Optimal stopping of integral functionals and a "no-loss" free boundary formulation

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Abstract

This paper is concerned with a modification of the classical formulation of the free boundary problem for the optimal stopping of integral functionals in nonregular one-dimensional diffusions. This modification was introduced in a recent paper of Rüschendorf and Urusov (2007). As main result of that paper a verification theorem was established. Solutions of the modified free boundary problem imply solutions of the optimal stopping problem. The main contribution of this paper is to establish the converse direction. Solutions of the optimal stopping problem necessarily also solve the modified free boundary problem. Thus the modified free boundary problem is also necessary and does not 'lose' solutions. In the final part of this paper we discuss related questions for the viscosity approach and describe an advantage of the modified free boundary formulation.

Key words and phrases: optimal stopping, free boundary problem, one-dimensional diffusion, Engelbert–Schmidt conditions, local times, occupation times formula, Itô–Tanaka formula, viscosity solution of a one-dimensional ODE of second order.

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1 Introduction

An effective method for solving optimal stopping problems for regular diffusions is to develop a connection with some related free boundary problems of Stefan type. A verification theorem implies that solving the free boundary problem with smooth fit (or related conditions) allows to establish explicit solutions of the optimal stopping problem in certain cases. Many examples of this type are presented in Peskir and Shiryaev (2006). In the other direction for some classes of optimal stopping problems for regular diffusion processes with smooth coefficients general existence and regularity results for the corresponding free boundary problems have been established. Some of the results in this direction are quite recent.

The present paper is concerned with optimal stopping problems for integral functionals in nonregular one-dimensional diffusions with discontinuous coefficients. More precisely we consider the following situation. Let $X = (X_t)_{t\in[0,\infty)}$ be a (possibly, explosive) one-dimensional diffusion with state space $J = (\ell, r)$, $-\infty \le \ell < r \le \infty$, that is a weak solution of the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \tag{1}$$

and with $\mathsf{P}_x(X_0 = x) = 1$, $x \in J$. Here W is a Brownian motion and μ and σ are Borel functions $J \to \mathbb{R}$ specified below. We adopt the convention that X stays in the additional state Δ after the explosion time ζ, i.e., ζ is a $[0,\infty]$-valued random variable and
(i) X is J-valued and continuous on $[0, \zeta)$;
(ii) if $\zeta < \infty$, then $X \equiv \Delta$ on $[\zeta, \infty)$ and either $\lim_{t\uparrow\zeta} X_t = \ell$ or $\lim_{t\uparrow\zeta} X_t = r$.

We consider optimal stopping problems of integral functionals of the form

$$V^*(x) = \sup_{\tau\in\mathfrak{M}} \mathsf{E}_x\left[\int_0^\tau e^{-\Lambda_u} f(X_u)\,du\right], \quad x \in J. \tag{2}$$

Here E_x denotes the expectation under the measure P_x,

$$\Lambda_t = \int_0^t \lambda(X_u)\,du, \quad t \in [0,\infty), \tag{3}$$

f (resp., λ) is a Borel function $J \to \mathbb{R}$ (resp., $J \to [0,\infty)$) specified below, and $\mathfrak{M}$ is the class of stopping times τ of X satisfying

$$\mathsf{E}_x\left[\int_0^\tau e^{-\Lambda_u} f^+(X_u)\,du\right] < \infty \text{ or } \mathsf{E}_x\left[\int_0^\tau e^{-\Lambda_u} f^-(X_u)\,du\right] < \infty. \tag{4}$$

We use the following convention. For any function $g\colon J \to \mathbb{R}$, we define $g(\ell) = g(r) = 0$. In particular, this concerns the functions f and λ in (2) and (3).

One encounters concrete stopping problems of type (2) in the literature. For example, Graversen, Peskir, and Shiryaev (2000) reduce the problem of stopping a Brownian motion as close as possible to its maximum to a problem of type (2). In order to solve a stochastic control problem Karatzas and Ocone (2002) study a stopping problem, that can be reduced to a problem of integral type. It is interesting to note that in Karatzas and Ocone (2002) the diffusion X is explosive.

In Rüschendorf and Urusov (2007) (in the following denoted RU (2007)) problem (2) is studied via a suitable free boundary approach for functions f having the following form (see Figure 1): there exist points $x_{1\ell} \le x_{1r} < x_{2\ell} \le x_{2r}$ in J such that $f = 0$ on $[x_{1\ell}, x_{1r}] \cup [x_{2\ell}, x_{2r}]$, $f > 0$ on $(x_{1r}, x_{2\ell})$, and $f < 0$ on $(\ell, x_{1\ell}) \cup (x_{2r}, r)$. In what follows, we say that *f has a two-sided form* for such functions f. For many stopping problems of type (2) in the literature, f has a two-sided form. This is e.g. the case for the problems of Graversen, Peskir, and Shiryaev (2000) and of Karatzas and Ocone (2002) mentioned above whenever their parameters belong to some domain (see Remark (ii) in Section 2.1 of RU (2007) for a more detailed discussion).

For $a, b \in J$, $a < b$, we denote by $\tau_{a,b}$ the stopping time

$$\tau_{a,b} = \inf\{t \in [0,\infty)\colon X_t \le a \text{ or } X_t \ge b\} \tag{5}$$

(as usual, $\inf\emptyset := \infty$). If f has a two-sided form, one may expect that problem (2) has a *two-sided optimal stopping time*, i.e., an optimal stopping time of the form τ_{α^*,β^*}.

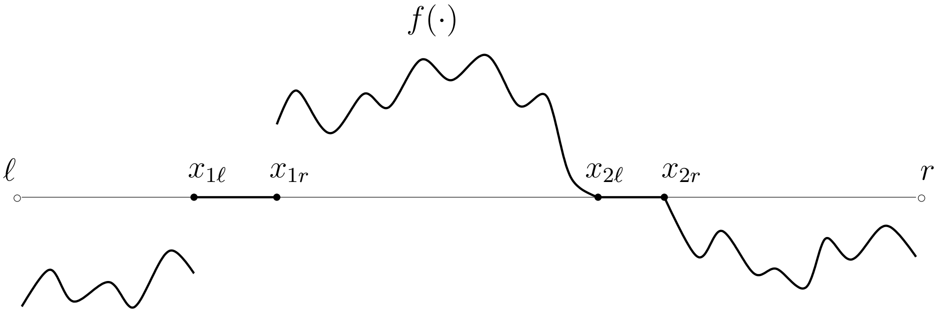

Figure 1: f has a two-sided form

One can formulate a classical free boundary problem on the triplet (V^*, α^*, β^*) in order to find the value function in (2) and the boundaries α^* and β^* of the stopping region. However, in our situation the functions μ, σ, f, and λ are allowed to be irregular (e.g. discontinuous), hence, the standard free boundary formulation can *lose* a two-sided solution of the stopping problem in the following sense: it can happen that there exists a two-sided optimal stopping time but the related free boundary problem has no solutions.

For this class of stopping problems in nonregular diffusions a modified version of the free boundary has been introduced in the recent paper RU (2007). The main result in that paper is a verification theorem for this version stating as in the classical regular case the sufficiency of the modified free boundary version. A result in the converse (necessary) direction is announced there. The main contribution of this paper is a general result stating the necessity of the modified free boundary formulation, i.e. the modified free boundary problem does not 'lose' solutions of the stopping problem. The proof of this result is technically involved and needs to develop some new tools and techniques which might be of use in further extensions. Both directions together imply that as in the case of regular diffusions the modified free boundary problem is completely adequate for this type of stopping problems in nonregular diffusions. As important practical consequence it is without loss to concentrate on the modified PDE formulation for the solution of this kind of stopping problems. For some explicit examples and a general class of explicitly solvable cases we refer to RU (2007).

The literature on optimal stopping for diffusions is very rich, even for the one-dimensional case that we consider here. We mention the monographs Shiryaev (1973), Krylov (1980), and Peskir and Shiryaev (2006) for the general theory. We would like also to mention some related papers of Salminen (1985), Beibel and Lerche (2000), Dayanik and Karatzas (2003), Dayanik (2003), and Lamberton and Zervos (2006), where stopping problems of the form "$\sup_\tau \mathsf{E}_x[e^{-\Lambda_\tau} g(X_\tau) I(\tau < \infty)]$" are studied. These authors allow the coefficients of X as well as the payoff function to be irregular (e.g. discontinuous). In the first four of these papers different approaches are introduced, some general results are obtained, and several examples are treated explicitly. Neither of these approaches is based on the free boundary method—the one we use here. Another difference with our paper is that we consider optimal stopping of *integral* functionals. Lamberton and Zervos (2006) prove that the value function in their problem is the difference of two convex functions and satisfies a certain variational inequality. In our situation the value function is even more regular (though the functions μ, σ, f, and λ can be irregular): it is differentiable and its derivative is absolutely continuous (see Theorem 2.1). This agrees with the intuition: the integral functionals $\mathsf{E}_x[\int_0^\tau e^{-\Lambda_u} f(X_u)\,du]$ are "more regular" than the functionals $\mathsf{E}_x[e^{-\Lambda_\tau} g(X_\tau) I(\tau < \infty)]$. Hence, it is natural to expect that value functions for integral functionals should also be "more regular".

This paper has the following structure. In Section 2 we introduce our assumptions, formulate the main result (Theorem 2.1), and describe its relation to some results of RU (2007). Related issues on the viscosity approach are discussed in Section 4. Loosely speaking, the standard free boundary formulation understood in the viscosity sense is not "no-loss" in our setting, which is due to possible discontinuities in the functions μ, σ, f, and λ. If we weaken the notion of viscosity solution a bit (we call this $*$-*viscosity solution*), then we obtain the "no-loss" property but the verification theorem is not true for $*$-viscosity solutions (cp. this with the fact that for our modified free boundary formulation we have both the "no-loss" property and the verification theorem; see Theorems 2.1 and 2.2 below). Finally, Theorem 2.1 is proved in Section 3. In the appendix we prove a technical lemma used in the proof of that theorem.

2 Main result

At first we describe our assumptions on the functions μ, σ, f, and λ.

Assumption 1. The coefficients μ and σ of the SDE (1) satisfy the Engelbert–Schmidt conditions

$$\sigma(x) \neq 0 \quad \forall x \in J,$$
$$\frac{1+|\mu|}{\sigma^2} \in L^1_{\text{loc}}(J),$$

where $L^1_{\text{loc}}(J)$ denotes the class of functions $J \to \mathbb{R}$ that are *locally integrable* on J, i.e., integrable on compact subintervals of J.

Assumption 2. The functions $f\colon J \to \mathbb{R}$ in (2) and $\lambda\colon J \to [0,\infty)$ in (3) satisfy the conditions

$$f/\sigma^2 \in L^1_{\text{loc}}(J),$$
$$\lambda/\sigma^2 \in L^1_{\text{loc}}(J).$$

Note that under Assumption 1 we have $\mathsf{E}_x \int_0^{\tau_{a,b}} |g(X_u)|\,du < \infty$, $x, a, b \in J$, $a < b$, whenever $g\colon J \to \mathbb{R}$ is a Borel function such that $g/\sigma^2 \in L^1_{\text{loc}}(J)$. In particular, $\tau_{a,b} \in \mathfrak{M}$ for all $a, b \in J$, $a < b$. For the proof of this, see Lemma A.3 in RU (2007). Assumption 1 guarantees that the SDE (1) has a unique in law (possibly, explosive) $J \cup \{\Delta\}$-valued weak solution for any starting point $X_0 = x \in J$ (see Engelbert and Schmidt (1985, 1991) or Karatzas and Shreve (1991, Ch. 5, Th. 5.15)). The Engelbert–Schmidt conditions are reasonable weak conditions. For example, if μ is locally bounded on J and σ is locally bounded away from zero on J, then they are satisfied. Further we note that due to local integrability of the function $1/\sigma^2$ Assumption 2 holds whenever f and λ are locally bounded on J. Set $F_t = \int_0^t f(X_u)\,du$. Using the occupation times formula (see Revuz and Yor (1999, Ch. VI, Cor. (1.6))) one can prove that $f/\sigma^2 \in L^1_{\text{loc}}(J)$ (resp., $\lambda/\sigma^2 \in L^1_{\text{loc}}(J)$) if and only if the process (F_t) (resp., (Λ_t)

defined in (3)) is well defined and finite a.s. on the stochastic interval $[0, \zeta)$. Therefore, Assumption 2 is reasonable.

In what follows, ν_L denotes the Lebesgue measure on J. The main subject in this paper is the following modified free boundary problem, which was introduced in RU (2007).

Modified free boundary problem (MFBP):

$$V' \text{ is absolutely continuous on } [\alpha, \beta]; \tag{6}$$

$$\frac{\sigma^2(x)}{2} V''(x) + \mu(x)V'(x) - \lambda(x)V(x) = -f(x) \text{ for } \nu_L\text{-a.a. } x \in (\alpha, \beta); \tag{7}$$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta); \tag{8}$$

$$V'_+(\alpha) = V'_-(\beta) = 0. \tag{9}$$

We say that a triplet (V, α, β) is a *solution* of (6)–(9) if $\alpha, \beta \in J$, $\alpha < \beta$, $V \in C^1([\alpha, \beta])$, and the triplet (V, α, β) satisfies (6)–(9). In (9), V'_+ and V'_- denote respectively right and left derivatives of V. Formally, under $V'(\alpha)$ and $V'(\beta)$ in (6) one should understand respectively right and left derivatives. However, (8) and (9) imply that the two-sided derivatives exist at both points.

The main contribution of this paper is the following theorem, which is proved in the next section. It implies that the modified free boundary formulation is also necessary and does not 'lose' two-sided solutions. It is of interest to note that this theorem does not need the assumption that the gain function f is of two-sided form.

Theorem 2.1. *Suppose that Assumptions 1 and 2 hold. If there exist $\alpha^*, \beta^* \in J$, $\alpha^* < \beta^*$, such that the stopping time τ_{α^*,β^*} is optimal in* (2)*, then (V^*, α^*, β^*) is a solution of the modified free boundary problem* (6)*–*(9)*.*

In the sequel, we denote by $B_{\text{loc}}(x)$ the class of functions that are bounded in a sufficiently small neighborhood of x and by $B_{\text{loc}}(J)$ the class of locally bounded functions on J. Theorem 2.1 above complements the following verification theorem, which extends Theorem 2.1 in RU (2007), who consider the case of constant functions λ. Introducing the condition that $\lambda \in B_{\text{loc}}(J)$ all arguments of proof in that paper can be extended to this more general case.

Theorem 2.2 (Verification theorem)**.** *Suppose that Assumptions 1 and 2 hold and that f has a two-sided form (see Figure 1). Additionally assume that $\lambda \in B_{\text{loc}}(J)$ and $1/f \in B_{\text{loc}}(x)$ for all $x \in J \setminus ([x_{1\ell}, x_{1r}] \cup [x_{2\ell}, x_{2r}])$. If (V, α, β) is a non-trivial solution of the modified free boundary problem* (6)*–*(9) *(i.e., $V \not\equiv 0$), then it is the unique non-trivial solution, V is the value function in* (2)*, i.e., $V = V^*$, and $\tau_{\alpha,\beta}$ is the unique optimal stopping time in* (2)*.*

Remarks. (i) The assumption that f has a two-sided form is essential for Theorem 2.2; see the remark after Theorem 2.3 in RU (2007). As in RU (2007), Theorem 2.3, also an extended version of the verification Theorem 2.2 for 'four-sided' functions with two positivity intervals and corresponding stopping times with two disjoint stopping intervals can be proved for the more general case of discounting as in Theorem 2.2. We remark that Theorem 2.1 of this paper was announced (without proof and details) for the special case of constant discounts λ in RU (2007).

(ii) If $x_{1\ell} < x_{1r}$ or $x_{2\ell} < x_{2r}$, then (6)–(9) has trivial solutions $(0, \alpha, \beta)$ for $\alpha < \beta$ belonging to $[x_{1\ell}, x_{1r}]$ or to $[x_{2\ell}, x_{2r}]$. Therefore, we deal only with non-trivial solutions of (6)–(9) in Theorem 2.2.

(iii) It is interesting to note that we have always *strict* inequalities $\alpha < x_{1\ell}$ and $\beta > x_{2r}$ no matter how large negative values the function f takes to the left from $x_{1\ell}$ or to the right from x_{2r} (see Proposition 2.9 in RU (2007)).

3 Proof of Theorem 2.1

At first we need several lemmas.

Lemma 3.1. *Let $\alpha, \beta \in J$, $\alpha < \beta$. There exists a function $U\colon [\alpha, \beta] \to \mathbb{R}$ such that $U \in C^1([\alpha, \beta])$, U' is absolutely continuous on $[\alpha, \beta]$,*

$$\frac{\sigma^2(x)}{2} U''(x) + \mu(x) U'(x) - \lambda(x) U(x) = -f(x) \text{ for } \nu_L\text{-a.a. } x \in [\alpha, \beta], \tag{10}$$

and $U(\alpha) = U(\beta) = 0$.

This lemma is proved in the appendix. Let in the following L_t^y denote the local time of X at time t and level y.

Lemma 3.2. *Let $\mu \equiv 0$, $\varepsilon > 0$, $x \in J$, $a_\varepsilon := x - \varepsilon \in J$, and $b_\varepsilon := x + \varepsilon \in J$. Then we have*

$$\mathsf{E}_x L^y_{\tau_{a_\varepsilon, b_\varepsilon}} = \varepsilon - |x - y|, \quad y \in (a_\varepsilon, b_\varepsilon), \tag{11}$$

and

$$\mathsf{E}_x (L^x_{\tau_{a_\varepsilon, b_\varepsilon}})^2 \le c_0 \, \varepsilon^2 \tag{12}$$

for some constant c_0 that does not depend on ε and x.

Proof. Since $(X_{t \wedge \tau_{a_\varepsilon, b_\varepsilon}})$ is a bounded P_x-martingale, it follows $\mathsf{E}_x X_{\tau_{a_\varepsilon, b_\varepsilon}} = x$. Applying the fact that $\tau_{a_\varepsilon, b_\varepsilon} < \infty$ P_x-a.s., we get

$$\mathsf{P}_x(X_{\tau_{a_\varepsilon, b_\varepsilon}} = a_\varepsilon) = \mathsf{P}_x(X_{\tau_{a_\varepsilon, b_\varepsilon}} = b_\varepsilon) = \frac{1}{2}.$$

For any $y \in (a_\varepsilon, b_\varepsilon)$, the Tanaka formula under P_x implies

$$|X_{\tau_{a_\varepsilon, b_\varepsilon}} - y| = |x - y| + \int_0^{\tau_{a_\varepsilon, b_\varepsilon}} \operatorname{sgn}(X_u - y) \sigma(X_u) \, dW_u + L^y_{\tau_{a_\varepsilon, b_\varepsilon}} \quad \mathsf{P}_x\text{-a.s.}, \tag{13}$$

where

$$\operatorname{sgn} y = \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y \le 0. \end{cases}$$

The process $M_t := \int_0^{t \wedge \tau_{a_\varepsilon, b_\varepsilon}} \operatorname{sgn}(X_u - y) \sigma(X_u) \, dW_u$ is a P_x-square integrable martingale. Indeed,

$$[M]_\infty = \int_0^{\tau_{a_\varepsilon, b_\varepsilon}} \sigma^2(X_u) \, du = [X]_{\tau_{a_\varepsilon, b_\varepsilon}} \quad \mathsf{P}_x\text{-a.s.}$$

and $\mathsf{E}_x[X]_{\tau_{a_\varepsilon,b_\varepsilon}} < \infty$ because $(X_{t\wedge\tau_{a_\varepsilon,b_\varepsilon}})$ is a bounded P_x-martingale. Hence, $\mathsf{E}_x M_{\tau_{a_\varepsilon,b_\varepsilon}} = 0$ and (11) follows by computing the expectations in (13). Further, it follows from (13) and the Burkholder–Davis–Gundy inequality that

$$\begin{aligned}\mathsf{E}_x(L^x_{\tau_{a_\varepsilon,b_\varepsilon}})^2 &= \varepsilon^2 + \mathsf{E}_x M^2_{\tau_{a_\varepsilon,b_\varepsilon}} = \varepsilon^2 + \mathsf{E}_x[X]_{\tau_{a_\varepsilon,b_\varepsilon}}\\ &\le \varepsilon^2 + c_1 \mathsf{E}_x \sup_{t\le\tau_{a_\varepsilon,b_\varepsilon}} |X_t - x|^2 = (1+c_1)\varepsilon^2\end{aligned}$$

for some universal constant c_1. We obtain (12) with $c_0 = 1 + c_1$. □

In the sequel, we shall often use the following observation without further explanations: if $A \subseteq J$ with $\nu_L(A) = 0$, then

$$\nu_L(\{t\in[0,\infty)\colon X_t \in A\}) = 0 \quad \mathsf{P}_x\text{-a.s.}, \quad x \in J.$$

This can be easily proved using the occupation times formula.

Proof of Theorem 2.1. 1) At first we additionally assume that $\mu \equiv 0$. Let $U\colon [\alpha^*,\beta^*] \to \mathbb{R}$ be a solution of (10) as in Lemma 3.1 with $\alpha = \alpha^*$ and $\beta = \beta^*$ and denote

$$V(x) = \begin{cases} U(x) & \text{if } x \in (\alpha^*,\beta^*),\\ 0 & \text{if } x \in J\setminus(\alpha^*,\beta^*).\end{cases}$$

Our aim is to prove that $V = V^*$. If $x \in J\setminus(\alpha^*,\beta^*)$, then $V(x) = V^*(x) = 0$. Suppose that $x\in(\alpha^*,\beta^*)$. Since U' is absolutely continuous (hence, of bounded variation), we can apply the Itô–Tanaka formula to $U(X_{t\wedge\tau_{\alpha^*,\beta^*}})$ (see Revuz and Yor (1999, Ch. VI, Th. (1.5))). We obtain

$$\begin{aligned}U(X_{t\wedge\tau_{\alpha^*,\beta^*}}) = U(x) &+ \int_0^{t\wedge\tau_{\alpha^*,\beta^*}} U'(X_u)\sigma(X_u)\,dW_u\\ &+ \frac{1}{2}\int_J L^y_{t\wedge\tau_{\alpha^*,\beta^*}}\, U'(dy) \quad \mathsf{P}_x\text{-a.s.}, \quad t\in[0,\infty).\end{aligned} \tag{14}$$

Since U' is absolutely continuous, we have $U'(dy) = U''(y)\,dy$. Applying the occupation times formula to the last term at the right-hand side of (14), we obtain

$$\begin{aligned}U(X_{t\wedge\tau_{\alpha^*,\beta^*}}) = U(x) &+ \int_0^{t\wedge\tau_{\alpha^*,\beta^*}} U'(X_u)\sigma(X_u)\,dW_u\\ &+ \int_0^{t\wedge\tau_{\alpha^*,\beta^*}} \frac{\sigma^2(X_u)}{2} U''(X_u)\,du \quad \mathsf{P}_x\text{-a.s.}, \quad t\in[0,\infty).\end{aligned} \tag{15}$$

By the product rule, this implies

$$\begin{aligned}e^{-\Lambda_{t\wedge\tau_{\alpha^*,\beta^*}}} U(X_{t\wedge\tau_{\alpha^*,\beta^*}}) = U(x) + M_t &+ \int_0^{t\wedge\tau_{\alpha^*,\beta^*}} e^{-\Lambda_u}\Bigg[\frac{\sigma^2(X_u)}{2}U''(X_u)\\ &- \lambda(X_u)U(X_u)\Bigg]\,du \quad \mathsf{P}_x\text{-a.s.}, \quad t\in[0,\infty),\end{aligned} \tag{16}$$

where

$$M_t = \int_0^{t\wedge\tau_{\alpha^*,\beta^*}} e^{-\Lambda_u} U'(X_u)\sigma(X_u)\,dW_u.$$

The function U' is continuous. We have

$$[M]_\infty \le \text{const} \int_0^{\tau_{\alpha^*,\beta^*}} \sigma^2(X_u)\,du = \text{const}[X]_{\tau_{\alpha^*,\beta^*}} \quad \mathsf{P}_x\text{-a.s.}$$

Since $(X_{t\wedge\tau_{\alpha^*,\beta^*}})$ is a bounded P_x-martingale, we get $\mathsf{E}_x[M]_\infty \le \text{const}\,\mathsf{E}_x[X]_{\tau_{\alpha^*,\beta^*}} < \infty$. Hence, $\mathsf{E}_x M_{\tau_{\alpha^*,\beta^*}} = 0$. Applying (10) and $U(\alpha^*) = U(\beta^*) = 0$, we obtain from (16) that

$$U(x) = \mathsf{E}_x \int_0^{\tau_{\alpha^*,\beta^*}} e^{-\Lambda_u} f(X_u)\,du.$$

Then $V(x) = V^*(x)$ because $V(x) = U(x)$ and τ_{α^*,β^*} is an optimal stopping time in (2).

2) We continue to work under the assumption $\mu \equiv 0$. It remains to prove that the boundary condition (9) is fulfilled for $V = V^*$. Note that the dynamic programming principle implies that for any $x, a, b \in J$, $a < b$, we have

$$V(x) \ge \mathsf{E}_x\left[e^{-\Lambda_{\tau_{a,b}}} V(X_{\tau_{a,b}}) + \int_0^{\tau_{a,b}} e^{-\Lambda_u} f(X_u)\,du\right]. \tag{17}$$

Alternatively, one can prove (17) directly using the strong Markov property. The right derivative $V'_+(\alpha^*)$ exists and is finite because $V = U$ on $[\alpha^*, \beta^*]$. Since $V(\alpha^*) = 0 \le V^*(x) = V(x)$, $x \in J$, we obtain $V'_+(\alpha^*) \ge 0$. Assuming that $V'_+(\alpha^*) > 0$ we will show that we obtain a contradiction to (17).

For $\varepsilon > 0$, we set $a_\varepsilon = \alpha^* - \varepsilon$, $b_\varepsilon = \alpha^* + \varepsilon$ and work below with sufficiently small ε so that $a_\varepsilon \in J$ and $b_\varepsilon < \beta^*$. Applying the Itô–Tanaka formula under P_{α^*} and proceeding as in (14)–(16), we obtain (note that $V(\alpha^*) = 0$)

$$\begin{aligned} e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}} V(X_{\tau_{a_\varepsilon,b_\varepsilon}}) = M_{\tau_{a_\varepsilon,b_\varepsilon}} + c\int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u}\,dL_u^{\alpha^*} + \int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u}\bigg[\frac{\sigma^2(X_u)}{2} V''(X_u) \\ - \lambda(X_u)V(X_u)\bigg]\,du \quad \mathsf{P}_{\alpha^*}\text{-a.s.}, \end{aligned} \tag{18}$$

where $c = V'_+(\alpha^*)/2 > 0$ and

$$M_{\tau_{a_\varepsilon,b_\varepsilon}} = \int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u} V'_-(X_u)\sigma(X_u)\,dW_u.$$

As earlier, $\mathsf{E}_{\alpha^*} M_{\tau_{a_\varepsilon,b_\varepsilon}} = 0$. The term $c\int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u}\,dL_u^{\alpha^*}$ in (18) appears due to the fact that the function V' has discontinuities at α^* and β^*. Hence, the measure $V'(dy)$ (with the distribution function V'_+) appearing in the Itô–Tanaka formula has the form

$$V'(dy) = V''(y)\,dy + V'_+(\alpha^*)\delta_{\alpha^*} - V'_-(\beta^*)\delta_{\beta^*},$$

where δ_x denotes the unit measure concentrated at the point x. Note that the local time at level β^* does not appear in (18) because $L_\cdot^{\beta^*} = 0$ P_{α^*}-a.s. on $[0, \tau_{a_\varepsilon,b_\varepsilon}]$ due to $b_\varepsilon < \beta^*$.

We have $(\sigma^2/2)V'' - \lambda V = -fI_{(\alpha^*,\beta^*)}$ ν_L-a.e. on J. Hence, it follows from (18) that

$$\begin{aligned} &\mathsf{E}_{\alpha^*}\left[e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}} V(X_{\tau_{a_\varepsilon,b_\varepsilon}}) + \int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u} f(X_u)\,du\right] \\ &\quad = \mathsf{E}_{\alpha^*}\left[c\int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u}\,dL_u^{\alpha^*} + \int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u} h(X_u)\,du\right], \end{aligned} \tag{19}$$

where $h := f I_{J\setminus(\alpha^*,\beta^*)}$. Putting $g := |h|$, we get

$$\int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u} h(X_u)\,du \geq -\int_0^{\tau_{a_\varepsilon,b_\varepsilon}} g(X_u)\,du = -\int_{a_\varepsilon}^{b_\varepsilon} \frac{g(y)}{\sigma^2(y)} L^y_{\tau_{a_\varepsilon,b_\varepsilon}}\,dy \quad \mathsf{P}_{\alpha^*}\text{-a.s.},$$

where the last equality follows from the occupation times formula. It follows from (19), $g \geq 0$, and Lemma 3.2 that

$$\begin{aligned}
&\mathsf{E}_{\alpha^*}\left[e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}} V(X_{\tau_{a_\varepsilon,b_\varepsilon}}) + \int_0^{\tau_{a_\varepsilon,b_\varepsilon}} e^{-\Lambda_u} f(X_u)\,du\right] \\
&\quad \geq c\,\mathsf{E}_{\alpha^*}\left[e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}} L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}}\right] - \int_{a_\varepsilon}^{b_\varepsilon} \frac{g(y)}{\sigma^2(y)} \mathsf{E}_{\alpha^*} L^y_{\tau_{a_\varepsilon,b_\varepsilon}}\,dy \\
&\quad \geq c\,\varepsilon - c\,\mathsf{E}_{\alpha^*}\left[(1 - e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}}) L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}}\right] - \varepsilon \int_{a_\varepsilon}^{b_\varepsilon} \frac{g(y)}{\sigma^2(y)}\,dy \\
&\quad = \left(c - \int_{a_\varepsilon}^{b_\varepsilon} \frac{g(y)}{\sigma^2(y)}\,dy\right)\varepsilon - c\,\mathsf{E}_{\alpha^*}\left[(1 - e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}}) L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}}\right].
\end{aligned} \tag{20}$$

Now we fix an arbitrary $\delta \in (0,1)$ and define

$$T_\delta = \inf\{t \in [0,\infty) \colon \Lambda_t > -\ln(1-\delta)\}$$

($\inf \emptyset := \infty$). By the occupation times formula,

$$\Lambda_t = \int_0^t \lambda(X_u)\,du = \int_J \frac{\lambda(y)}{\sigma^2(y)} L^y_t\,dy < \infty \ \mathsf{P}_{\alpha^*}\text{-a.s. on } \{t < \zeta\}$$

because $\lambda/\sigma^2 \in L^1_{\mathrm{loc}}(J)$ and P_{α^*}-a.s. on $\{t < \zeta\}$ the function $y \mapsto L^y_t$ has a compact support in J and is bounded due to the fact that it is càdlàg (recall that ζ denotes the explosion time of X). Since $\zeta > 0$ P_{α^*}-a.s., we obtain that for P_{α^*}-a.a. elementary outcomes ω there exists $t_0 = t_0(\omega)$ such that $\Lambda_{t_0} < \infty$. Due to the continuity of $(\Lambda_u)_{u\in[0,t_0]}$ we get $\Lambda_t \downarrow 0$ P_{α^*}-a.s. as $t \downarrow 0$, hence, $T_\delta > 0$ P_{α^*}-a.s..

We have

$$(1 - e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}}) L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}} \leq \delta L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}} + L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}} I(\tau_{a_\varepsilon,b_\varepsilon} > T_\delta).$$

Hence, by Lemma 3.2,

$$\mathsf{E}_{\alpha^*}\left[(1 - e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}}) L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}}\right] \leq \delta\varepsilon + \mathsf{E}_{\alpha^*}\left[L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}} I(\tau_{a_\varepsilon,b_\varepsilon} > T_\delta)\right]. \tag{21}$$

By the Cauchy–Bunyakovski–Schwarz inequality and (12),

$$\left\{\mathsf{E}_{\alpha^*}\left[L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}} I(\tau_{a_\varepsilon,b_\varepsilon} > T_\delta)\right]\right\}^2 \leq c_0\,\varepsilon^2\,\mathsf{P}_{\alpha^*}(\tau_{a_\varepsilon,b_\varepsilon} > T_\delta). \tag{22}$$

It follows from (21), (22), $\tau_{a_\varepsilon,b_\varepsilon} \downarrow 0$ P_{α^*}-a.s. as $\varepsilon \downarrow 0$, and $T_\delta > 0$ P_{α^*}-a.s. that

$$\mathsf{E}_{\alpha^*}\left[(1 - e^{-\Lambda_{\tau_{a_\varepsilon,b_\varepsilon}}}) L^{\alpha^*}_{\tau_{a_\varepsilon,b_\varepsilon}}\right] \leq 2\delta\varepsilon$$

for sufficiently small $\varepsilon > 0$. Since $\delta \in (0,1)$ is arbitrary and $g/\sigma^2 \in L^1_{\mathrm{loc}}(J)$, the right-hand side of (20) is strictly positive for sufficiently small $\varepsilon > 0$. This contradicts (17) with $x = \alpha^*$. Hence, $V'_+(\alpha^*) = 0$. Similarly, $V'_-(\beta^*) = 0$.

3) In the final step we prove the result without the assumption $\mu \equiv 0$. For some fixed $c \in J$ we consider the scale function of the process X

$$p(x) = \int_c^x \exp\left(-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}\,dz\right)dy, \quad x \in J.$$

We define the process $\widetilde{X}_t = p(X_t)$, $p(\Delta) := \Delta$, with the state space $\widetilde{J} \cup \{\Delta\}$, $\widetilde{J} = (\widetilde{\ell}, \widetilde{r}) := (p(\ell), p(r))$. Then we have

$$d\widetilde{X}_t = \widetilde{\sigma}(\widetilde{X}_t)\,dW_t$$

with $\widetilde{\sigma}(x) = (p'\sigma) \circ p^{-1}(x)$, $x \in \widetilde{J}$. We shall use the alternative notation $\widetilde{P}_x$ for the measure $P_{p^{-1}(x)}$ so that $\widetilde{P}_x(\widetilde{X}_0 = x) = 1$. Consider now the stopping problem

$$\widetilde{V}^*(x) = \sup_{\tau \in \mathfrak{M}} \widetilde{\mathsf{E}}_x \left[\int_0^\tau e^{-\widetilde{\Lambda}_u} \widetilde{f}(\widetilde{X}_u)\,du\right], \quad x \in \widetilde{J},$$

where $\widetilde{f} = f \circ p^{-1}$, $\widetilde{\lambda} = \lambda \circ p^{-1}$, and $\Lambda_t = \int_0^t \widetilde{\lambda}(\widetilde{X}_u)\,du$. This stopping problem is a reformulation of problem (2) in the sense that $\widetilde{V}^* = V^* \circ p^{-1}$, and a stopping time τ^* is optimal in problem $V^*(x)$ if and only if it is optimal in problem $\widetilde{V}^*(p(x))$. Note that Assumptions 1 and 2 for the functions $\widetilde{\mu} \equiv 0$, $\widetilde{\sigma}$, $\widetilde{f}$, and $\widetilde{\lambda}$ are satisfied (one should replace J with $\widetilde{J}$ in these conditions). One can easily verify that the triplet (V, α, β) is a solution of (6)–(9) if and only if the triplet $(\widetilde{V}, \widetilde{\alpha}, \widetilde{\beta}) := (V \circ p^{-1}, p(\alpha), p(\beta))$ is a solution of the modified free boundary problem

$$\begin{aligned}
&\widetilde{V}' \text{ is absolutely continuous on } [\widetilde{\alpha}, \widetilde{\beta}];\\
&\frac{\widetilde{\sigma}^2(x)}{2}\widetilde{V}''(x) - \widetilde{\lambda}(x)\widetilde{V}(x) = -\widetilde{f}(x) \text{ for } \nu_L\text{-a.a. } x \in (\widetilde{\alpha}, \widetilde{\beta});\\
&\widetilde{V}(x) = 0, \quad x \in \widetilde{J} \setminus (\widetilde{\alpha}, \widetilde{\beta});\\
&\widetilde{V}'_+(\widetilde{\alpha}) = \widetilde{V}'_-(\widetilde{\beta}) = 0.
\end{aligned}$$

Now the result follows from parts 1) and 2). $\square$

4 Viscosity approach

The reason why we consider the modification (6)–(9) of the classical free boundary problem is that we want to allow discontinuous μ, σ, f, and λ. Hence, the value function in (2) is not enough regular to be a solution of the free boundary problem in the classical form. A usual approach to handle this problem is to consider viscosity solutions (see Crandall, Ishii, and Lions (1992)). It is usually proved that value functions of certain classes of (multidimensional) stopping problems satisfy corresponding variational inequalities in the viscosity sense (see e.g. Øksendal and Reikvam (1998) or Øksendal and Sulem (2005)). However, the diffusion coefficients and the payoff are usually assumed to be continuous. In this paper, we consider a modified free boundary formulation rather than variational inequalities. So it is interesting to see, what we obtain for our stopping problem (2), when we consider viscosity solutions for the classical free boundary problem.

Thus, we consider the following free boundary problem

$$\frac{\sigma^2(x)}{2}V''(x) + \mu(x)V'(x) - \lambda(x)V(x) = -f(x), \quad x \in (\alpha, \beta); \tag{23}$$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta); \tag{24}$$

$$V'_+(\alpha) = V'_-(\beta) = 0 \tag{25}$$

and define its viscosity solution as follows (cp. with Definition 2.1 in Øksendal and Reikvam (1998) for the case of variational inequalities).

Definition 4.1. A *viscosity solution of* (23)–(25) is a triplet (V, α, β) such that $\alpha, \beta \in J$, $\alpha < \beta$, V is a continuous function $J \to \mathbb{R}$ satisfying (24) and (25), and V is both a *viscosity subsolution* and a *viscosity supersolution of* (23) in the sense of the following definition.

Definition 4.2. Let $\alpha, \beta \in J$, $\alpha < \beta$. Set $I = (\alpha, \beta)$. A continuous function $V\colon I \to \mathbb{R}$ is a *viscosity subsolution of* (23) if for each $\psi \in C^2(I)$ and each $y_0 \in I$ such that $\psi \geq V$ on I and $\psi(y_0) = V(y_0)$ we have

$$-\frac{\sigma^2(y_0)}{2}\psi''(y_0) - \mu(y_0)\psi'(y_0) + \lambda(y_0)\psi(y_0) - f(y_0) \leq 0. \tag{26}$$

A continuous function $V\colon I \to \mathbb{R}$ is a *viscosity supersolution of* (23) if for each $\phi \in C^2(I)$ and each $y_0 \in I$ such that $\phi \leq V$ on I and $\phi(y_0) = V(y_0)$ we have

$$-\frac{\sigma^2(y_0)}{2}\phi''(y_0) - \mu(y_0)\phi'(y_0) + \lambda(y_0)\phi(y_0) - f(y_0) \geq 0. \tag{27}$$

Now we consider the following question.

Question 1. Is it true that under the assumptions of Theorem 2.1 the triplet (V^*, α^*, β^*) is a viscosity solution of (23)–(25)?

The answer is **No**. The reason for that are possible discontinuities in μ, σ, f, and λ. Indeed, if we change the functions μ, σ, f, and λ on sets of ν_L-measure 0, then X remains a solution of (1) and problem (2) does not change. A viscosity solution of (23)–(25), however, can lose this property under such a transformation (see (26) and (27)). Suppose that the answer to Question 1 is Yes. Then considering any μ, σ, f, and λ such that there exists a two-sided optimal stopping time τ_{α^*,β^*} in (2), we obtain that (V^*, α^*, β^*) is a viscosity solution of (23)–(25). Then we take any appropriate pair (ψ, y_0) of Definition 4.2 and modify f only at point y_0 in order to violate (26). Hence, (V^*, α^*, β^*) is no more a viscosity solution of (23)–(25) with the modified function f. This contradicts our assumption that the answer to Question 1 is Yes.

Thus, in order to obtain a positive result in this direction it is natural to define $*$-*viscosity solutions of* (23)–(25) through $*$-*viscosity subsolutions* and $*$-*viscosity supersolutions* as in Definition 4.1. Here $*$-*viscosity subsolutions* and $*$-*viscosity supersolutions* are defined as follows:

Definition 4.3. Let $\alpha, \beta \in J$, $\alpha < \beta$. Set $I = (\alpha, \beta)$. A continuous function $V\colon I \to \mathbb{R}$ is a $*$-*viscosity subsolution of* (23) if for ν_L-a.a. $y_0 \in I$ the following condition holds: For each $\psi \in C^2(I)$ such that $\psi \geq V$ on I and $\psi(y_0) = V(y_0)$, condition (26) is satisfied.

A $*$-*viscosity supersolution of* (23) is defined in a symmetric way.

Now the positive answer to the modification of Question 1 with $*$-viscosity solutions instead of viscosity solutions follows directly from Theorem 2.1.

Corollary 4.4. *Suppose that Assumptions 1 and 2 hold. If there exist $\alpha^*, \beta^* \in J$, $\alpha^* < \beta^*$, such that the stopping time τ_{α^*,β^*} is optimal in* (2)*, then (V^*, α^*, β^*) is a $*$-viscosity solution of* (23)*–*(25)*.*

Then it is natural to pose the following question.
Question 2. Suppose that the assumptions of Theorem 2.2 are satisfied and (V, α, β) is a non-trivial $*$-viscosity solution of (23)–(25).
(a) Does this imply that $V = V^*$?
(b) Does this imply that $\tau_{\alpha,\beta}$ is an optimal stopping time in (2)?

In the case of positive answers one could use the standard free boundary (23)–(25) (understood in the $*$-viscosity sense) to solve (2) even with irregular (e.g. discontinuous) functions μ, σ, f, and λ. Unfortunately, the answer is **No** both to (a) and to (b) as the following examples show.

Example 4.5. We set $\mu \equiv 0$, $\lambda \equiv 0$ and consider σ and f satisfying the assumptions of Theorem 2.2 such that there exists a non-trivial solution (V, α, β) of (6)–(9) (see Section 3 of RU (2007) for necessary and sufficient conditions). Then by Theorem 2.2, V is the value function in (2), i.e., $V = V^*$. We take any continuous function $h\colon J \to \mathbb{R}$ such that

$$h = 0 \text{ on } J \setminus (\alpha, \beta),$$

$$h'_+(\alpha) = h'_-(\beta) = 0,$$

$$h' = 0 \ \nu_L\text{-a.e. on } (\alpha, \beta),$$

$$\int_\alpha^\beta h(x)\, dx = 0,$$

and h is not absolutely continuous on $[\alpha, \beta]$ (such a function h can be easily constructed through the Cantor staircase function) and set

$$\widetilde{V}(y) = V(y) + \int_\ell^y h(x)\, dx, \quad y \in J.$$

Then the triplet $(\widetilde{V}, \alpha, \beta)$ satisfies (7)–(9), hence, $(\widetilde{V}, \alpha, \beta)$ is a non-trivial $*$-viscosity solution of (23)–(25) but $\widetilde{V} \neq V = V^*$. As consequence, we obtain a negative answer to part (a) of Question 2.

Example 4.6. As earlier, we set $\mu \equiv 0$, $\lambda \equiv 0$ and consider σ and f satisfying the assumptions of Theorem 2.2 such that $x_{1\ell} < x_{1r}$ and that there exists a non-trivial solution (V, α, β) of (6)–(9). Then by Theorem 2.2, $\tau_{\alpha,\beta}$ is the unique optimal stopping time in (2). However, for any $\widetilde{\alpha} < \widetilde{\beta}$ in $[x_{1\ell}, x_{1r}]$, we can construct a non-trivial $*$-viscosity solution $(\widetilde{V}, \widetilde{\alpha}, \widetilde{\beta})$ of (23)–(25) using exactly the same idea as in the previous example. We have $\alpha < \widetilde{\alpha} < \widetilde{\beta} < \beta$ (see Remark (iii) after Theorem 2.2). Hence, the stopping time $\tau_{\widetilde{\alpha},\widetilde{\beta}}$ is not optimal in (2). Thus, we obtain a negative answer also to part (b) of Question 2.

Appendix

Here we outline a probabilistic proof of Lemma 3.1. Without loss of generality we assume that $\mu \equiv 0$. First, we consider the homogenous ODE

$$\frac{\sigma^2(x)}{2} U''(x) - \lambda(x) U(x) = 0. \tag{28}$$

Our aim is to prove that there exist two differentiable functions ϕ and ψ on $[\alpha, \beta]$ with absolutely continuous derivatives that solve (28) ν_L-a.e. on $[\alpha, \beta]$ and satisfy the inequality

$$\psi(\alpha)\phi(\beta) - \psi(\beta)\phi(\alpha) > 0. \tag{29}$$

Under the additional assumption that X is non-explosive (and some technical assumptions on the function λ), this result can be found in Johnson and Zervos (2007) or derived from Rogers and Williams (2000, Ch. V, Prop. (50.3)). Below we follow the lines of Rogers and Williams (2000) and concentrate mostly on the part of the proof that should be done differently (this difference occurs because X can explode). If $\lambda = 0$ ν_L-a.e. on $[\alpha, \beta]$, then we can put $\psi(x) = 1$ and $\phi(x) = x$. Therefore, below we suppose that $\nu_L(\{x \in [\alpha, \beta] \colon \lambda(x) > 0\}) > 0$. We take two additional points $\alpha' \in (\ell, \alpha)$ and $\beta' \in (\beta, r)$ and define

$$\begin{aligned} \psi(x) &= \mathsf{E}_x \left[e^{-\Lambda_{\tau_{\alpha,\beta'}}} \right], \quad x \in [\alpha, \beta'], \\ \phi(x) &= \mathsf{E}_x \left[e^{-\Lambda_{\tau_{\alpha',\beta}}} \right], \quad x \in [\alpha', \beta]. \end{aligned}$$

One can see that ψ and ϕ are continuous and satisfy (29). Consider now the function ψ in more detail. For any $x \in [\alpha, \beta']$ the Markov property implies

$$\begin{aligned} \mathsf{E}_x \left[e^{-\Lambda_{\tau_{\alpha,\beta'}}} \middle| \mathcal{F}_t \right] &= e^{-\Lambda_{\tau_{\alpha,\beta'}}} I(\tau_{\alpha,\beta'} \le t) + \mathsf{E}_x \left[e^{-\Lambda_{\tau_{\alpha,\beta'}}} I(\tau_{\alpha,\beta'} > t) \middle| \mathcal{F}_t \right] \\ &= e^{-\Lambda_{\tau_{\alpha,\beta'}}} I(\tau_{\alpha,\beta'} \le t) + I(\tau_{\alpha,\beta'} > t) e^{-\Lambda_t} \psi(X_t) \\ &= e^{-\Lambda_{\tau_{\alpha,\beta'} \wedge t}} \psi(X_{\tau_{\alpha,\beta'} \wedge t}). \end{aligned}$$

Hence, the process

$$M_t = e^{-\Lambda_{\tau_{\alpha,\beta'} \wedge t}} \psi(X_{\tau_{\alpha,\beta'} \wedge t}) \tag{30}$$

is a uniformly integrable P_x-martingale, $x \in [\alpha, \beta']$. Consequently, for any $x, a, b \in [\alpha, \beta']$ such that $a < x < b$ it holds that $\mathsf{E}_x M_{\tau_{a,b}} = \psi(x)$. Now we are able to prove convexity of ψ. Indeed,

$$\begin{aligned} \psi(x) &= \mathsf{E}_x \left[e^{-\Lambda_{\tau_{a,b}}} \psi(X_{\tau_{a,b}}) \right] \le \mathsf{E}_x \left[\psi(X_{\tau_{a,b}}) \right] \\ &= \psi(a) \frac{b-x}{b-a} + \psi(b) \frac{x-a}{b-a}. \end{aligned}$$

Applying the Itô–Tanaka formula to (30), one derives that ψ is differentiable on $[\alpha, \beta']$ and ψ' is absolutely continuous on $[\alpha, \beta']$. Moreover, it holds

$$\frac{\sigma^2}{2} \psi''(x) - \lambda(x) \psi(x) = 0$$

for ν_L-a.a. $x \in [\alpha, \beta']$. The analogous result (with $[\alpha, \beta']$ replaced by $[\alpha', \beta]$) can be proved for ϕ as well.

We turn to the inhomogeneous equation

$$\frac{\sigma^2(x)}{2} U''(x) - \lambda(x) U(x) = -f(x) \tag{31}$$

considered ν_L-a.e. on $[\alpha, \beta]$. One can see that the function $\psi'\phi - \psi\phi'$ is constant on $[\alpha, \beta]$ (cp. with formula (26) in Johnson and Zervos (2007)). We denote this constant by A and note that $A \neq 0$. (Indeed, we can choose β' close enough to β so that ψ attains its mimimum at a point $c \in [\alpha, \beta]$. Then $\psi'(c)\phi(c) - \psi(c)\phi'(c) = -\psi(c)\phi'(c) \neq 0$.) A special solution of (31) can be found in the form

$$U_0(x) = \frac{2\phi(x)}{A} \int_\alpha^x \frac{\psi(y) f(y)\, dy}{\sigma^2(y)} + \frac{2\psi(x)}{A} \int_x^\beta \frac{\phi(y) f(y)\, dy}{\sigma^2(y)}, \quad x \in [\alpha, \beta]$$

(cp. with formula (31) in Johnson and Zervos (2007). Note that since $\alpha > \ell$ and $\beta < r$, both integrals are finite). One can see that the function U_0 is differentiable everywhere on $[\alpha, \beta]$ and its derivative is absolutely continuous. Finally, due to (29) we can find constants C_1 and C_2 such that the function

$$U(x) = U_0(x) + C_1\psi(x) + C_2\phi(x), \quad x \in [\alpha, \beta]$$

satisfies $U(\alpha) = U(\beta) = 0$. This function U is what we need.

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