$$T_v(t) := \inf\{\tau_k > t : \ Y_k > v(\tau_k)\}, \quad \inf \emptyset := 1,$$

for $t \in [0, 1)$, the Poisson assumption allows us to calculate the joint distribution of $(T_v(t), \overline{Y}_{T_v(t)})$. By standard arguments for Poisson processes we obtain

Lemma 2.4 *Let N satisfy (B). Then the distribution of $(T_v(t), \overline{Y}_{T_v(t)})$ has on $[0,1)\times\mathbb{R}$ the Lebesgue density*

$$F_t(s,z) := \begin{cases} e^{-\mu((t,s]\times\mathbb{R}\cap M_v)} g(s,z)\chi_{M_v}(s,z), & \text{if } s\in(t,1),\\ 0, & \text{if } s\in[0,t]. \end{cases} \tag{2.6}$$

In the sequel we will need the following *differentiability condition* on the density g of μ:

(D) Assume that there exists a version of the density g of μ on M_f such that the *intensity function*

$$G(t,y) := \int_y^\infty g(t,z)\,dz \tag{2.7}$$

is continuous on $M_f\cap[0,1)\times\mathbb{R}$. Furthermore we assume that $\lim_{y\to\infty} y\,G(t,y)=0$ for all $t\in[0,1)$.

Based on Lemma 2.4 we next prove that for $x\in[c,\infty)$ the *optimality equation* for a threshold function v

$$\begin{aligned} v(t) &= E[\overline{Y}_{T_v(t)}\vee x], \quad t\in[0,1),\\ v(1) &= x, \end{aligned} \tag{2.8}$$

which by Theorem 2.2 is fulfilled in particular for the optimal stopping curve $u(\cdot,x)$ is essentially equivalent to a differential equation of first order:

$$\begin{aligned} v'(t) &= -\int_{v(t)}^\infty G(t,y)\,dy, \quad t\in[0,1),\\ v(1) &= x. \end{aligned} \tag{2.9}$$

To be able to apply the differentiation and integration rules needed in the proof of this equivalence we assume the differentiability condition (D). In the following we give a simplified proof of Theorem 2.5 in [KR] (2000a) and add essential information on the important case that $c=-\infty$. In this case a solution of the differential equation (2.9) does not need to satisfy the optimality equation (2.8), but we give a formula for the difference between $v(t)$ and the expected value $E\overline{Y}_{T_v(t)}$, which will later be used to derive uniqueness results for the differential equation (2.9).

Proposition 2.5 (Equivalence of optimality equation and differential equation) *Let N fulfill (B) and (D), let $v:[0,1]\to\mathbb{R}\cup\{-\infty\}$ be continuous with $v>f$ on $[0,1)$ and $x\in[c,\infty)$.*

(a) If v satisfies the optimality equation (2.8)*, then it satisfies also the differential equation* (2.9)*.*

(b) If $x\in\mathbb{R}$ and v satisfies the differential equation (2.9)*, then v satisfies the optimality equation* (2.8)*.*

(c) If $x = -\infty$ and v satisfies the differential equation (2.9), then for $t \in [0,1)$

$$v(t) - \lim_{s \uparrow 1} v(s) e^{-\mu((t,s]\times\mathbb{R}\cap M_v)} = E\overline{Y}_{T_v(t)}. \tag{2.10}$$

As v is assumed to be continuous and $v(1) = -\infty$, this implies $v(t) \le E\overline{Y}_{T_v(t)}$.

Proof: We shall make use of the partial integration formula stating for $p \ge 0$ measurable and satisfying $\int_a^\infty z p(z)\, dz < \infty$ and $\lim\limits_{y\to\infty} y \int_y^\infty p(z)\, dz = 0$ for $a \in \mathbb{R}$ that

$$\int_a^\infty z p(z)\, dz = a \int_a^\infty p(z)\, dz + \int_a^\infty \int_y^\infty p(z)\, dz\, dy. \tag{2.11}$$

(a) For $t < 1$ holds with $T(t) := T_v(t)$

$$v(t) = E[\,\overline{Y}_{T(t)} \vee x] = E[\,\overline{Y}_{T(t)} \chi_{\{T(t)<1\}}] + xP(T(t) = 1). \tag{2.12}$$

Since $v(t) > -\infty$ for $t < 1$ we obtain $P(T(t) = 1) = 0$ if $x = -\infty$. W.l.g. in case $x > -\infty$ we can assume that $x = 0$ (by shifting the point process and v by $-x$). In consequence we obtain by (2.11)

$$\begin{aligned} v(t) = E[\overline{Y}_{T(t)} \chi_{\{T(t)<1\}}] &= \int_t^1 \int_{v(s)}^\infty z g(s,z)\, dz e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds \\ &= \int_t^1 \left(v(s)h(s) + \int_{v(s)}^\infty \int_y^\infty g(s,z)\, dz\, dy \right) e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds, \end{aligned} \tag{2.13}$$

where $h(s) := \int_{v(s)}^\infty g(s,z)\, dz = G(s, v(s))$.

By (D) and continuity of v we get that h is continuous. For $f(s,t)$ differentiable in s and continuous in t holds

$$\frac{d}{dt} \int_t^1 f(s,t)\, ds = \int_t^1 \frac{d}{dt} f(s,t)\, ds - f(t,t).$$

Since $-\mu((t,s]\times\mathbb{R}\cap M_v) = -\int_t^s \int_{v(r)}^\infty g(r,z)\, dz dr = -\int_t^s h(r) dr$ is differentiable in t with derivative $h(t)$, we obtain

$$v'(t) = v(t)h(t) - \left(v(t)h(t) + \int_{v(t)}^\infty \int_y^\infty g(t,z)\, dz\, dy \right) = -\int_{v(t)}^\infty G(t,y)\, dy.$$

(b) Let $t < r \le 1$. Similarly as in (a) and using condition (D) we obtain

$$\begin{aligned} &E[\overline{Y}_{T(t)} \chi_{\{T(t)<r\}}] \\ &\quad = \int_t^r \big(v(s)h(s) + \int_{v(s)}^\infty \int_y^\infty g(s,z)\, dz\, dy \big) e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds \end{aligned}$$

$$
\begin{aligned}
&= \int_t^r \Big(v(s)h(s) - v'(s) \Big) e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds \\
&= \int_t^r v(s)h(s) e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds \\
&\quad - \Big([v(s)e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}]_t^r + \int_t^r v(s)h(s)e^{-\mu((t,s]\times\mathbb{R}\cap M_v)}\, ds \Big) \\
&= v(t) - v(r)e^{-\mu((t,r]\times\mathbb{R}\cap M_v)}.
\end{aligned}
$$

With $r = 1$ and using that $P(T(t) = 1) = P(N((t,1] \times \mathbb{R} \cap M_v) = 0) = e^{-\mu((t,1]\times\mathbb{R}\cap M_v)}$ we obtain (b) as $v(1) = x$.

(c) For $t < 1$ we obtain as in (b) using the Fatou Lemma

$$-\infty < v(t) \le \limsup_{r\uparrow 1} E[\overline{Y}_{T(t)}\chi_{\{T(t)<r\}}] \le E[\overline{Y}_{T(t)}\chi_{\{T(t)<1\}}].$$

Thus $\overline{Y}_{T(t)}\chi_{\{T(t)<1\}}$ is integrable. This implies uniform integrability of $\{\overline{Y}_{T(t)}\chi_{\{T(t)<r\}} \mid 0 < r < 1\}$, and thus convergence of expectations. Further from the proof of (b) follows by condition (B) that $\mu((t,1] \times \overline{R} \cap M_v) = \infty$ as $v(r) \downarrow -\infty$ for $r \uparrow 1$. As $P(T(t) = 1) = P(N((t,1] \times \overline{R} \cap M_v) = 0) = e^{-\mu((t,1]\times\overline{\mathbb{R}}\cap M_v)} = 0$ we obtain $T(t) < 1$ P-a.s. and thus

$$v(t) \le v(t) - \lim_{s\uparrow 1} v(s)e^{-\mu((t,s]\times\mathbb{R}\cap M_v)} = E\overline{Y}_{T_v(t)}. \qquad \square$$

Based on the equivalence in Proposition 2.5 we now establish that the optimal stopping curve can be described as solution of the differential equation (2.9) of first order. We will see that the separation condition (S) is equivalent to the existence of a solution $v > f$ on $[0,1)$ with $v(1) = c$. This also holds true in the case $c = -\infty$. At first we treat the case of a finite guarantee value.

Proposition 2.6 *Let $x \in [c,\infty) \cap \mathbb{R}$ and let N satisfy (B) and (D).*

(a) If v_1, $v_2 : [0,1] \to \mathbb{R}$ with $v_i > f$ on $[0,1)$ are solutions of the differential equation (2.9), then $v_1 = v_2$.

(b) If a solution $v : [0,1] \to \mathbb{R}$ of (2.9) exists such that $v > f$ on $[0,1)$, then $u(\cdot,x) = v$. In particular, if $x = c$, then the separation condition (S) is fulfilled.

Proof:

(a) see [KR] (2000a, p. 310)

(b) By Proposition 2.5 (b) v satisfies the optimality equation (2.8). By definition of $u(\cdot,x)$ follows $u(t,x) \ge E[\overline{Y}_{T_v(t)} \vee x] = v(t) > f(t)$ for all $t \in [0,1)$. By Theorem 2.2 and Proposition 2.5 (a) $u(\cdot,x)$ solves (2.9) and by part (a) we thus obtain $u(\cdot,x) = v$. $\square$

In contrast to the case of a finite guarantee value x uniqueness of solutions of (2.9) does not hold for $x = -\infty$. The following theorem identifies the optimal stopping curve in the set of all solutions of (2.9) as the largest one. It also gives a criterion for uniqueness.

Theorem 2.7 *Let $c = x = -\infty$ and let N satisfy (B) and (D). Also let $v : [0,1] \to \mathbb{R} \cup \{-\infty\}$ be a solution of the differential equation (2.9) with $v > f$ on $[0,1)$. Then*

(a) $v \le u$. In particular, the separation condition (S) is fulfilled and so u is also a solution of (2.9).

(b) If for some function $b : [0,1) \to \mathbb{R}$ we have $u \le b$ (as e.g. for $b(t) := E[\sup_{\tau_k > t} Y_k]$) and
$$\liminf_{t \uparrow 1} \frac{v(t)}{b(t)} < \infty, \tag{2.14}$$
then $u = v$. If (2.14) holds true with v replaced by f, then the solution of (2.9) is uniquely determined.

(c) Let u_s denote the optimal stopping curve of $N_s := N(\cdot \cap [0,s] \times \overline{\mathbb{R}} \cap M_f)$, let $b : [0,1) \to \mathbb{R}$ satisfy $u \le b$ and assume that for every $s \in (1-\varepsilon, 1)$ with some $\varepsilon > 0$ there exists a function $a_s : [0,s) \to \mathbb{R}$ with $f < a_s \le u_s$ on $[0,s)$ such that
$$\liminf_{t \uparrow 1} \frac{\limsup_{s \uparrow 1} a_s(t)}{b(t)} < \infty. \tag{2.15}$$
Then the solution of (2.9) is unique.

Proof:

(a) follows from Proposition 2.5 (c).

(b) By Lemma 2.1 u is continuous and therefore u solves equation (2.9). If $v \neq u$ then as in the proof of Proposition 2.6 it follows that $u > v$ and $u' \ge v'$ on $[0,1)$. With $w(s) := u(s) - v(s)$ for $s \in [0,1)$ we have
$$w'(s) = u'(s) - v'(s) = \int_{v(s)}^{u(s)} \int_y^\infty g(s,x)\,dx\,dy \ge w(s) \int_{u(s)}^\infty g(s,x)\,dx$$
and thus
$$\frac{\partial}{\partial s} \log(w(s)) = \frac{w'(s)}{w(s)} \ge \int_{u(s)}^\infty g(s,x)\,dx.$$
By integration we get
$$w(t) \ge w(0) e^{\mu([0,t] \times \mathbb{R} \cap M_u)}. \tag{2.16}$$
Since for $v = u$ equality holds in (2.10) we obtain
$$\lim_{t \uparrow 1} u(t) e^{-\mu([0,t] \times \mathbb{R} \cap M_u)} = 0 \tag{2.17}$$
and thus
$$\frac{v(t)}{u(t)} - 1 = -\frac{w(t)}{u(t)} \ge w(0) \frac{1}{-u(t) e^{-\mu([0,t] \times \mathbb{R} \cap M_u)}} \to \infty \text{ for } t \uparrow 1.$$

Since $u(t) \le b(t)$ it follows that

$$\frac{v(t)}{b(t)} \ge \frac{v(t)}{u(t)} \to \infty \text{ as } t \uparrow 1$$

contradicting our assumption. Thus v is the optimal stopping curve. If (2.14) holds for f then it holds also for any solution $v > f$. Thus uniqueness follows.

(c) For the proof of (c) see [F] (2009). □

To verify the separation condition (S) and thus the existence of a solution of (2.9) (which is an assumption of Theorem 2.7) or to construct the functions a_s used in part (c) of the theorem, one can use a comparison argument given in the following proposition.

Proposition 2.8 *Let N, N_* be Poisson processes on M_f which satisfy (B), (D), with intensity functions G, G_* and optimal stopping curves $u(t,x)$ and $u_*(t,x)$. Further let N_* satisfy (S) and let $u(\cdot,x) > f$ for all $x > c$. Then for any $s \in [0,1)$, $G \ge G_*$ on $[s,1)\times\mathbb{R}\cap M_f$ implies that $u(t,x) \ge u_*(t,x)$ for all $(t,x) \in [s,1]\times[c,\infty]$. In particular, if $G \ge G_*$, then (S) is also satisfied for N.*

Proof: Assume first that $x \in \mathbb{R}$. For any $t \in [s,1)$ with $u(t,x) < u_*(t,x)$ holds $u'(t,x) \le u'_*(t,x)$ since

$$\begin{aligned} u'(t,x) &= -\int_{u(t,x)}^{\infty} G(t,y)\,dy \le -\int_{u_*(t,x)}^{\infty} G(t,y)\,dy \\ &\le -\int_{u_*(t,x)}^{\infty} G_*(t,y)\,dy = u'_*(t,x). \end{aligned}$$

Assume that for some $r \in [s,1)$ holds $u(r,x) < u_*(r,x)$.

Since $u(1,x) = u_*(1,x) = x$, there exists a $t_0 \in (r,1]$ such that $u(t_0,x) = u_*(t_0,x)$ and $u(t,x) < u_*(t,x)$ for all $t \in [r,t_0)$. This implies

$$u(t_0,x) - u(r,x) = \int_r^{t_0} u'(t,x)dt \le \int_r^{t_0} u'_*(t,x)dt = u_*(t_0,x) - u_*(r,x),$$

and thus $u(r,x) \ge u_*(r,x)$.

In case $x = -\infty$ we obtain from Lemma 2.3 $u \ge u_* > f$ on $[s,1)$. □

For some applications of this comparison principle see [F] (2009).

Example 2.9 *Let the intensity function be of the form $G(t,y) = A(t)e^{-B(t)y}$ on $[0,1)\times(-\infty,\infty]$ with continuous functions $A, B : [0,1) \to \mathbb{R}$ such that $A \ge 0$, $A(t) > 0$ for t large, $\int_0^1 A(t)dt < \infty$, and $B > 0$ bounded such that $\liminf_{t\uparrow 1} B(t) > 0$. Then we can compare G, G_**

$$G(t,y) \ge G_*(t,y) := \begin{cases} A(t)e^{-My}, & \text{if } y \ge 0, \\ A(t)e^{-my}, & \text{if } y < 0, \end{cases}$$

where $M := \sup B$, $m := \inf B$. *Thus by Proposition 2.8 and using the terminology of Theorem 2.7(c) we obtain*

$$u_s(t) \geq a_s(t) := u_{*,s}(t) = \frac{1}{m} \log \left(\frac{1}{d} \left(1 - e^{-d \int_t^s A(r) dr} \right) \right)$$

with $d := 1 - \frac{m}{M} > 0$ *for* $t < s$ *large enough. Similarly by estimating* G *from above*

$$u(t) \leq b(t) := \frac{1}{M} \log \left(\frac{1}{d'} \left(1 - e^{-d' \int_t^1 A(r) dr} \right) \right)$$

with $d' := 1 - \frac{M}{m} < 0$. *This implies condition* (2.15) *as* $\lim_{x \downarrow 0} \log(1 - e^{-x}) / \log(e^{vx} - 1) < \infty$ *for any* $v > 0$. *As consequence uniqueness of the solutions of* (2.9) *follows.*

With $w(t) := e^{u(t)}$ *we obtain as a particular consequence that the differential equation*

$$\begin{aligned} w'(t) &= -A(t) w(t)^{1-B(t)}, \qquad t \in [0,1), \\ w(1) &= 0 \end{aligned} \tag{2.18}$$

has a unique solution w *such that* $w > 0$ *on* $[0, 1)$.

3 Explicit solutions of the optimality equation

In Section 2 the optimal threshold function has been characterized by the differential equation (2.9), which by the results in Section 2 we therefore also call *optimality equation*. In the case that the intensity function G is separable, i.e. $G(t, y) = a(t)H(y)$ a characterization and existence of solutions is given by a classical result on differential equations in separate variables (see [KR] (2000a, Prop. 2.6)). Note however that even in this case the characterization is in general far from giving an 'explicit' form of the solution. The second main point in this paper is the introduction of some new classes of intensity functions $G(t, y)$ which allow to establish 'explicit' solutions of the optimality equation (2.7). An important class of applications of this development is given in Section 5 of this paper to optimal stopping of i.i.d. sequences with discount and observation costs. Further interesting applications of this development to a general treatment of multistopping problems are given in [FR] (2010).

In the following we will introduce two classes of intensity functions, which allow to give an explicit form of the solutions of (2.9). These intensity functions are of form

$$G(t, y) = H \left(\frac{y}{v(t)} \right) \frac{v'(t)}{v(t)} \tag{3.1}$$

resp.

$$G(t, y) = H(y - v(t)) v'(t). \tag{3.2}$$

with functions H, v. It should be noted that the functions v used here are different from the threshold functions used in the last paragraph. The function v in the representation of $G(t, y)$ is in general not unique. The main point of exhibiting these kind of representations in (3.1), (3.2) is that they imply that v is up to a normalization concerning the boundary value a solution of the optimality equation (2.9). Thus (3.1), (3.2) are a particular useful representation of G concerning solutions of (2.9). To see this connection note that the optimality equation for u can be written by substitution in the equivalent forms:

$$\begin{aligned}u'(t) &= -\int_{u(t)}^{\infty} G(t,y)\,dy = -\int_{1}^{\infty} G(t,yu(t))u(t)\,dy \\ &= -\int_{0}^{\infty} G(t,y+u(t))\,dy.\end{aligned}$$

In both representations (3.1), (3.2) v is therefore verified to be up to a normalizing constant and up to the boundary condition a solution of equation (2.9). This crucial observation motivates the representation in (3.1), (3.2). We will see in the following that for both forms (3.1), (3.2) under some conditions on H, v explicit solutions of the essential differential equation (2.9) can be found for any boundary value. This needs a detailed study of several cases.

Let $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ be a continuous lower boundary function and as before $c := f(1)$. Let N be a Poisson process on M_f with intensity function G which fulfils (B), (D).

3.1 First class of intensity functions

Let $f = av$ on $[0,1)$ with $a \in \mathbb{R} \cup \{-\infty\}$ and a monotone C^1-function $v : [0,1) \to \mathbb{R}$, $v > 0$, and assume that G is of the form

$$G(t,y) = H\left(\frac{y}{v(t)}\right)\frac{|v'(t)|}{v(t)} \tag{3.3}$$

with some monotone nonincreasing continuous function $H : (a,\infty] \to \mathbb{R}$, $H \geq 0$. Assume that $\int_a^\infty H(y)\,dy > 0$ and let v be not constant in $(1-\varepsilon, 1)$ for some $\varepsilon > 0$, so that 1 is the critical point for c. Define $v(1) := \lim_{t\uparrow 1} v(t) \in \overline{\mathbb{R}}$. The following example shows how the case of separate variables fits into the form in (3.3).

Example 3.1 *Let $a : [0,1] \to [0,\infty]$ be continuous and integrable and define $A(t) := \int_t^1 a(s)\,ds$.*

(a) For $\alpha > 1$ the intensity function $G(t,y) = a(t)y^{-\alpha}$ on $[0,1) \times (0,\infty]$ is of the form in (3.3) with $H(y) = (\alpha-1)y^{-\alpha}$, $v(t) = \left(\frac{\alpha}{\alpha-1}A(t)\right)^{\frac{1}{\alpha}}$.

(b) For $\alpha > 0$ the intensity function $G(t,y) = a(t)(-y)^{\alpha}$ for $y \leq 0$ and $G(t,y) = 0$ for $y > 0$ on $[0,1) \times (-\infty,\infty]$ is of the form in (3.3) with $H(y) = 0$ for $y > 0$ and $H(y) = (\alpha+1)(-y)^{\alpha}$ for $y \leq 0$, $v(t) = \left(\frac{\alpha}{\alpha+1}A(t)\right)^{-\frac{1}{\alpha}}$.

Representation (3.3) is in general even for fixed H not unique since v only has to satisfy a differential equation without initial value. It depends on H for which initial values solutions can be found. We distinguish three cases[2]: $v(1) = 0$, $v(1) = 1$, and $v(1) = \infty$. The cases $v(1) = 0$ and $v(1) = \infty$ lead to a simpler structure of solutions.

For v monotonically nonincreasing we define

$$R(x) := x - \int_x^\infty H(y)\,dy, \quad x \in [a, \infty). \tag{3.4}$$

Then $R : [a, \infty) \to \overline{\mathbb{R}}$ is concave, monotonically nondecreasing.

For v monotonically nondecreasing we define

$$\begin{aligned} R(x) &:= x + \int_x^\infty H(y)\,dy, \quad x \in (a, \infty) \\ R(a) &:= \lim_{x \downarrow a} R(x). \end{aligned} \tag{3.5}$$

In this case $R : [a, \infty) \to \overline{\mathbb{R}}$ is convex.

The form of solutions of the optimality equation (2.9) with boundary value x depends critically on the existence and on the number of zero points of R. In the following we reduce the problem of solving the optimality equation (2.9) for all boundary values x to finding solutions Φ of the differential equation

$$\Phi'(x) = \frac{\Phi(x)}{R(x)} \neq 0. \tag{3.6}$$

Solutions of (3.6) are given e.g. by

$$\Phi(x) = \exp\left(\int_{x_0}^x \frac{1}{R(y)}\,dy \right), \tag{3.7}$$

with x_0 chosen such that the integral exists. The inverse mapping ϕ of Φ exists and solves the equation

$$\phi'(z) = \frac{R(\phi(z))}{z}. \tag{3.8}$$

The definition of R in (3.4), (3.5) and solutions of (3.6), (3.8) will allow us in the following to construct solutions of the optimality equation (2.9) for any boundary values and to verify the separation condition (S).

In the following we omit some of the simple calculations. We distinguish four cases:

[2] If $v(1) = d \in (0, \infty)$, consider $\tilde{v}(t) := v(t)/d$ and absorb the d into H by considering $\tilde{H}(x) := H(x/d)$.

C1) v monotonically nonincreasing, $v(1) = 0$
Then $c = 0$. Let $a \geq 0$ and assume $R(r) = 0$ for some $r > a$. Then the separation condition (S) is fulfilled and the optimal stopping curves are given by

$$u(t,x) = \phi\left(\frac{x}{v(t)}\right) v(t), \quad (t,x) \in [0,1) \times [0,\infty), \tag{3.9}$$

where $\phi : [0,\infty) \to [r,\infty)$ is the inverse to

$$\Phi : [r,\infty) \to [0,\infty), \quad \Phi(x) = x \exp\left(-\int_x^\infty \left(\frac{1}{R(y)} - \frac{1}{y}\right) dy\right).$$

In particular the optimal solution u is given explicitly as $u(t) = rv(t)$.

Proof: Φ satisfies the differential equation (3.6), ϕ satisfies (3.8). We first establish that u satisfies the optimality equation (2.9). By definition

$$u(t,x) = \phi\left(\frac{x}{v(t)}\right) \frac{v(t)}{x} \cdot x \stackrel{t\to 1}{\longrightarrow} x,$$

since $\lim_{y\to\infty} \frac{\phi(y)}{y} = \lim_{x\to\infty} \frac{x}{\Phi(x)} = 1$. Thus the boundary condition is fulfilled. Further

$$\begin{aligned}
\frac{\partial}{\partial t} u(t,x) &= \phi'\left(\frac{x}{v(t)}\right) \frac{-xv'(t)}{v(t)^2} \cdot v(t) + \phi\left(\frac{x}{v(t)}\right) v'(t) \\
&= \frac{R(\phi(\frac{x}{v(t)}))}{\frac{x}{v(t)}} \frac{-xv'(t)}{v(t)} + \phi\left(\frac{x}{v(t)}\right) v'(t) \\
&= \left(-\phi\left(\frac{x}{v(t)}\right) + \int_{\phi(\frac{x}{v(t)})}^\infty H(y)\,dy\right) v'(t) + \phi\left(\frac{x}{v(t)}\right) v'(t) \\
&= \int_{\phi(\frac{x}{v(t)})v(t)}^\infty H\left(\frac{y}{v(t)}\right) \frac{v'(t)}{v(t)}\,dy = -\int_{u(t,x)}^\infty G(t,y)\,dy.
\end{aligned} \tag{3.10}$$

Thus $u(\cdot,x)$ solves the optimality equation (2.9).

It remains to show that $\int_x^\infty \left(\frac{1}{R(y)} - \frac{1}{y}\right) dy < \infty$ for $x \in (r,\infty)$. With $I(y) := \int_y^\infty H(x)\,dx$ we have

$$\frac{1}{R(y)} - \frac{1}{y} = \frac{I(y)}{y(y - I(y))} = \frac{1}{y^2} \frac{I(y)}{1 - \frac{I(y)}{y}} \leq C \frac{1}{y^2}$$

and thus the integral is finite. □

C2) v monotonically nondecreasing, $v(1) = \infty$
In this case we have $c = -\infty$. Let $a < 0$ and assume $R(r) = 0$ for some $a < r < 0$. We assume in this case also that

$$\int_0^\infty H(x)\,dx = 0 \quad \text{and} \quad \int_y^0 \frac{H(x)}{-x}\,dx < \infty \quad \text{for } y < 0.$$

Under this assumption the separation condition (S) is fulfilled and the optimal stopping curves are given for $(t, x) \in [0, 1) \times \overline{\mathbb{R}}$ by

$$u(t,x) = \begin{cases} x, & \text{if } x \geq 0, \\ \phi\left(\dfrac{x}{v(t)}\right) v(t), & \text{if } x < 0, \end{cases} \tag{3.11}$$

with $\phi : [-\infty, 0] \to [r, 0]$ the inverse of

$$\Phi : [r, 0] \to [-\infty, 0], \quad \Phi(x) := x \exp\left(\int_x^0 \left(\frac{1}{y} - \frac{1}{R(y)}\right) dy\right).$$

In particular, the optimal solution u is given by $u(t) = rv(t)$.

Proof: Φ solves the differential equation (3.6), while the inverse ϕ solves (3.8). We have to establish that $\int_x^0 (\frac{1}{y} - \frac{1}{R(y)})\, dy < \infty$ for $x \in (r, 0)$. Again with $I(y) := \int_y^0 H(x)\, dx$ we obtain the estimate

$$\frac{1}{y} - \frac{1}{R(y)} = \frac{I(y)}{y(y + I(y))} = \frac{1}{-y} \frac{\frac{I(y)}{-y}}{1 - \frac{I(y)}{-y}} \leq \frac{1}{-y} \frac{H(y)}{1 - H(y)} \leq C \frac{H(y)}{-y}$$

for $y < 0$ with $H(y) \leq 1 - \frac{1}{C} < 1$. By assumption this is integrable. As in case C1) we find by similar calculations that $u(t, x)$ satisfies the optimality equation (2.9). Thus the result follows. □

The following cases are derived in a similar way as in C1) and C2). We therefore only state the results.

C3) v monotonically nonincreasing, $v(1) = 1$
Then $c = a$. Let $r > c$ such that $R(r) = 0$. We assume that $\int_c^x R(y)^{-1}\, dy > -\infty$ for some $x \in (c, r)$. This is e.g. the case when $c \in \mathbb{R}$. Under this assumption the separation condition (S) is fulfilled and the optimal stopping curves are given for $(t, x) \in [0, 1) \times [c, \infty)$ by

$$u(t,x) = \begin{cases} \phi_1\left(\frac{\Phi_1(x)}{v(t)}\right) v(t), & \text{if } r < x < \infty, \\ xv(t), & \text{if } x = r, \\ \phi_2\left(\frac{\Phi_2(x)}{v(t)}\right) v(t), & \text{if } c \leq x < r, \end{cases} \tag{3.12}$$

where $\Phi_1 : (r, \infty) \to \mathbb{R}$ and $\Phi_2 : [c, r) \to \mathbb{R}$ are solutions of (3.6) and ϕ_1, ϕ_2 are the inverses. Φ_2 can be chosen as $\Phi_2(x) := \exp\left(\int_c^x R(y)^{-1}\, dy\right)$.

The boundary case $x = r$ is particularly simple here.

C4) v monotonically nondecreasing, $v(1) = 1$
Then $c = a$. We have to distinguish three cases. In each of these cases the solution is similar to C3) and will therefore be omitted.

As result one sees that in the cases $v(1) = 0$ and $v(1) = \infty$ a solution of a simpler structure compared to $v(1) = 1$ can be given. Thus a representation of G as in (3.3) – which is not unique – is preferable if it leads to one of the first two cases.

3.2 Second class of intensity functions

Let $f = a+v$ on $[0,1)$ with $a \in \mathbb{R}\cup\{-\infty\}$ and a monotone C^1-function $v : [0,1) \to \mathbb{R}$ with $v(1) := \lim_{t\uparrow 1} v(t)$.

We consider intensity functions on $M_f \cap [0,1) \times \overline{\mathbb{R}}$ of the form

$$G(t,y) = H(y - v(t))|v'(t)| \tag{3.13}$$

with a continuous monotonically nonincreasing function $H : (a,\infty] \to \mathbb{R}, H \geq 0$ such that $\int_a^\infty H(y)\,dy > 0$. We assume that v is not constant in $(1-\varepsilon, 1)$ for some $\varepsilon > 0$, so that 1 is the critical point of c.

As an example let $a : [0,1] \to [0,\infty]$ be continuous and integrable and let, as in Example 3.1, $A(t) := \int_t^1 a(s)\,ds$. Then the intensity function $G(t,y) = a(t)e^{-y}$ on $[0,1) \times (-\infty,\infty]$ is a case of separate variables. It fits into the form (3.13) with $H(y) = e^{-y}$, $v(t) = \log A(t)$.

If v is monotonically nonincreasing, then we define

$$R(x) := 1 - \int_x^\infty H(y)\ dy \quad \text{ for } x \in [a,\infty).$$

$R : [a,\infty) \to [-\infty, 1)$ is concave, monotonically nondecreasing. If v is monotonically nondecreasing, then define

$$R(x) := 1 + \int_x^\infty H(y)\,dy \quad \text{for } x \in [a,\infty).$$

In this case $R : [a,\infty) \to [1,\infty)$ is convex and monotonically nonincreasing.

We construct optimal stopping curves by means of solutions of the equation

$$\Phi'(x) = \frac{1}{R(x)}. \tag{3.14}$$

(3.14) is solved e.g. by $\Phi(x) = \int_{x_0}^x \frac{1}{R(y)}\,dy$, where x_0 is chosen such that the integral exists. The inverse function ϕ of Φ satisfies

$$\phi'(z) = R(\phi(z)). \tag{3.15}$$

Similarly to the examples in C1)–C4) we obtain explicit forms of the solution $u(t,x)$ of the optimal stopping curves. The arguments are similar and we therefore essentially only state the results.

D1) v monotonically nonincreasing, $v(1) = -\infty$
Then $c = -\infty$. Assume that $R(r) = 0$ for some $r > a$. We further assume here that

$$\int_z^\infty \int_y^\infty H(x)\,dx\,dy < \infty \qquad \text{for } z > r.$$

Then the separation condition (S) is fulfilled and the optimal stopping curves for $(t,x) \in [0,1) \times \overline{\mathbb{R}}$ are given by

$$u(t,x) = \phi(x - v(t)) + v(t), \tag{3.16}$$

where $\phi : \overline{\mathbb{R}} \to [r,\infty]$ is the inverse of $\Phi : [r,\infty] \to \overline{\mathbb{R}}$

$$\Phi(x) := x - \int_x^\infty \Big(\frac{1}{R(y)} - 1\Big)\,dy.$$

The optimal stopping curve u is given by $u(t) = r + v(t)$.

For the proof note that Φ solves the differential equation $\Phi'(x) = \frac{1}{R(x)}$, ϕ solves $\phi'(z) = R(\phi(z))$. In consequence as in case C1) we find that $u(\cdot,x)$ solves the optimality equation (2.9) with boundary value x. We still need to show that $\int_x^\infty \big(\frac{1}{R(y)} - 1\big)\,dy < \infty$ for $x \in (r,\infty)$. With $I(y) := \int_y^\infty H(x)\,dx$ we obtain the bound

$$\frac{1}{R(y)} - 1 = \frac{I(y)}{1 - I(y)} \le C I(y).$$

The last term is integrable by assumption.

The next two cases allow similar explicit solutions but are not used in the applications in Section 5 and therefore are not explicitly stated. For details see [F] (2009).

D2) v monotonically decreasing, $v(1) = 0$
D3) v monotonically increasing, $v(1) = 0$

Remark 3.2 *One can extend the class of intensity functions for which solutions can be given in a simple way by translations. Let N satisfy conditions (B), (S), and (D) with intensity function G on M_f. For $d \in \mathbb{R}$ consider the intensity function*

$$G_d(t,y) := G(t, y-d), \quad (t,y) \in M_{f+d}.$$

Then the optimal stopping curves $u_d(\cdot,x)$ w.r.t. G_d and $c_d := c + d$ are given by

$$u_d(t,x) = u(t, x-d) + d \text{ for } (t,x) \in [0,1] \times [c_d,\infty]. \tag{3.17}$$

For $x \in \mathbb{R}$ (3.17) follows by simple calculation. For $x = -\infty$ this follows by means of Lemma 2.3.

As application of Remark 3.2 we consider the following example which is relevant in Section 5 for the stopping of i.i.d. sequences with discount and observation costs.

Example 3.3 *In this example $c \in \mathbb{R}$ denotes a real constant and the guarantee value is $-\infty$. Consider on $[0,1) \times \overline{\mathbb{R}}$*

$$G_{c,d}(t,y) = \begin{cases} 0, & \text{if } \frac{y}{v(t)} \geq d, \\ \frac{1}{t}\left(-\frac{y}{v(t)} + d\right)^{\alpha}, & \text{if } \frac{y}{v(t)} < d, \end{cases} \tag{3.18}$$

with $v(t) := t^{c-\frac{1}{\alpha}}$, where $\alpha > 0$ and $c, d \in \mathbb{R}$ with $c \neq \frac{1}{\alpha}$. These intensity functions fulfill (3.3) with

$$H(x) := \begin{cases} 0, & \text{if } x \geq d, \\ \frac{\alpha}{|1-c\alpha|}(-x+d)^{\alpha}, & \text{if } x < d. \end{cases}$$

By cases C3) and C4) the optimal stopping curve $u_{c,d}$ of the Poisson process $N = N_{c,d}$ with intensity function $G = G_{c,d}$ where $u_{c,d}(t) := u_{c,d}(t, -\infty)$ is given by

$$u_{c,d}(t) = \phi_{c,d}\left(\frac{1}{v(t)}\right) v(t). \tag{3.19}$$

Here $\phi_{c,d}$ is the inverse of

$$\Phi_{c,d}(x) := \exp\left(\int_{-\infty}^{x} \frac{1}{R_{c,d}(y)}\, dy\right).$$

$\Phi_{c,d}$ is defined on $[-\infty, r]$, where r is the smallest zero point of

$$R_{c,d}(x) := \begin{cases} x, & \text{if } x \geq d, \\ x - \frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}(-x+d)^{\alpha+1}, & \text{if } x < d, \end{cases}$$

resp. $r := \infty$, if no zeros exist. $\phi_{c,d}$ is defined on $[0,1]$, if $c < \frac{1}{\alpha}$, and on $[1,\infty]$, if $c > \frac{1}{\alpha}$.

For $d = 0$ all functions can be calculated explicitly. The primitive of $\frac{1}{y-c(-y)^{\alpha+1}}$ is given by $-\frac{1}{\alpha}\log|\frac{1}{c}(-y)^{-\alpha}+1|$ and as consequence one obtains

$$u_{c,0}(t) = -\left(\tfrac{\alpha}{\alpha+1}\tfrac{1}{1-c\alpha}(1-t^{1-c\alpha})\right)^{-\frac{1}{\alpha}},$$

$$u_{c,0}(t,x) = \begin{cases} x, & \text{if } x \geq 0, \\ -\left(\frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}(1-t^{1-c\alpha}) + (-x)^{-\alpha}\right)^{-\frac{1}{\alpha}}, & \text{if } x < 0. \end{cases}$$

For $d \neq 0$ and general α, $\Phi_{c,d}$ and $\phi_{c,d}$ cannot be calculated explicitly. One can, however, derive the following bounds (see [F] (2009) for details).

If $c > \frac{1}{\alpha}$, $d > 0$ or if $c < \frac{1}{\alpha}$, $d < 0$, then for all $(t,x) \in [0,1] \times \overline{\mathbb{R}}$ holds:

$$u_{c,0}(t, x - dv(t)) + dv(t) \leq u_{c,d}(t,x) \leq u_{c,0}(t, x-d) + d. \tag{3.20}$$

In the other cases $c > \frac{1}{\alpha}$, $d < 0$ or $c < \frac{1}{\alpha}$, $d > 0$ holds:

$$u_{c,0}(t, x-d) + d \leq u_{c,d}(t,x) \leq u_{c,0}(t, x - dv(t)) + dv(t). \tag{3.21}$$

In particular we obtain in all cases

$$\lim_{t\uparrow 1}(u_{c,d}(t) - u_{c,0}(t)) = d. \tag{3.22}$$

Equation (3.22) opens up another way of calculating $u_{c,d}$ numerically. In the first step one should solve the differential equation for $u_{c,d} - u_{c,0}$. This is relieved by the fact that the initial value d is finite. In the second step we just add the explicitely known constant $u_{c,0}$ to obtain $u_{c,d}$.

For the proof of (3.21) *assume that $d < 0$ and $c < \frac{1}{\alpha}$ and thus that v is monotonically nonincreasing. The other cases follow similarly. Choose $t_1 \in [0,1)$. Then for $t \in [t_1, 1)$, $\varepsilon := v(t_1) \geq v(t) > 1$ and we have*

$$G_{c,d}(t,y) \geq G_{c,0}(t, y - \varepsilon d),$$

By the comparison result (Proposition 2.8) and Remark 3.2 we obtain for the optimal stopping curves

$$u_{c,d}(t,x) \geq u_{c,0}(t, x - \varepsilon d) + \varepsilon d \text{ for } t \geq t_1.$$

This holds in particular for $t = t_1$. In the opposite direction we have for all t

$$G_{c,d}(t,y) \leq G_{c,0}(t, y - d),$$

and thus

$$u_{c,d}(t,x) \leq u_{c,0}(t, x - d) + d.$$

4 Approximation of optimal stopping problems

In this section we state an extension of the approximation result in [KR] (2004, Theorem 2.1) for optimal stopping problems for dependent sequences. In particular we add essential information on the important case $c = -\infty$. We also extend the approximation result to general filtrations which is useful when dealing with dependent sequences. In the subsequent Section 5 we give an application of this extended approximation result and the development in the previous sections to the optimal stopping of i.i.d. sequences with discount and observation costs.

Let N be as in Section 2 a Poisson process on M_f, let the intensity measure μ have Lebesgue density g, let $c := f(1)$ and $t_0(c) = 1$. $u(t,x)$, $u(t)$ denote the optimal stopping curves of N, $T(t,x)$, $\overline{Y}_{T(t,x)}$ denote the optimal stopping times and rewards and generally we assume conditions (B), (S).

Let for $n \in \mathbb{N}$, $X_1^n, \ldots, X_n^n$ be real random variables with $E(X_i^n)^+ < \infty$ adapted to a filtration $\mathcal{F}^n = (\mathcal{F}_i^n)_{0 \leq i \leq n}$ and such that $\mathcal{F}^n_{\lfloor tn \rfloor} \subset \mathcal{F}^{n+1}_{\lfloor t(n+1) \rfloor}$. For the embedded point process

$$N_n := \sum_{i=1}^{n} \delta_{(\frac{i}{n}, X_i^n)} \tag{4.1}$$

in $[0,1] \times \overline{\mathbb{R}}$ we define the optimal stopping curve w.r.t. $\mathcal{F}^n$ with guarantee value $x \in [c, \infty]$ by

$$\begin{aligned} u_n(t,x) &:= W_{\mathcal{F}^n}(X^n_{\lfloor tn \rfloor+1} \vee x, \dots, X^n_n \vee x), \quad t \in [0,1), \\ u_n(1,x) &:= x. \end{aligned} \tag{4.2}$$

Here $W_{\mathcal{F}^n}$ denotes the optimal stopping value over all $\mathcal{F}^n$-stopping times. In detail (4.2) is given by

$$\begin{aligned} u_n(t,x) &= \operatorname{ess\,sup}\big\{E[X^n_T \vee x \,|\, \mathcal{F}^n_{\lfloor tn \rfloor}] : T > tn,\ T \text{ is an } \mathcal{F}^n\text{-stopping time}\big\} \\ &= E[X^n_{T_n(t,x)} \vee x \,|\, \mathcal{F}^n_{\lfloor tn \rfloor}] \quad P\text{-a.s.} \end{aligned}$$

with optimal stopping times

$$\begin{aligned} T_n(t,x) &:= \min\{tn < i \le n : X^n_i > u_n(\tfrac{i}{n}, x)\}, \\ T_n(1,x) &:= n. \end{aligned}$$

$u_n(\cdot, x)$ is a right-continuous piecewise constant curve. It is monotone in the sense that for $0 \le s \le t \le 1$

$$u_n(s,x) \ge E\left[u_n(t,x) \,|\, \mathcal{F}^n_{\lfloor sn \rfloor}\right] \quad P\text{-a.s.} \tag{4.3}$$

In the other direction we get inductively by the recursive definition of optimal thresholds for $0 \le s \le t \le 1$

$$u_n(s,x) \le E\big[\max_{s < \frac{i}{n} \le t} X^n_i \vee u_n(t,x) \,|\, \mathcal{F}^n_{\lfloor sn \rfloor}\big] \quad P\text{-a.s.} \tag{4.4}$$

An important condition in the dependent case is the *asymptotic independence condition*

(A) For $0 \le s < t \le 1$

$$P(\max_{s < \frac{i}{n} \le t} X^n_i \vee f(s) \le x \,|\, \mathcal{F}^n_{\lfloor sn \rfloor}) \xrightarrow{P} P(\sup_{s < \tau_k \le t} Y_k \vee f(s) \le x) \quad \forall x \in \mathbb{R}.$$

We need the uniform integrability condition:

(U) M_n^+, with $M_n := \max_{1 \le i \le n} X^n_i$, is uniformly integrable and $E[\limsup_{n\to\infty} M_n^+] < \infty$.

The addition $E[\limsup_n M_n^+] < \infty$ can be omitted when $\mathcal{F}^n$ is the canonical filtration and $N_n \xrightarrow{d} N$ on $([0,1] \times \overline{\mathbb{R}}) \setminus \operatorname{graph}(f)$. In this case the Skorohod theorem is applicable and the above additional condition is a consequence of condition (B) for N which is assumed throughout this paper. To ensure in case $c = -\infty$ uniform integrability, the following *uniform integrability condition from below* is postulated which is a functional version of the corresponding condition in [KR] (2004):

(L) For some sequence $(v_n)_{n\in\mathbb{N}}$ of monotonically nonincreasing functions $v_n : [0,1] \to \mathbb{R} \cup \{-\infty\}$ with $v_n \to u$ pointwise, for all $t \in [0,1)$ and the corresponding threshold stopping times

$$\hat{T}_n(t) := \min\{tn < i \leq n : X_i^n > v_n(\tfrac{i}{n})\}$$

holds

$$\lim_{s\uparrow 1} \limsup_{n\to\infty} E[X^n_{\hat{T}_n(t)} \chi_{\{\hat{T}_n(t)>sn\}}] = 0. \tag{4.5}$$

Conditions (L), (U) imply uniform integrability of $(X^n_{\hat{T}_n(t)})_{n\in\mathbb{N}}$ (see [F] (2009, p. 30)).

For notational convenience we write

$$T_n := T_n(0,c) \quad \text{and} \quad T := T(0,c).$$

Theorem 4.1 (Approximation of stopping problems) *Assume* $N_n \xrightarrow{d} N$ *on* M_f *and conditions (A) and (U). If* $c = -\infty$*, then we additionally assume condition (L).*

1. *For all* $(t,x) \in [0,1] \times [c,\infty)$ *holds*

$$u_n(t,x) \xrightarrow{P} u(t,x). \tag{4.6}$$

If $c \in \mathbb{R}$ *and assuming* $\mu(M_u) = \infty$ *or* $X_n^n \xrightarrow{P} c$ *then*

$$(\tfrac{T_n}{n}, X^n_{T_n}) \xrightarrow{d} (T, \overline{Y}_T \vee c). \tag{4.7}$$

2. *If* $c \in \mathbb{R}$ *and* $X_n^n \xrightarrow{L^1} c$*, then* $\hat{T}_n := \min\left\{1 \leq i \leq n : X_i^n > u\left(\frac{i}{n}\right)\right\}$ *is an asymptotically optimal sequence of stopping times, i.e.*

$$EX^n_{\hat{T}_n} \to u(0).$$

If $c = -\infty$*, then* $\hat{T}_n := \min\left\{1 \leq i \leq n : X_i^n > v_n\left(\frac{i}{n}\right)\right\}$*, with* v_n *from condition (L) is an asymptotically optimal sequence of stopping times and* $EX^n_{\hat{T}_n} \to u(0)$.

For the detailed proof of this extended approximation result we refer to [F] (2009, Satz 1.20).

5 Optimal stopping of i.i.d. sequences with discount and observation costs

Based on the results in Sections 2–4 we are able to give a fairly complete treatment of the optimal stopping problem of i.i.d. sequences with discount and observation costs. Some particular instances of this problem were established in [KR] (2000b). The problem goes back to Kennedy and Kertz (1990) and Kennedy and Kertz (1991) in the i.i.d. case.

Let (Z_i) be an i.i.d. sequence with d.f. F in the domain of attraction of an extreme value distribution G, thus for some constants $a_n > 0$, $b_n \in \mathbb{R}$

$$n(1 - F(a_n x + b_n)) \to -\log G(x), \quad x \in \mathbb{R}. \tag{5.1}$$

Consider $X_i = c_i Z_i + d_i$ the sequence with discount and observation factors, $c_i > 0$, $d_i \in \mathbb{R}$ and both sequences monotonically nondecreasing or nonincreasing. For convergence of the corresponding imbedded point processes

$$\hat{N}_n = \sum_{i=1}^n \delta_{\left(\frac{i}{n}, \frac{X_i - \hat{b}_n}{\hat{a}_n}\right)} \tag{5.2}$$

the following choices of $\hat{a}_n$, $\hat{b}_n$ turn out to be appropriate:

$$\begin{aligned} &\hat{a}_n := c_n a_n,\ \hat{b}_n := 0 && \text{for } F \in D(\Phi_\alpha) \text{ or } F \in D(\Psi_\alpha), \\ &\hat{a}_n := c_n a_n,\ \hat{b}_n := c_n b_n + d_n && \text{for } F \in D(\Lambda), \end{aligned} \tag{5.3}$$

where Φ_α, Ψ_α, Λ are the Fréchet, Weibull, and Gumbel distributions and a_n, b_n are the corresponding normalizations in (5.1). We give further conditions on c_i, d_i to establish point process convergence in (5.2). Related conditions are given in de Haan and Verkaade (1987) in the treatment of i.i.d. sequences with trends resp. in [KR] (2000b).

In the following c denotes some general constant and not as before the guarantee value. The guarantee value of N is in case Φ_α given by 0 and in cases Ψ_α, Λ given generally by $-\infty$. This application shows the importance of treating the case with lower boundary $-\infty$ as in Sections 2–4 of this paper. We state the optimality results for all three cases.

Theorem 5.1 *Let $F \in D(\Phi_\alpha)$ with $\alpha > 1$ and $F(0) = 0$. Also let $b_n = 0$ and assume that*

$$\frac{d_n}{c_n a_n} \to d, \quad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \quad \forall t \in [0, 1]$$

for constants c, $d \in \mathbb{R}$ *as well that* c_n *does not converge to 0. Then*

$$\frac{EX_{T_n}}{\hat{a}_n} \to \begin{cases} \infty, & \text{if } c \le -\frac{1}{\alpha}, \\ d \exp\left(\int_d^\infty \frac{1}{x} - \frac{1}{R(x)}\, dx\right), & \text{if } c > -\frac{1}{\alpha}, d > 0, \\ \left(\frac{\alpha}{\alpha-1}\frac{1}{1+c\alpha}\right)^{\frac{1}{\alpha}}, & \text{if } c > -\frac{1}{\alpha}, d = 0, \\ \exp\left(\int_1^\infty \frac{1}{x} - \frac{1}{R(x)}\, dx - \int_d^1 \frac{1}{R(x)}\, dx\right), & \text{if } c > -\frac{1}{\alpha}, d < 0, r = \infty, \\ 0, & \text{if } c > -\frac{1}{\alpha}, d < 0, r < \infty, \end{cases} \tag{5.4}$$

where $r > d$ *is the smallest zero point of* $R(x) := x + \frac{\alpha}{\alpha-1}\frac{1}{1+c\alpha}(x-d)^{-\alpha+1}$ *,* $x \in (d, \infty)$*, resp.* $r := \infty$*, if R has no zero point* $> d$*.*

For the values of c, d *where the limit in* (5.4) *is not* 0 *or* ∞ *we determine asymptotically optimal sequences of stopping times: For* $c > -\frac{1}{\alpha}$ *define*

$$u(t) := \phi\left(\frac{1}{v(t)}\right) v(t) \tag{5.5}$$

with $v(t) := t^{c+1/a}$ *and* ϕ *the inverse of* $\Phi : [d, r] \to [1, \infty]$*, given by*

$$\Phi(x) := \exp\left(\int_d^x \frac{1}{R(y)}\, dy\right).$$

Then $\hat{T}_n := \min\{1 \le i \le n : X_i > \hat{a}_n u(\frac{i}{n})\}$ *is an asymptotically optimal sequence of stopping times, i.e. the sequence of normalized expectations has the same limit as in* (5.4)*.*

Theorem 5.2 *Let* $F \in D(\Psi_\alpha)$ *with* $\alpha > 0$ *and* $F(0) = 1$*. Also let* $a_n \downarrow 0$ *and* $b_n = 0$*, and assume that*

$$\frac{d_n}{c_n a_n} \to d, \quad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \quad \forall t \in [0,1]$$

for constants c, $d \in \mathbb{R}$*. If* $d_n > 0$*, then assume that either* $(d_n)_{n\in\mathbb{N}}$ *is monotonically nondecreasing or* $c_n a_n$ *does not converge to 0. Then*

$$\frac{EX_{T_n}}{\hat{a}_n} \to \begin{cases} \infty, & \text{if } c < \frac{1}{\alpha}, d > 0, \\ -\left(\frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}\right)^{-\frac{1}{\alpha}}, & \text{if } c < \frac{1}{\alpha}, d = 0, \\ d \exp\left(-\int_{-\infty}^d \frac{1}{R(x)}\, dx\right), & \text{if } c < \frac{1}{\alpha}, d < 0, \\ 0, & \text{if } c = \frac{1}{\alpha}, \\ d \exp\left(-\int_{-\infty}^d \frac{1}{R(x)}\, dx\right), & \text{if } c > \frac{1}{\alpha}, r = \infty (\Rightarrow d > 0), \\ 0, & \text{if } c > \frac{1}{\alpha}, r < \infty, \end{cases} \tag{5.6}$$

where r is the smallest zero point of

$$R(x) := \begin{cases} x, & \text{if } x \ge d, \\ x - \frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}(-x+d)^{\alpha+1}, & \text{if } x < d. \end{cases}$$

resp. $r := \infty$, *if* R *has no zero point.*

For the values of c, d *where the limit of* (5.6) *is not* 0 *or* ∞ *we can construct asymptotically optimal sequences of stopping times: Let* $(w_n)_{n\in\mathbb{N}}$ *be monotonically nondecreasing sequence of negative constants with* $\lim_{n\to\infty} n(1-F(w_n)) = \frac{\alpha+1}{\alpha}$, *as e.g.* $w_n := -\left(\frac{\alpha+1}{\alpha}\right)^{\frac{1}{\alpha}} a_n$. *For* $c \neq \frac{1}{\alpha}$ *let* $u_{c,d}$ *be the solutions derived in* (3.19) *and define*

$$v_n(t) := \frac{u_{c,0}(t)}{u_{0,0}(t)} \frac{w_{\lfloor (1-t)n \rfloor}}{a_n} + u_{c,d}(t) - u_{c,0}(t) \quad \text{for } t \in [0,1), \qquad v_n(1) := -\infty,$$

where $u_{c,0}(t) = -\left(\frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}(1-t^{1-c\alpha})\right)^{-\frac{1}{\alpha}}$ *and* $u_{0,0}(t) = -\left(\frac{\alpha}{\alpha+1}(1-t)\right)^{-\frac{1}{\alpha}}$.

Then $\hat{T}_n := \min\{1 \leq i \leq n : X_i > \hat{a}_n v_n(\frac{i}{n})\}$ *is an asymptotically optimal sequence of stopping times.*

Theorem 5.3 *Let* $F \in D(\Lambda)$ *and assume that*

$$\frac{b_n}{a_n}\left(1 - \frac{c_{\lfloor tn \rfloor}}{c_n}\right) \to c\log(t), \qquad \frac{d_n - d_{\lfloor tn \rfloor}}{c_n a_n} \to d\log(t) \quad \forall t \in [0,1] \tag{5.7}$$

for constants c, $d \in \mathbb{R}$. *Assume that* $(c_n)_{n\in\mathbb{N}}$ *and* $(d_n)_{n\in\mathbb{N}}$ *are monotonically nondecreasing. Then*

$$\frac{EX_{T_n} - \hat{b}_n}{\hat{a}_n} \to \begin{cases} \infty, & \text{if } c+d \geq 1, \\ \log\left(\frac{1}{1-(c+d)}\right), & \text{if } c+d < 1. \end{cases} \tag{5.8}$$

For $c + d < 1$ *let* $(w_n)_{n\in\mathbb{N}}$ *be monotonically nondecreasing with* $\lim_{n\to\infty} n(1 - F(w_n)) = 1$, *as e.g.* $w_n := b_n$. *Let* $u(t) := \log(\frac{1}{1-(c+d)}(1 - t^{1-(c+d)}))$ *and*

$$v_n(t) := \frac{w_{\lfloor (1-t)n \rfloor} - b_n}{a_n} + u(t) - \log(1-t).$$

Then $\hat{T}_n := \min\{1 \leq i \leq n : X_i > \hat{a}_n v_n\left(\frac{i}{n}\right) + \hat{b}_n\}$ *is an asymptotically optimal sequence of stopping times.*

Remark 5.4 *If* F *is the distribution function of the standard normal distribution* $N(0,1)$ *then* $F \in D(\Lambda)$ *and normalization constants fulfilling condition* (5.7) *are given by*

$$a_n = \frac{1}{\sqrt{2\log n}}, \quad b_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}. \tag{5.9}$$

Then possible choices of constants c_n, d_n *which fulfill* (5.7) *are*

$$c_n := C(\log n)^A, \quad d_n := D(\log n)^B$$

with A, B, C, $D \in \mathbb{R}$, $A \geq 0$, $C > 0$, $B \leq A + \frac{1}{2}$. *The limit constants* c, d *from* (5.7) *are given in this case by* $c = -2A$ *and* $d = -\sqrt{2}\frac{BD}{C}$ *if* $B = A + \frac{1}{2}$ *resp.* $d = 0$ *if* $B < A + \frac{1}{2}$.

Proof of Theorem 5.1: By rearrangement

$$\frac{X_i - \hat{b}_n}{\hat{a}_n} = \frac{c_i}{c_n}\left(\frac{Z_i - b_n}{a_n}\right) - \frac{b_n}{a_n}\left(1 - \frac{c_i}{c_n}\right) + \frac{d_i}{c_n a_n}$$

and thus

$$\hat{N}_n := \sum_{i=1}^n \delta_{\left(\frac{i}{n}, \frac{X_i - \hat{b}_n}{\hat{a}_n}\right)} = \sum_{i=1}^n \delta_{R_n\left(\frac{i}{n}, \frac{Z_i - b_n}{a_n}\right)}$$

with the transformation

$$R_n(t,y) := \left(t, \frac{c_{\lfloor tn \rfloor}}{c_n} y - \frac{b_n}{a_n}(1 - \frac{c_{\lfloor tn \rfloor}}{c_n}) + \frac{d_{\lfloor tn \rfloor}}{c_n a_n}\right) \longrightarrow R(t,y) := \left(t, t^c y + d t^{c+\frac{1}{\alpha}}\right) \tag{5.10}$$

For (5.10) note that $\frac{a_{\lfloor tn \rfloor}}{a_n} \to t^{\frac{1}{\alpha}}$ (see Resnick (1987, (0.18))) and thus

$$\frac{d_{\lfloor tn \rfloor}}{c_n a_n} = \frac{d_{\lfloor tn \rfloor}}{c_{\lfloor tn \rfloor} a_{\lfloor tn \rfloor}} \frac{c_{\lfloor tn \rfloor}}{c_n} \frac{a_{\lfloor tn \rfloor}}{a_n} \longrightarrow d t^{c+\frac{1}{\alpha}}.$$

Monotonicity of the constants implies that R_n converges to R uniformly on compact subsets in $(0,1] \times \mathbb{R}$ and R maps $[0,1] \times (0,\infty]$ to $M_{\hat{f}}$ with $\hat{f}(t) := dt^{c+\frac{1}{d}}$. The continuous mapping principle implies convergence of the point processes $\hat{N}_n$ to a Poisson process $\hat{N}$ on $M_{\hat{f}}$, where $\hat{N}$ has the intensity function

$$\hat{G}(t,z) = G(R^{-1}(t,z)) = t^{c\alpha}(z - dt^{c+\frac{1}{\alpha}})^{-\alpha} \text{ on } M_{\hat{f}},$$

where here $G(t,y) = y^{-\alpha}$ for $(t,y) \in [0,1] \times (0,\infty]$.

$\hat{G}$ can be represented for $c + \frac{1}{\alpha} \neq 0$ in the form

$$\hat{G}(t,z) = H\left(\frac{z}{v(t)}\right)\frac{v'(t)}{v(t)} \tag{5.11}$$

with $v(t) := t^{c+\frac{1}{\alpha}}$ and $H(x) := \frac{\alpha}{\alpha c+1}(x-d)^{-\alpha}$ for $x > d$. Theorem 5.1 implies convergence of the optimal stopping curves and stopping times of $\hat{N}_n$ to those of $\hat{N}$. The optimal stopping curve u of $\hat{N}$ for the guarantee value has by the results in Section 3 for $c \neq -\frac{1}{\alpha}$ the form in (5.5). Thus we obtain

$$u(0) = \lim_{t \downarrow 0} \phi\left(\frac{1}{v(t)}\right) v(t) = \lim_{y \uparrow r} \frac{y}{\Phi(y)} = \lim_{y \uparrow r} y \exp\left(-\int_d^y \frac{1}{R(z)}\,dz\right).$$

This implies Theorem 5.1 considering all cases one by one. □

Proof of Theorem 5.2: As in the previous proof we calculate the intensity function of the limiting process as

$$\hat{G}(t,z) = t^{-c\alpha}(-z + dt^{c-\frac{1}{\alpha}})^{\alpha}$$

for $z < dt^{c-\frac{1}{\alpha}}$ and 0 else (which equals (3.18)).

To check the uniform integrability condition (L) we use the functions v_n and their associated threshold stopping times

$$\hat{T}_n(t) := \min\left\{tn < i \le n : \frac{X_i}{\hat{a}_n} > v_n(\tfrac{i}{n})\right\}, \quad \hat{T}_n := \hat{T}_n(0).$$

Convergence of $\lambda_n(t) := \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c$ and $\mu_n(t) := \frac{d_{\lfloor tn \rfloor}}{c_n a_n} \to dt^{c+\frac{1}{\alpha}}$ are by the monotonicity conditions and by continuity of the limiting functions uniform on each interval $[t,1]$, $t > 0$. Further $\lim_{t\uparrow 1} \frac{u_{c,0}(t)}{u_{0,0}(t)} = 1$ and by (3.22) $\lim_{t\uparrow 1} u_{c,d}(t) - u_{c,0}(t) = d$. This is the basic tool for establishing the uniform integrabilty condition (L). For details of the proof see [F] (2009). □

Proof of Theorem 5.3: The proof is analogously to the previous proofs. With the constants $\hat{b}_n := c_n b_n + d_n$ we obtain in the limit the transformation $R(t,y) = (t, y - c\log t - d\log t)$. Thus $R^{-1}(t,z) = (t, z + (c+d)\log t)$ and the intensity function of the limit process $\hat{N}$ is given by

$$\hat{G}(t,z) = G(R^{-1}(t,z)) = e^{-z} t^{-(c+d)},$$

where here $G(t,y) = e^{-y}$. The optimal stopping curve of $\hat{N}$ is given by $u(t) = \log\left(\frac{1}{1-(c+d)}(1 - t^{1-(c+d)})\right)$. For details of the proof of the uniform integrability condition (L) we refer to [F] (2009). □

References

F. T. Bruss and L. C. G. Rogers. Embedding optimal selection problems in a Poisson process. *Stochastic Processes Appl.*, 38(2):267–278, 1991.

Y. S. Chow, H. Robbins, and D. Siegmund. *Great Expectations – The Theory of Optimal Stopping.* Houghton Mifflin, Boston, 1971.

L. de Haan and E. Verkaade. On extreme-value theory in the presence of a trend. *Journal of Applied Probability*, 24:62–76, 1987.

A. Faller. *Approximative Lösungen von Mehrfachstoppproblemen.* Dissertation, University of Freiburg, 2009.

A. Faller and L. Rüschendorf. On approximative solutions of multistopping problems. *Annales of Applied Probability*, to appear, 2010.

A. Gnedin and M. Sakaguchi. On a best choice problem related to the Poisson process. *Contemporary Mathematics*, 125:59–64, 1992.

D. P. Kennedy and R. P. Kertz. Limit theorems for threshold-stopped random variables with applications to optimal stopping. *Advances in Applied Probability*, 22(2):96–411, 1990.

D. P. Kennedy and R. P. Kertz. The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d. random variables. *The Annals of Probability*, 19 (1):329–341, 1991.

R. Kühne and L. Rüschendorf. Approximation of optimal stopping problems. *Stochastic Processes and Their Applications*, 90(2):301–325, 2000a.

R. Kühne and L. Rüschendorf. Optimal stopping with discount and observation costs. *Journal of Applied Probability*, 37:64–72, 2000b.

R. Kühne and L. Rüschendorf. Optimal stopping and cluster point processes. *Statistics and Decisions*, 21:261–282, 2003.

R. Kühne and L. Rüschendorf. Approximate optimal stopping of dependent sequences. *Theory of Probability and Its Applications*, 48(3):465–480, 2004.

S. I. Resnick. *Extreme Values, Regular Variation and Point Processes.* Springer, New York, 1987.

Andreas Faller
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany

Ludger Rüschendorf
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany
ruschen@stochastik.uni-freiburg.de