Ordering of insurance risk

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Comparing and ordering of risks is a basic problem of actuarial theory and practice. Risks are generally modeled by random variables or distribution functions. There is a great variety of stochastic models in use which reflect the diversity of insurances like theft insurance, car insurance, liability insurance, etc. The diversity of insured populations is reproduced in the stochastic models by introducing individual or collective risk models which include internal, external, and group risk factors. There are risk models with rare extreme events, and on the other hand models with moderate or even bounded risks. A particular problem in risk theory is dependence in a portfolio of insurance policies which may lead to a drastic increase of the risk of the portfolio.

Ordering of risks gives a guideline to many of the basic tasks of risk theory like measuring of risk or equivalently to the choice and analysis of risk premium principles. It is also a basic tool for estimating the ruin probability, the effect of various bonus–malus and credibility systems, the confidence of statistical estimates forcasting the total of claims, etc. The ordering approach is an extension of the classical mean-variance approach of Markovitz to obtain a more specific analysis of essential risk features. Premium principles and risk measures should be consistent w.r.t. natural risk orderings.

In the following we shall concentrate on two of the most important orderings.

1. Stochastic order and the stop loss order The basic question is: when does a risk X represent a riskier situation than another risk Y? The answer to this question depends on the attitude towards risk aversion. The most simple and obvious postulate is that stochastically larger risks describe more dangerous situations. Here the stochastic ordering $X \leq_{\mathrm{st}} Y$ is defined by the postulate that the expectation of all increasing functions is larger for Y than for X, i.e.

$$Ef(X) \leq Ef(Y), \text{ for all } f \in \mathcal{F}, \tag{1}$$

where $\mathcal{F} = \mathcal{F}_i$ is the set of increasing functions. Equivalent to condition (1) is that the distribution functions F_X, F_Y of X, Y are comparable in the sense that

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &\geq P(Y \leq x) = F_Y(x) \end{aligned} \tag{2}$$

for all x. For most of the usual models in insurance it is not difficult to check whether $X \leq_{\mathrm{st}} Y$. In particular pointwise comparison of risks $X \leq Y$ implies stochastic ordering $X \leq_{\mathrm{st}} Y$ (see Figures 1, 2).

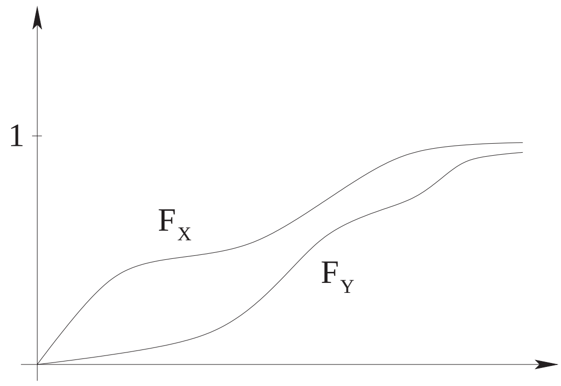

Figure 1 stochastic ordering

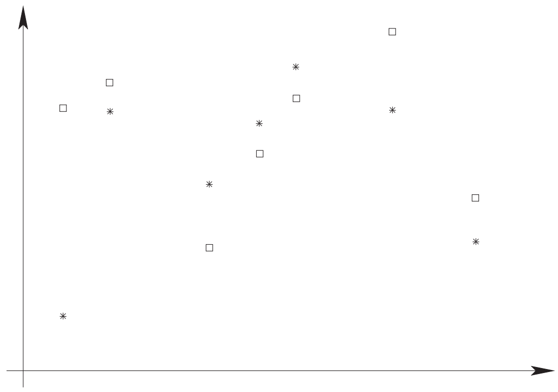

Figure 2 typical sample where $X \leq_{\mathrm{st}} Y$, $* \sim X$, $\square \sim Y$

A common way to describe risk aversion behavior is in terms of utility functions. Risk X is preferred to risk Y by all risk averse agents (traders, persons) if inequality (1) holds true for all $f \in \mathcal{F} = \mathcal{F}_{\mathrm{icx}}$, the class of increasing convex functions. In this case we write

$$X \leq_{\mathrm{icx}} Y \;\; \textit{increasing convex order}. \tag{3}$$

Increasing convex order combines the increase in stochastic order with an increase of the diffusiveness of risks. It is equivalently described by the comparison of the expectation of the angle (call) functions . For all real a holds

$$E(X-a)_+ \leq E(Y-a)_+, \tag{4}$$

i.e. the expected stop loss risks are bigger for Y than for X. Therefore stop loss contracts for the risk Y should have a higher premium than those for X. Equation (4) defines the *stop loss ordering*

$$X \leq_{\mathrm{sl}} Y. \tag{5}$$

The equivalence to the increasing convex order $\leq_{\mathrm{icx}}$ is a basic justification for considering the stop loss order. For risks X, Y with the same expectation $EX = EY$ a sufficient condition for the stop loss order in (5) is the Karlin–Novikov cut criterion saying that the distribution functions F_X, F_Y cross exactly one time, i.e., for some x_0 holds

$$\begin{aligned} & F_X(x) \leq F_Y(x) \quad \text{for } x < x_0 \\ \text{and } & F_X(x) \geq F_Y(x) \quad \text{for } x > x_0. \end{aligned} \tag{6}$$

(6) is a consequence of twice crossing of the densities f_X, f_Y (see Figures 3, 4).

2. Applications

a) Individual and collective risk model The classical individual model of risk theory has the form

$$X_{\mathrm{ind}} = \sum_{i=1}^{n} b_i I_i, \tag{7}$$

where $I_i \sim B(1, p_i)$ are independent Bernoulli distributed random variables. With probability p_i contract i will yield a claim of size $b_i \geq 0$ for any of the n policies. Replacing the claims $b_i I_i$ by $b_i N_i$ with Poisson-distributed $N_i \sim \mathrm{Poisson}(\lambda_i)$ we obtain the classical approximation of the individual risk model by the collective model

$$X_{\mathrm{coll}} = \sum_{i=1}^{n} b_i N_i \tag{8}$$

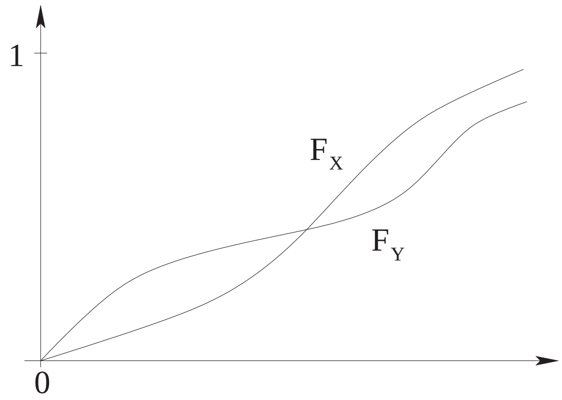

Figure 3 cut criterion

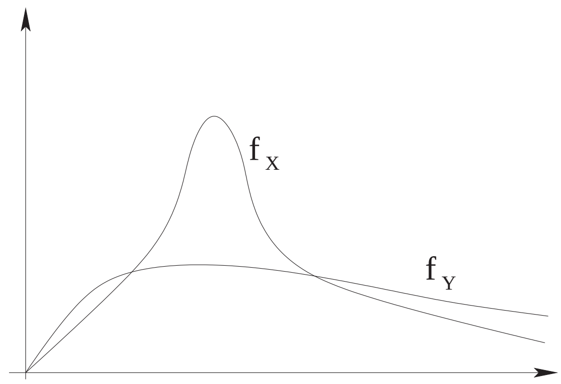

Figure 4

X_{coll} is called *collective model* since it has a representation of the form

$$X_{\text{coll}} = \sum_{i=1}^{N} X_i \tag{9}$$

with $N \sim \text{Poisson}(\lambda)$, $\lambda = \sum_{i=1}^n \lambda_i$ where (X_i) are independent identically distributed with point masses $\frac{\lambda_i}{\lambda}$ at b_i, $1 \leq i \leq n$. If we choose $\lambda_i = p_i$, then the expected payments coincide $EX_{\text{coll}} = EX_{\text{ind}}$ since $EI_i = p_i = EN_i$.

As an application of stochastic and stop loss ordering we get that the collective risk model X_{coll} leads to an overestimate of the risks and, therefore, also to an increase of the corresponding risk premiums for the whole portfolio

$$X_{\text{ind}} \leq_{\text{sl}} X_{\text{coll}}. \tag{10}$$

From the cut criterion it follows that $I_i \leq_{\text{sl}} N_i$ and, therefore, by convolution stability of the stop loss order we obtain the comparison in (10). Choosing the parameter λ_i in the collective model as $\lambda_i = -\log(1 - p_i) > p_i$ we obtain a collective model, which is even more on the safe side for the insurer. With this choice even stochastic ordering holds

$$X_{\text{ind}} \leq_{\text{st}} X_{\text{coll}}. \tag{11}$$

b) Reinsurance contracts As a second application of the stop loss ordering we consider reinsurance contracts $I(X)$ for a risk X, where $0 \leq I(X) \leq X$ is the reinsured part of the risk X and $X - I(X)$ is the retained risk of the insurer. Consider the stop loss reinsurance contract $I_a(X) = (X - a)_+$, where a is chosen such that $EI_a(X) = EI(X)$. Then it follows from the cut criterion (6) that for any reinsurance contract $I(X)$

$$X - I_a(X) \leq_{\text{sl}} X - I(X), \tag{12}$$

holds, i.e. the stop loss contract $I_a(X)$ minimizes the retained risk of the insurer. Thus it is the *optimal reinsurance contract* for the insurer in the class of all contracts $I(X)$ which have the same expected risk.

c) Diversification of risks The stop loss ordering also gives a clue to the *diversification of risk* problem. Let $X_1, \dots, X_n$ be n independent and identically distributed risks. For any diversification strategy (p_i), $0 \leq p_i$ with $\sum_{i=1}^n p_i = 1$ we obtain a diversified risk portfolio $\sum_{i=1}^n p_i X_i$ with p_i relative shares of the i-th risk. Then by stochastic ordering techniques it is easy to establish that under all diversification schemes

(p_i) the *uniform diversification is optimal*, i.e., it has the lowest risk

$$\frac{1}{n}\sum_{i=1}^{n} X_i \leq_{\rm sl} \sum_{i=1}^{n} p_i X_i. \qquad (13)$$

In fact, since the expectations of both sides in (13) coincide we even get the ordering in the stronger sense that (1) holds for $\mathcal{F} = \mathcal{F}_{\rm cx}$, the *convex ordering*. Thus it allows in particular also comparison of the angle (put) function $(a-x)_+$.

d) Dependent portfolios increase risk The following example demonstrates the strong influence that dependence between individual risks may have on the risk of the joint portfolio. Let $X_i = \Theta Y_i + (1-\Theta)Z_i$, $1 \leq i \leq 10^5$, be a mixed model for a large portfolio with Bernoulli distributed $Y_i \sim B(1, \frac{1}{100})$, $Z_i \sim B(1, \frac{1}{1100})$, and $\Theta \sim B(1, \frac{1}{100})$, where all Θ,Y_i, Z_i are independent. It is easy to calculate that $X_i \sim B(1, \frac{1}{1000})$, i.e. each individual contract X_i yields a unit loss with small probability $\frac{1}{1000}$. The presence of Θ in the model implies an increase of the risk of all contracts(*positive dependence*) in rare cases. Thus typically, $\Theta = 0$ and the risks in the portfolio produce independently with small probability $\frac{1}{1100}$ a unit loss to the insurer. With small probability however a bad event $\Theta = 1$ happens which causes that all contracts undergo an increase in risk which now yields with probability $\frac{1}{100}$ a unit loss independently for any of the contracts.

The common risk factor Θ introduces a small positive correlation of magnitude $\frac{1}{1000}$ between the individual risks. It is interesting to compare the total risk $T_n = \sum_{i=1}^n X_i$ in the mixed model (X_i) with the total risk $S_n = \sum_{i=1}^n W_i$ in an independent portfolio model (W_i), where $W_i \sim B(1, \frac{1}{1000})$ are distributed identical to X_i. Then we obtain from stochastic ordering results as above that the risk of T_n is bigger than that of S_n

$$S_n \leq_{\rm sl} T_n. \qquad (14)$$

In fact, also $S_n \leq_{\rm cx} T_n$ since $ES_n = ET_n = 100$. What is the magnitude of this difference? By the central limit theorem S_n is approximately normal distributed with mean $\mu = 100$ and dispersion $\sigma = 10$. Thus $t = \mu + 5\sigma = 150$ is a *safe* retention limit for S_n. $P(S_n > t)$ is extremely small and the net premium is approximatively

$$E(S_n - t)_+ \approx 2.8 \cdot 10^{-8}. \qquad (15)$$

The positive dependence in the mixed model (X_i) which is small in terms of correlation causes a big increase of risk. For the mixed model we get using the same retention limit t as in the independent model the considerable stop loss premium

$$E(T_n - t)_+ \approx 8.5. \qquad (16)$$

The presence of positive dependence in the mixed model implies that with probability about $\frac{1}{100}$ the risk of the joint portfolio T_n is greater than 800. Thus neglecting the common risk factor Θ and basing the calculation of premiums on the incorrect independent model (W_i) would lead to a disaster for the insurance company. The effect demonstrated in this example is in similar form present in many related mixture models and clearly shows the necessity and importance of correct modelling of risks and also the necessity to use more advanced stochastic ordering tools going beyond mean variance analysis.

3. An outlook Many further tools and orderings have been investigated in the literature to describe for specific classes of risk models how they compare w.r.t.

different kinds of risk measures and in which sense one distribution represents a more dangerous situation than another distribution. Several orderings have been developed for risk vectors or risk portfolios in particular for measuring the degree of positive dependence in multivariate portfolios and its influence on various risk functionals. Important examples of positive dependence orderings are the supermodular, the directionally convex, the Δ-monotone orderings, and the positive orthant dependence orderings. In various circumstances and for various risk measures results of the type *more positive dependence implies higher risk* have been established.

A main actual topic of ordering of risks is to obtain sharp bounds on the risk based on partial knowledge of the dependence structure. For detailed exposition of ordering of insurance risks we refer to the following references.

References

[1] R. Kaas, M. Goovaerts, J. Dhaene, and M. Denuit (2001). *Modern Actuarial Risk Theory.* Kluwer Academic Publishers

[2] A. Müller and D. Stoyan (2002). *Comparison Methods for Stochastic Models and Risks.* Wiley

[3] M. Denuit, J. Dhaene, M. Goovaerts, and R. Kaas (2005). *Actuarial Theory for Dependent Risks. Measures, Orders, and Models.* Wiley

[4] L. Rüschendorf (2005). Stochastic ordering of risks, influence of dependence, and a.s. constructions. In: *Advances on Models, Characterizations, and Applications*, eds. N. Balakrishnan et al., 15–56, Chapman & Hall

[5] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels (1998). *Stochastic Processes for Insurance and Finance.* Wiley