Optimal stopping and cluster point processes

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Abstract

In some recent work it has been shown how to solve optimal stopping problems approximatively for independent sequences and also for some dependent sequences, when the associated embedded point processes converge to a Poisson process. In this paper we extend these results to the case where the limit is a Poisson cluster process with random or with deterministic cluster. We develop a new method of directly proving convergence of optimal stopping times, stopping curves, and values and to identify the limiting stopping curve by a unique solution of some first order differential equation. In the random cluster case one has to combine the optimal stopping curve of the underlying *hidden* Poisson process with a statistical prediction procedure for the maximal point in the cluster. We study in detail some finite and infinite moving average processes.

1 Introduction

The theory of optimal stopping of independent and dependent sequences $X_1, \dots, X_n$ is a classical subject of probability theory which still has a lot of open problems and of new applications as for instance in the context of exotic options in financial mathematics. In some recent work a new approach has been developed in order to solve approximatively optimal stopping problems for independent and for some types of dependent sequences $X_1, \dots, X_n$ (see Kühne and Rüschendorf (2000a, 2000b), in the following abbreviated by KR). The basic assumption in this approach is the convergence of the imbedded planar point process

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \to N \tag{1.1}$$

to some Poisson process N. Here $X_{n,i} = \frac{X_i - b_n}{a_n}$ is a normalization of X_i induced from the central limit theorem for maxima. For the limiting Poisson process N, which has accumulation points along a lower boundary curve, an optimal stopping problem in continuous time can be formulated. It is shown in KR (2000b) that the optimal solution of this stopping problem for the Poisson process is given by a threshold stopping time

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where the threshold function is determined by a differential equation of first order. This is in analogy to stopping problems for diffusion processes which typically lead to free boundary value problems with differential equations of second order for the stopping curve (Stefan free boundary problem). The differential equations for the optimal threshold function in the Poisson case can be solved in several cases explicitely or numerically. Furthermore, it has been shown in KR (2000a, 2000b) that some further uniform integrability and separation conditions, a differentiability condition of the intensity measure of N as well as an asymptotic independence condition in the dependent case ensure, that the optimal stopping problem for $X_1, \ldots, X_n$ can be approximated by the optimal stopping problem for the limiting Poisson process.

This approach has been applied to obtain approximative solutions to a large class of stopping problems of iid sequences with observation costs and/or with discount factors (see KR (2000b)). It also has been applied to independent (non iid) sequences (KR (2000d)), to the optimal stopping of some types of moving average processes, to hidden Markov chains and to max-autoregressive sequences (KR (2000a, 2000b)) as well as to best choice problems (KR (2000c)) and to optimal two stopping problems, where one is allowed to stop two times and to choose the best of both values (KR (1999)). It is clear from these examples that the approach should also be applicable to further related stopping problems as for example to m-stopping problems. It is of particular interest that in this way one not only gets structural results for these stopping problems but typically one gets explicit (approximative) optimal stopping values, optimal stopping times and in some cases even optimal stopping distributions.

In this paper we develop an extension of the optimal stopping approach as described above to the case where the limiting process is a Poisson cluster process. In contrast to the previous approach we do not in the first step solve the optimal stopping problem for the limiting Poisson cluster process and then in a second step establish an approximation property for the optimal stopping problem of $X_1, \ldots, X_n$ as in the Poisson process case.

In contrast we directly identify any limiting curve u of the finite optimal stopping curves as unique solution of a differential equation which can be solved in explicit form. This implies in particular convergence of the optimal stopping curves. We also construct explicitly an asymptotically optimal stopping sequence. In the Poisson cluster process case one cannot directly infer from the limiting stopping problem asymptotically optimal stopping sequences since in the limit one looses the time structure for the points in the cluster.

We consider in detail the optimal stopping of moving average sequences. In section 2 we deal with some examples which lead to a limiting Poisson cluster process with a random cluster. In detail we discuss the example where $X_i = Y_i + Y_{i-1}$ for an iid sequence (Y_i). For some type of distribution in the domain of attraction of the extreme value distribution $\Lambda(x) = e^{-e^{-x}}$, $x \in \mathbb{R}^1$, it was shown in Davis and Resnick (1988) that the point process N_n in (1.1) converges to a Poisson cluster process with random clusters of size 2. We then relate the optimal stopping curve for the (X_i) to that of the underlying Poisson process in the limit and we use the cluster structure to determine an asymptotically optimal stopping sequence by a statistical prediction argument for estimating the maximal point in the cluster. We also obtain approximatively the optimal stopping value in explicit form. The method of proof can be extended to more general random cluster cases as long as one can construct similar predictions.

In section 3 we consider infinite moving average sequences with polynomial tails, where the underlying distribution F is in the domain of attraction – $F \in D(\Phi_\alpha)$ – of the extreme value $\Phi_\alpha(x) = e^{-x^\alpha}, x > 0$. The proof of the main theorem in this case can be based upon similar ideas as in the limiting Poisson process case together with an identification procedure for the maximal cluster points which is simpler than the prediction rule in the random cluster case.

It seems possible to extend the methods for optimal stopping of dependent sequences as introduced in this paper as well as in KR (2000a, 2000b) to further interesting classes of dependent sequences which exhibit a similar limiting structure. For further examples we refer to the thesis of Kühne (1997) on which this paper is based.

2 Optimal stopping in the random cluster case

For finite or infinite moving average processes of the form $X_i = \sum_{j=1}^k c_j Y_{i-j}$ (resp. $\sum_{j=1}^\infty c_j Y_{i-j}$) where (Y_i) are iid with df F in the domain of attraction of an extreme value distribution $\Lambda(x) = e^{-e^{-x}}$, $\Phi_\alpha(x) = e^{-x^{-\alpha}}$, $x \geq 0$ resp. $\Psi_\alpha(x) = e^{-(-x)^\alpha}$, $x < 0$ there has been developed an extensive literature on extreme value theory and related point process convergence (see Durrett and Resnick (1978), Davis and Resnick (1988), Davis and Resnick (1991), Resnick (1987), or Rootzen (1986)). Under various conditions it is established in these papers that the embedded point process

$$N_n = \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)} \to N \tag{2.1}$$

converges in distribution to some Poisson process or some Poisson cluster process with random or with deterministic cluster. In KR (2000a, 2000b) it has been shown how to use this point process convergence in order to solve approximatively the related optimal stopping problem for $X_1, \dots, X_n$ in the case that N is a Poisson process. In this section we consider the more involved case of Poisson random cluster processes. In contrast to the previous approach we will not solve in the first step a related stopping problem for the limiting cluster process N and then prove in a second step convergence of the optimal stopping times and values. Instead we use several achievements on convergence of threshold stopping times etc. from KR (2000a, 2000b) and establish a direct approximation argument.

For the case where $F \in D(\Lambda)$ we use the following point process convergence result from Davis and Resnick (1988). Define for a distribution function F with right endpoint of support $\omega_F = \infty$:

$F \in S_r(\gamma)$ for some $\gamma \geq 0$ if

$$\begin{aligned} \lim_{x\to\infty} \frac{1 - F * F(x)}{1 - F(x)} &= d \in (0, \infty) \\ \text{and} \quad \lim_{x\to\infty} \frac{1 - F(x-y)}{1 - F(x)} &= e^{\gamma y}, \quad y \in \mathbb{R}^1. \end{aligned} \tag{2.2}$$

$S_r(0)$ contains the log normal distribution as well as e.g. the distribution with $1-F(x) = e^{-\frac{x}{(\log x)^2}}$, $x > 1$, $\alpha > 0$. More generally suppose that $1 - F_{\alpha,p}(x) \sim Kx^\alpha e^{-x^p}$, $p > 0, \alpha \in \mathbb{R}$ as $x \to \infty$. Then for $p = 1$ and $\alpha < -1$ holds $F = F_{\alpha,p} \in S_r(1)$ while for $p \in (0,1)$ holds $F \in S_r(0)$ (see Durrett and Resnick (1978)).

Davis and Resnick (1988, Prop. 3.1) and Rootzen (1986) have established interesting results in this framework with convergence in (2.1) to a Poisson process. These results have been used in KR (2000a) for related optimal stopping problems. We shall next consider in detail the following special case leading to a random cluster process.

Let $F \in D(\Lambda) \cap S_r(1)$ with left endpoint of support $\alpha_F = 0$ and consider the moving average process $X_i = Y_i + Y_{i-1}$, $i \geq 1$ then by Davis and Resnick (1988)

$$N_n = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, X_i - a_n)} \to N = \sum_{k=1}^{\infty} \sum_{i=1}^{2} \varepsilon_{(\tau_k, y_k + Z_k^i)} \tag{2.3}$$

where $N' = \sum \varepsilon_{(\tau_k, y_k)}$ is the corresponding underlying Poisson process such that the normalized embedded (Y_i) process N'_n with additive normalizations (a_n) converges to N',

$$N'_n = \sum \varepsilon_{(\frac{i}{n}, Y_i - a_n)} \to N' \tag{2.4}$$

and $(Z_k^i)_{i=1,2}$ are independent and independent of (y_k) with $Z_k^i \stackrel{d}{=} Y_1$. So we obtain random clusters of size 2. For the optimal stopping of N' the threshold stopping time with critical function

$$u^\Lambda(t) = \log(1-t) \tag{2.5}$$

has been shown to be optimal in KR (2000b). For the random cluster situation it turns out that the optimal stopping curves for the finite stopping problem of $X_1, \ldots, X_n$ converge to a curve u which is identical to u^Λ if modified by just a constant

$$a := \log \int (e^{-x} E((x + Y_1) \vee E(x + Y_2)_+))dx, \tag{2.6}$$

$$u(t) = u^\Lambda(t) + a. \tag{2.7}$$

Define for random variables Z_1, Z_2

$$Ef(Z_1 + \dot{Z}_2) = \int f(Z_1 + x) dP^{Z_2}(x) \tag{2.8}$$

i.e. the integral is taken w.r.t. the rv with a dot.

Let T_n denote the optimal stopping time of $X_1, \ldots, X_n$ and let $c_n \to 0$ satisfy $P(X_1 \wedge \cdots \wedge X_{\sqrt{n}} \geq c_n) \to 0$ assuming that the left endpoint of the support is zero, $\alpha_F = 0$, and the right endpoint $\omega_F = \infty$. Furthermore, define

$$w_n := F^{-1}\left(1 - \frac{e^a}{n}\right), \qquad \text{and} \qquad \widehat{m}_n := \min\{j \leq \sqrt{n} : X_j = X_1 \wedge \ldots \wedge X_{\sqrt{n}}\}.$$

From the structure of $X_i = Y_i + Y_{i-1}$ it is quite natural to use the alternating sum of the last $\widehat{m}_n$ alternating terms X_j to predict the unobserved Y_i.

$$\widehat{Y}_i := X_i - X_{i-1} + X_{i-2} - \ldots \pm X_{\widehat{m}_n \wedge (i-1)}. \tag{2.9}$$

$\widehat{Y}_i$ predicts the unobserved Y_i.

For the following theorem we define for $0 < \varepsilon < 1$ stopping times

$$T_n^\varepsilon = \min\{T_{n,1}^\varepsilon; T_{n,2}^\varepsilon\} \tag{2.10}$$

where

$$T_{n,1}^\varepsilon := \min\left\{i > n - [n\varepsilon]; X_{\widehat{m}_n} > c_n,\ X_i \geq w_{n-i}\right\},$$
$$T_{n,2}^\varepsilon := \min\left\{i : \sqrt{n} < i \leq n - [n\varepsilon], X_{\widehat{m}_n} \leq c_n \text{ and } \left[\left(\widehat{Y}_i > \widehat{Y}_{i-1} \text{ and } X_i - a_n \geq E\left(\left(\widehat{Y}_i - a_n + \dot{Z}_1^1\right) \vee u\left(\frac{i}{n}\right)\right)\right) \text{ or } \left(\widehat{Y}_i \leq \widehat{Y}_{i-1} \text{ and } X_i - a_n \geq u\left(\frac{i}{n}\right)\right)\right]\right\}.$$

Remark 2.1 The heuristic for the construction of T_n^ε is the following. In the Poisson cluster process points appear pairwise. If one reaches for some $X_i - a_n$ approximatively the first point of a *large* cluster then one has to compare it with the expected value of the second point i.e. with $E((Y_i - a_n + \dot{Z}_1^i) \vee u(\frac{i}{n}))$. Since Y_i is not observed one has to estimate this quantity by replacing Y_i by the predictor $\widehat{Y}_i$. For $i > n - [n\varepsilon]$ we stop by a different simple threshold rule in order to guarantee uniform integrability of the stopped variables.

Theorem 2.2 (Optimal stopping of $\boldsymbol{Y_{i-1} + Y_i}$) *Assume that $F \in D(\Lambda) \cap S_r(1)$ and $\alpha_F = 0, \omega_F = \infty$. For the optimal stopping of $X_i = Y_{i-1} + Y_i$ and the optimal stopping times T_n for $X_1, \ldots, X_n$ we obtain*

1. $EX_{T_n} - a_n \to a$
2. $\lim_{\varepsilon \to 0} \lim_{n \to \infty} (EX_{T_n^\varepsilon} - a_n) = a$

i.e. T_n^ε is an asymptotically optimal double sequence of stopping times.

Proof: In the first part of the proof we consider the modified stopping problem w.r.t. the enriched filtration $\mathcal{F}_i^e = \sigma(Y_0, \ldots, Y_i)$. We make use of several arguments and notations of the proof of Theorem 2.2 in KR (2000a), but since the limiting point process in (2.2) is a Poisson cluster process (with random cluster) we have to choose a new strategy of proof. For the preparation of the proof we first establish two conditions.

Since $X_1, X_3, \ldots$ are iid with $F_{X_1} \in D(\Lambda)$ we obtain that the following lower curve condition (L) holds:

$$\text{Condition (L)} \qquad \liminf_{n \to \infty} E\gamma_{n,[nt]} > -\infty, \forall t < 1 \tag{2.11}$$

where $\gamma_{n,k} = \operatorname{ess\,sup}\{E(X_\tau|\mathcal{F}_k^e), k \le \tau \le n\}$ is the optimal stopping value after time k.

Since $\{X_{2k-1}, k \in \mathbb{N}\}, \{X_{2k}; k \in \mathbb{N}\}$ both are iid with $F_{X_i} \in D(\Lambda)$, the normalized maxima of both sequences are uniformly integrable and therefore this holds also for the normalized maxima of the whole sequence: i.e., we obtain that the uniform integrability condition (G) holds:

Condition (G) $\quad \{(M_n - a_n)^+, n \in \mathbb{N}\}$ is uniformly integrable.

The next step of the proof is essential. We establish directly a weak subsequence compactness of the normalized stopping times T_n/n and the normalized stopping values $X_{T_n} - a_n$. Since $X_{n+2}, \dots$ and $\mathcal{F}_n^e$ are independent we obtain

$$E(\gamma_{n,[nt+2]}|\mathcal{F}_{[nt]}^e) = E\gamma_{n,[nt+2]} =: u_{n,[nt+2]} \tag{2.12}$$

is the optimal stopping value after time $[nt+2]$. There exists by condition(L) a subsequence $(n') \subset \mathbb{N}$ such that for all $t \in [0,1)$ $u_{n,[nt+2]} - a_n \to u(t)$ for some limit function u; w.l.g. let $(n') = \mathbb{N}$. Finally using the methods in Davis and Resnick (1988) and Resnick (1987, p. 144) (see also KR (2000b)) we may assume that for some version of the process a.s. convergence of the points holds i.e. for some $m_{n,k}^j = m_{n,k}^j(\omega)$, $j = 1, 2$, $k = 1, \dots, n, \dots$ holds P a.s. $\left(\frac{m_{n,k}^j}{n}, X_{m_{n,k}^j} - a_n\right) \to \left(\tau_k, y_k + Z_k^j\right)$. Here $m_{n,i}^1 + 1 = m_{n,i}^2$, and we even obtain

$$\left(\frac{m_{n,k}^j}{n}, X_{m_{n,k}^j} - a_n, Y_{m_{n,k}^1} - a_n\right) \to \left(\tau_k, y_k + Z_k^j, y_k\right), \tag{2.13}$$

and $Y_{m_{n,k}^1+1} - a_n \to -\infty$, $Y_{m_{n,k}^1-1} - a_n \to -\infty$.

Step 1: In the first main step of this part of the proof we consider an analogue $\widetilde{T}_n$ of T_n^ε in the enriched framework and establish convergence of $\left(\frac{\widetilde{T}_n}{n}, X_{\widetilde{T}_n} - a_n\right)$ to some limiting stopping time and value $(T, \widehat{y}_{K^T})$ defined in terms of the limiting process.

Define

$$\begin{aligned}\widetilde{T}_n \quad := \quad \inf \Bigg\{ i : &\left(X_i - a_n > u\left(\frac{i}{n}\right),\ i \in \{m_{n,k}^2\}_{k\in\mathbb{N}}\right) \text{ or} \\ &\left(X_i - a_n > E\left(\left(Y_i - a_n + \dot{Z}_1^1\right) \vee u\left(\frac{i}{n}\right)\right),\ i \in \{m_{n,k}^1\}_{k\in\mathbb{N}}\right)\Bigg\}.\end{aligned} \tag{2.14}$$

It is easy to see that no stopping point lies on the curve u. First $P(y_k + Z_k^2 = u(\tau_k)) = 0$ since $\sum_k \varepsilon_{(\tau_k, y_k + Z_k^2)}$ is a Poisson process with intensity $\lambda_{[0,1]} \otimes \nu_1$ with $\nu_1[x,\infty) = \int e^{-(x-z)} P^{Z_k^2}(dz)$. Second $P(y_k + Z_k^1 \ge E((y_k + \dot{Z}_1^1) \vee u(\tau_k))) = 0$ since $\sum_k \varepsilon_{(\tau_k, y_k + Z_k^1)}$ is a Poisson process with Lebesgue continuous intensity. Further $f(y) = E((y + \dot{Z}_1^1) \vee u(\tau_k))$ is continuous, monotonically nondecreasing and $y_1 > y_2$ implies $f(y_1) - f(y_2) > y_1 - y_2$. Therefore, $f(y) = y$ holds for at most one $y \in \mathbb{R}$.

With $\widetilde{T}_n^2 := \inf\{i : X_i - a_n > u\left(\frac{i}{n}\right), i \in \{m_{n,k}^2\}_k\}$ we have convergence of $\left(\frac{\widetilde{T}_n^2}{n}, X_{\widetilde{T}_n^2} - a_n\right)$ by KR (2000b, Proposition 2.4) on convergence of threshold stopping times. Similarly also convergence holds for $\widetilde{T}_n^1 := \inf\Big\{i : X_i - a_n > E\Big((Y_i - a_n + \dot{Z}_1^1) \vee u\left(\frac{i}{n}\right)\Big), i \in \{m_{n,k}^1\}_k\Big\}$ since the right side of the inequality is continuous in Y_i and $u\left(\frac{i}{n}\right)$. Further the Y_i converge along $\{m_{n,k}^1\}_k$ by (2.13). Define $T = \inf\{\tau_k : y_k + Z_k^2 > u(t)$ or $y_k + Z_k^1 > E((y_k + \dot{Z}_1^1) \vee u(t))\}$ and $\widehat{y}_{K^T}$ the corresponding stopping value at the stopping index induced by T, $k = K^T$:

$$\widehat{y}_{K^T} := \begin{cases} y_k + Z_k^1 & \text{if } y_k + Z_k^1 > E((y_k + \dot{Z}_1^1) \vee u(\tau_k)) \\ y_k + Z_k^2 & \text{if } y_k + Z_k^2 > u(\tau_k) \end{cases}$$

Then using $\widetilde{T}_n = \widetilde{T}_n^1 \wedge \widetilde{T}_n^2$ we obtain

$$\left(\frac{\widetilde{T}_n}{n}, X_{\widetilde{T}_n} - a_n\right) \to (T, \widehat{y}_{K^T}). \tag{2.15}$$

and as in KR (2000b) we see that $P(T < 1) = 1$.

Step 2: We next investigate the asymptotics of the optimal stopping curves of $X_1, ..., X_n$ and prove that asymptotically $\widetilde{T}_n$ approximates the optimal stopping times T_n in the sense that

$$P(T_n \neq \widetilde{T}_n) \to 0. \tag{2.16}$$

For the proof of convergence of the optimal stopping curves we first observe that

$$\begin{aligned} E\left(\gamma_{n,\widetilde{T}_n+1} \mid \mathcal{F}^e_{\widetilde{T}_n}\right) &= E\left(X_{\widetilde{T}_n+1} \vee E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right) \\ &= E\left(E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right) \\ &\quad + E\left(E\left(X_{\widetilde{T}_n+1} - E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right)\right)_+ \mid \mathcal{F}^e_{\widetilde{T}_n}\right) \\ &\geq E\left(\gamma_{n,\widetilde{T}_n+2} | \mathcal{F}^e_{\widetilde{T}_n}\right) \; = \; u_{n,\widetilde{T}_n+2}. \end{aligned} \tag{2.17}$$

Next we state that

$$E\left(X_{\widetilde{T}_n+1} - E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}_{\widetilde{T}_n+1}\right)\right)_+ \mathbb{1}_{\{\widetilde{T}_n \in \{m_{n,k}^2\}_k, \widetilde{T}_n \leq n - [n\varepsilon]\}} \to 0.$$

For the proof note that $\left(X_{\widetilde{T}_n+1} - a_n\right) \mathbb{1}_{\{\widehat{T}_n \in \{m_{n,k}^2\}\}} \xrightarrow{P} -\infty$. Further $\Big\{\left(X_{\widetilde{T}_n+1} - a_n\right) \mathbb{1}_{\{\widetilde{T}_n \in \{m_{n,k}^2\}\}}; n \in \mathbb{N}\Big\}$ is uniformly integrable by condition (G), $E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \geq E(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+1}) = u_{n,\widetilde{T}_n+3}$, and $(u_{n,\widetilde{T}_n+3} - a_n)\, \mathbb{1}_{\{\frac{\widetilde{T}_n+3}{n} \leq 1-\varepsilon\}}$ is bounded from below $\forall \varepsilon > 0$ as $\frac{\widetilde{T}_n}{n} \to T$ and $P(T < 1) = 1$. This implies

$$E\left(\left(X_{\widetilde{T}_n+1} - E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right)\right)_+ \mathbb{1}_{\{\widetilde{T}_n \in \{m_{n,k}^2\}\}} \mid \mathcal{F}^e_{\widetilde{T}_n}\right) \to 0$$

From (2.17) we obtain

$$\left(E\left(\gamma_{n,\widetilde{T}_n+1}|\mathcal{F}^e_{\widetilde{T}_n}\right) - E\gamma_{n,\widetilde{T}_n+2}\right)\mathbb{1}_{\{\widetilde{T}_n\in\{m^2_{n,k}\}\}} \xrightarrow{P} 0$$

and therefore

$$\left(E\left(\gamma_{n,\widetilde{T}_n+1} \mid \mathcal{F}^e_{\widetilde{T}_n}\right) - a_n - Eu\left(\tfrac{\widetilde{T}_n+2}{n}\right)\right)\mathbb{1}_{\{\widetilde{T}_n\in\{m^2_{n,k}\}\}} \xrightarrow{P} 0. \tag{2.18}$$

Similarly we obtain for the stopping curves on $\widetilde{T}_n \in \{m^1_{n,k}\}$

$$\begin{aligned}
&E\left(\gamma_{n,\widetilde{T}_n+1} \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&= E\left(X_{\widetilde{T}_n+1} \vee E\left(\gamma_{n,\widetilde{T}_n+2} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&= E\left(X_{\widetilde{T}_n+1} \vee E\left(X_{\widetilde{T}_n+2} \vee E\left(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+2}\right) \mid \mathcal{F}_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&\geq E\left(X_{\widetilde{T}_n+1} \vee E\left(E\left(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+2}\right) \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) |\mathcal{F}^e_{\widetilde{T}_n}\right)\\
&= E\left(X_{\widetilde{T}_n+1} \vee E\left(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&= E\left(X_{\widetilde{T}_n+1} \vee E\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&= E\left(\left(Y_{\widetilde{T}_n} + Y_{\widetilde{T}_n+1}\right) \vee E\gamma_{n,\widetilde{T}_n+3} \mid Y_0,\dots,Y_{\widetilde{T}_n}\right)\\
&= E\left(\left(Y_{\widetilde{T}_n} + \dot{Y}_{\widetilde{T}_n+1}\right) \vee E\gamma_{\widetilde{T}_n+3}\right).
\end{aligned}$$

Arguing as above this implies

$$\begin{aligned}
&\Big(E\left(X_{\widetilde{T}_n+1} \vee E\left(X_{\widetilde{T}_n+2} \vee E\left(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+2}\right) \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\\
&\quad - E\left(X_{\widetilde{T}_n+1} \vee E\left(E\left(\gamma_{n,\widetilde{T}_n+3} \mid \mathcal{F}^e_{\widetilde{T}_n+2}\right) \mid \mathcal{F}^e_{\widetilde{T}_n+1}\right) \mid \mathcal{F}^e_{\widetilde{T}_n}\right)\Big)\mathbb{1}_{\{\widetilde{T}_n\in\{m^1_{n,k}\}\}} \xrightarrow{P} 0.
\end{aligned}$$

Thus

$$\left(E\left(\gamma_{n,\widetilde{T}_n+1} - a_n|\mathcal{F}^e_{\widetilde{T}_n}\right) - E\left(\left(Y_{\widetilde{T}_n} + \dot{Y}_{\widetilde{T}_n+1} - a_n\right) \vee u\left(\tfrac{\widetilde{T}_n+3}{n}\right)\right)\right)\mathbb{1}_{\{\widetilde{T}_n\in\{m^1_{n,k}\}\}} \xrightarrow{P} 0. \tag{2.19}$$

The asymptotic equivalence of the optimal stopping curve and the stopping curve of $\widetilde{T}_n$ in (2.18), (2.19) implies as in the prove in KR(2000a, (2.19)) $P(T_n \neq \widetilde{T}_n) \to 0$, i.e., (2.16).

As consequence of (2.15) and (2.16) we conclude convergence of the normalized optimal stopping time and value

$$\left(\frac{T_n}{n}, X_{T_n} - a_n\right) \to (T, \widehat{y}_{K^T}). \tag{2.20}$$

Finally from conditions (G), (L) we obtain as in the proof in KR (2000b, Theorems 4.3–4.5, (4.34)) (see also KR (2000a, (2.12)))

$$E\widehat{y}_{K^{T\geq t}} = \lim_{n'\to\infty} EX_{T_{n'}^{\geq [n't]}} - a_{n'} = u(t). \tag{2.21}$$

Here $K^{T\geq t}$, $T_{n'}^{\geq [n't]}$ are the threshold stopping times and stopping index restricted to the time domain $\geq t$ resp. $\geq [n't]$. Thus we obtain along the subsequence (n') the asymptotics of the optimal stopping times and values (see (2.20), (2.15)).

Step 3: The second main part of the proof is to identify the limit point u as the unique solution of a first order differential equation. This part then implies convergence of the optimal stopping sequence as in (2.20), (2.21) along the whole sequence $\mathbb{N}$.

The argument for this step is an extension of the proof of the corresponding result for Poisson processes in KR (2000b, proof of Theorem 2.5). We first have to determine the distribution of the limiting Poisson cluster process in (2.2). To that purpose we introduce a distributional version of the limiting cluster point process of points above the threshold as in the definition of T (resp. (2.15)). Let $N^1 = \sum_k \varepsilon_{\tau_k^1}$ be a Poisson process with intensity measure μ_1, where

$$\mu_1(ds) = \int e^{-y} P(y+Y_2 \geq u(s) \text{ or } y+Y_1 \geq E((y+Y_2)\vee u(s)))\,dy\,ds. \tag{2.22}$$

Let $y_i^1, i\in\mathbb{N}$ be conditionally independent given N^1 with

$$P(y_i^1 \in dy | \tau_i^1 = s) = \frac{e^{-y} P(y+Y_2 \geq u(s) \text{ or } y+Y_1 \geq E((y+Y_2)\vee u(s)))}{\int e^{-y} P(y+Y_2 \geq u(s) \text{ or } y+Y_1 \geq E((y+Y_2)\vee u(s)))\,dy}. \tag{2.23}$$

Let further (z_i^1, z_i^2) be rv's conditionally independent given N^1 and $y_i^1, i\in\mathbb{N}$ such that

$$\begin{aligned} &P((z_i^1, z_i^2)\in \cdot \mid N^1) \\ &= P((Y_1,Y_2)\in\cdot \mid y_i^1+Y_2 \geq u(\tau_i^1) \text{ or } y_i^1 + Y_1 \geq E((y_i^1+\dot{Y}_2)\vee u(\tau_i^1))). \end{aligned} \tag{2.24}$$

Then $\sum_i \varepsilon_{(\tau_i^1, y_i^1+z_i^1, y_i^1+z_i^2)}$ is a Poisson process and

$$\sum_i \varepsilon_{(\tau_i^1, y_i^1+z_i^1, y_i^1+z_i^2)} \overset{d}{=} \sum_{\substack{k;\, y_k+Z_k^2\geq u(\tau_k) \text{ or} \\ y_k+Z_k^1 \geq E((y_k+\dot{Z}_1^1)\vee u(\tau_k))}} \varepsilon_{(\tau_k, y_k+Z_k^1, y_k+Z_k^2)}. \tag{2.25}$$

To prove (2.25) we first establish finiteness of μ_1 which implies a.s. finiteness of the number of points above the threshold. Let $\mu_4 := \lambda_{[0,1]}\otimes\nu\otimes P^{Y_1}\otimes P^{Y_1}$, let $M\subset\mathbb{R}^4$ be defined by

$$M := \{(t,y,z^1,z^2):\ y+z^2 \geq u(t) \text{ or } y+z^1 \geq E((y+Z_1^1)\vee u(t))\}$$

and define $\mu_4^u := \mu_4(\cdot \cap M)$ the restriction of μ_4 on M.

Then the projection $(\mu_4^u)^{\pi_1}([0,t]) < \infty, \forall t < 1$. To see this we introduce

$$\begin{aligned} M' &:= \{(t,y,z^1,z^2): \ y+z^2 \geq u(t) \ \text{ or } \ y+z^1 \geq u(t)\} \\ &= \{(t,y,z^1,z^2): \ y+z^2 \vee z^1 \geq u(t)\}. \end{aligned}$$

Then $M \subset M'$ and $\sum_{k: y_k + Z_k^1 \vee Z_k^2 \geq u(\tau_k)} \varepsilon_{\tau_k, y_k + Z_k^1 \vee Z_k^2}$ is by Davis and Resnick (1988) a Poisson process with intensity $\mu' = \lambda_{[0,1]} \times \nu'(\cdot \cap (\pi_1,\pi_2)(M'))$ where $\nu'[x,\infty) = \int e^{-(x-z)} P^{Z_k^1 \vee Z_k^2}(dz)$. Thus $N(\cdot \cap M')$ is a Poisson process with intensity μ' and $(\mu')^{\pi_1}([0,t]) < \infty$ for all $t < 1$ and thus finiteness follows for $(\mu_4^u)^{\pi_1}$. Define μ_2 on $[0,1] \times \mathbb{R}$ by

$$\mu_2(dt,dy) = e^{-y} P(y+Y_2 \geq u(t) \ \text{ or } \ y+Y_1 \geq E(Y_2 \vee u(t)))\, dt\, dy.$$

Then we obtain by direct calculation $\mu_2 = (\mu_4^u)^{(\pi_1,\pi_2)}$ and from the definition of μ_2 it follows that $\mu_1 = (\mu_4^u)^{\pi_1}$. In particular μ_1 defined above is finite, $\mu_1([0,t] < \infty, \forall t < 1$.

As consequence of the finiteness of μ_1 we can assume w.l.g. that $\tau_1^1 < \tau_2^1 < \dots$ are ordered and that $T = \tau_1^1$ a.s. Further as in KR (2000b, proof of Theorem 2.5) one finds that $N^2 = \sum_k \varepsilon_{(\tau_k^1, y_k^1)}$ is a Poisson process with intensity μ_2 and by an argument as above applied to N_2 and z_i^j one obtains the equality in (2.25).

To establish the differential equation for u we introduce $h(x,y) := E((x+Y_1) \vee E((x+Y_2) \vee y) - y)$. Then $h(x+z, y+z) = h(x,y)$, and thus

$$\begin{aligned} \int e^{-x} h(x,y)\, dx &= \int e^{-x} h(x-y,0)\, dx \\ &= \int e^{-(x+y)} h(x,0)\, dx \\ &= e^{-y} \int e^{-x} h(x,0)\, dx \\ &= e^{-y} e^a. \end{aligned} \tag{2.26}$$

The following identity will turn out to be useful. Let for random variables W_i, $V((W_i))$ denote the optimal stopping value for (W_i), then

$$\begin{aligned} &E\Big((y+Y_1)\mathbb{1}_{\{y+Y_1 \geq E((y+\dot{Y}_2) \vee u(s))\}} + (y+Y_2)\mathbb{1}_{\{y+Y_1 < E((y_k+\dot{Y}_2)\vee u(s))\}} - u(s)\Big)_+ \\ &= E\Big((\underbrace{y+Y_1}_{=:W_1})\mathbb{1}_{\{y+Y_1 \geq E((y+Y_2)\vee u(s))\}} \\ &\quad + (\underbrace{(y+Y_2) \vee u(s)}_{=:W_2})\mathbb{1}_{\{y+Y_1 < E((y_k+\dot{Y}_2)\vee u(s))\}}\Big) - u(s) \\ &= EW_S - u(s) \qquad \text{where } S := \inf\{i \in \{1,2\} : W_i \geq E((y+Y_2) \vee u(s))\} \\ &= V(W_1, W_2) - u(s) = V(y+Y_1, (y+Y_2) \vee u(s)) - u(s) \\ &\qquad \text{as } S \text{ is the optimal stopping time of } W_1, W_2 \\ &= E((y+Y_1) \vee E((y+Y_2) \vee u(s)) - u(s)). \end{aligned}$$

Let $\mathcal{T} := T^{\geq t} = \tau_{\inf\{i:\tau_i \geq t\}}$ denote the first stopping point of T after time t and let $y_{K^{\mathcal{T}}}$ denote the value of the (y_k) at the corresponding stopping index $K^{\mathcal{T}}$. Then we obtain using the previously established identities and the distributional properties of the point process

$$\begin{aligned}
u(t) =&\ E\widehat{y}_{K^{\mathcal{T}}} \\
=&\ \int_t^1 \int E(\widehat{y}_{K^{\mathcal{T}}} | \mathcal{T} = s, y_{K^{\mathcal{T}}} = y) dP^{y_{K^{\mathcal{T}}}, \mathcal{T}}(y, s) \\
=&\ \int_t^1 \Big(u(s) + \int E((y_{K^{\mathcal{T}}} + z_1^1) \mathbb{1}_{\{y_{K^{\mathcal{T}}} + z_1^1 \geq E((y_{K^{\mathcal{T}}} + \dot{z}_k^2) \vee u(\mathcal{T}))\}} \\
&\qquad + (y_{K^{\mathcal{T}}} + z_k^2) \mathbb{1}_{\{y_{K^{\mathcal{T}}} + z_k^1 < E((y_{K^{\mathcal{T}}} + \dot{z}_k^2) \vee u(\mathcal{T}))\}} \\
&\qquad - u(s) | \mathcal{T} = s, y_{K^{\mathcal{T}}} = y) dP^{y_{K^{\mathcal{T}}} | \mathcal{T} = s}(y) \Big) dP^{\mathcal{T}}(s) \\
\overset{(2.24)}{=}&\int_t^1 \Big(u(s) + \int E\Big((y + Y_1) \mathbb{1}_{\{y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s))\}} \\
&\qquad + (y + Y_2) \mathbb{1}_{\{y + Y_1 < E((y + \dot{Y}_2) \vee u(s))\}} \\
&\qquad - u(s) \mid y + Y_2 \geq u(s) \text{ or } y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s)) \Big) \\
&\qquad dP^{y_{K^{\mathcal{T}}} | \mathcal{T} = s}(y) \Big) dP^{\mathcal{T}}(s) \\
=&\int_t^1 \Big(u(s) + \int \frac{E((y + Y_1) \mathbb{1}_{\{y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s))\}} + (y + Y_2) \mathbb{1}_{\{y + Y_1 < E((y_k + \dot{Y}_2) \vee u(s))\}} - u(s))_+}{P(y + Y_2 \geq u(s) \text{ or } y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s)))} \\
&\qquad dP^{y_{K^{\mathcal{T}}} | \mathcal{T} = s}(y) \Big) dP^{\mathcal{T}}(s) \\
\overset{(2.23), (2.22), (2.25)}{=}& \int_t^1 \Big(u(s) + \int \frac{E((y + Y_1) \vee E((y + Y_2) \vee u(s)) - u(s))}{P(y + Y_2 \geq u(s) \text{ or } y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s)))} \\
&\qquad \cdot \frac{e^{-y} P(y + Y_2 \geq u(s) \vee y + Y_1 \geq E((y + \dot{Y}_2) \vee u(s))}{\frac{d\mu_1}{d\lambda}(s)} \Big) dP^{\mathcal{T}}(s) \\
=&\ \int_t^1 \Big(u(s) + \int e^{-y} \frac{E((y + Y_1) \vee E((y + \dot{Y}_2) \vee u(s)) - u(s))}{\frac{d\mu_1}{d\lambda}(s)} dy \Big) dP^{\mathcal{T}}(s) \\
\overset{(2.22)}{=}& \int_t^1 \left(u(s) + \int e^{-y} \frac{h(y, u(s))}{\frac{d\mu_1}{d\lambda}(s)} dy \right) \frac{d\mu_1}{d\lambda}(s) e^{-(\mu_1([0,s]) - \mu([0,t]))} ds \\
\overset{(2.26)}{=}& \int_t^1 \left(u(s) + \frac{e^{-u(s)} e^a}{\frac{d\mu_1}{d\lambda}(s)} \right) \frac{d\mu_1}{d\lambda}(s) e^{-(\mu_1([0,s]) - \mu_1([0,t]))} ds.
\end{aligned}$$

By differentiation w.r.t. t we obtain as in KR (2000b, proof of Theorem 2.5)

$$u'(t) = -e^{a - u(t)}, \qquad u(1) = -\infty. \tag{2.27}$$

This differential equation has by Proposition 2.6 in KR (2000b) the unique solution

$u(t) = a + \log(1-t)$. Since this holds for any limit of a convergent subsequence there exists a unique limit point of (u_n) given in (2.27) and thus u_n converges to u, $u_n \to u$.

Step 4: In the previous parts of the proof we considered stopping w.r.t. the filtration $\mathcal{F}_n^e$ generated by the Y_i. In the final step we show that the Y_i can be estimated by the X_i to a sufficient precise order so that one can use the canonical filtration $(\mathcal{F}_n)$ to obtain approximatively the same stopping behavior. Here we need the assumption $\alpha_F = 0$, i.e., the left endpoint of the support of F is zero. Note that for $j < i$

$$Y_i = \begin{cases} X_i - X_{i-1} + X_{i-2} - \cdots + X_j - Y_{j-1} & \text{if} \quad i-j = 0 \mod 2 \\ X_i - X_{i-1} + X_{i-2} - \cdots - X_j + Y_{j-1} & \text{if} \quad i-j = 1 \mod 2, \end{cases}$$

in particular if $\widehat{m}_n < i$

$$Y_i = X_i - X_{i-1} + X_{i-2} - \cdots \pm X_{\widehat{m}_n+1} \mp Y_{\widehat{m}_n}. \tag{2.28}$$

By assumption $P(X_1 \wedge \cdots \wedge X_{\sqrt{n}} \geq c_n) \to 0$ for some sequence $c_n \to 0$. As $\alpha_F = 0$ and $X_{\widehat{m}_n} \geq Y_{\widehat{m}_n} \geq 0$, we obtain $Y_{\widehat{m}_n} \xrightarrow{P} 0$. Therefore, from (2.28) we conclude that

$$\sup_{\sqrt{n} < i \leq n} \{|Y_i - \widehat{Y}_i|\} \xrightarrow{P} 0, \tag{2.29}$$

i.e., $\widehat{Y}_i$ uniformly in $\sqrt{n} \leq i \leq n$ estimates Y_i. We next define $\widetilde{T}_n'$ by plugging in $\widehat{Y}_i$ for Y_i in the definition of $\widetilde{T}_n$. The conditions $i \in \{m_{n,k}^j\}$ for $j = 1, 2$ are replaced by the conditions $\widehat{Y}_i > \widehat{Y}_{i-1}$ resp. $\widehat{Y}_i \leq \widehat{Y}_{i-1}$, which by (2.13) and (2.29) are asymptotically equivalent. Formally,

$$\begin{aligned} \widetilde{T}_n' \quad := \quad \inf \Bigg\{ i : & \left(\widehat{Y}_i > \widehat{Y}_{i-1}, X_i - a_n \geq E\left(\left(\widehat{Y}_i - a_n + \dot{Z}_1^1 \right) \vee u\left(\frac{i}{n}\right) \right) \right) \\ & \text{or} \quad \left(\widehat{Y}_i \leq \widehat{Y}_{i-1}, X_i - a_n \geq u\left(\frac{i}{n}\right) \right) \Bigg\}. \end{aligned}$$

The asymptotic equivalence of the stopping thresholds implies $P(\widetilde{T}_n' = \widetilde{T}_n) \to 1$. Therefore, by (2.16)

$$P(\widetilde{T}_n' = T_n) \to 1. \tag{2.30}$$

By definition of T_n^ε and $\widetilde{T}_n'$

$$T_n^\varepsilon \Big/_{\{\sqrt{n} < \widetilde{T}_n' \leq n(1-\varepsilon), X_{\widehat{m}_n} \leq c_n\}} = \widetilde{T}_n' \Big/_{\{\sqrt{n} < \widetilde{T}_n' \leq n(1-\varepsilon), X_{\widehat{m}_n} \leq c_n\}}.$$

Combining the convergence of the stopped variables, in (2.15), (2.20), (2.21) with (2.30) and using $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} E(X_{T_n^\varepsilon} - a_n)_- \mathbb{1}_{\{T_n^\varepsilon > n(1-\varepsilon)\}} \to 0$ we obtain finally $\lim_{\varepsilon \to 0} \lim_{u \to \infty} (EX_{T_n^\varepsilon} - a_n) = a$. □

Remark 2.3 1.) The proof of Theorem 2.2 also implies convergence in distribution of $\left(\frac{T_n}{n}, X_{T_n} - a_n\right)$. It seems however difficult to calculate the distribution of the stopped variables in the limiting process.

2.) In KR (2000a, 2000b) the essential idea was to solve the optimal stopping problem for the limiting process. Using this solution it was possible to introduce some finite stopping rules T_n' which yield asymptotically the optimal stopping value for the limiting problem. Then it was possible to establish that the optimal stopping value of the limiting problem is asymptotically an upper bound for the finite stopping problem and thus T_n' is asymptotically optimal.

In our present proof we use a weak compactness argument and the convergence of threshold stopping times for the optimal stopping sequence to obtain asymptotic equivalence of the optimal stopping sequence with a simple threshold stopping sequence whose definition depends on a special weak limit point u of the optimal threshold sequence (u_n) (cp. (2.14)). Then in a second step we identify the weak limit points by a differential equation and thus in particular get uniqueness and convergence of (u_n). Finally by a prediction procedure we can extend this result to the natural filtration $\mathcal{F}_n$ (instead of $\mathcal{F}_n^e$).

As mentioned in the introduction the method of proof of Theorem 2.2 works for more general moving average sequences. The essential point is to find consistent predictors of the Y_i. To consider a concrete example let $b \in (0,1)$ and let $(Y_i)_{i\geq -2}$ be iid random variables with df $F \in D(\Lambda) \cap S_r(1)$, $\alpha_F = 0$ and consider the MA-sequence of length 3.

$$X_i = Y_i + bY_{i-1} + Y_{i-2}, \quad i \in \mathbb{N}. \tag{2.31}$$

Define the analogue to the constant a in Theorem 2.2:

$$a_* := \log \int \left(e^{-x} E\left((x + bY_1 + Y_2) \vee E(x + bY_3 + Y_4)_+\right)\right) dx. \tag{2.32}$$

Let $c_n > 0$ be a sequence with $c_n \to 0$ and

$$P((X_1 \vee X_2) \wedge \cdots \wedge (X_{\sqrt{n}} \vee X_{\sqrt{n}-1}) \geq c_n) \to 0.$$

Define $u := u^\Lambda + a_*$ and $w_n := F^{-1}\left(1 - \frac{e^{a_*}}{n}\right)$, where $u^\Lambda(t) = \log(1-t)$ is the optimal stopping curve for the Poisson process in the corresponding iid case with $F \in D(\Lambda)$ (see KR (2000b, Theorem 4.3)). Further we introduce $\widehat{m}_n := \min\{2 \leq i \leq \sqrt{n} : X_i \vee X_{i-1} = \bigwedge_{j=2}^{\sqrt{n}} (X_j \vee X_{j-1})\}$, $a_i := (0\ 1)\mathcal{M}^{i-1}\binom{0}{1}$, $i \geq 1$, with $\mathcal{M} := \begin{pmatrix} 0 & -1 \\ 1 & -b \end{pmatrix}$, and the predictor $\widehat{Y}_i := \sum_{\ell=i-\widehat{m}_n+1}^{i} X_\ell\, a_{i-\ell+1}$ of the Y_i. Finally we introduce the analogue of the stopping times in (2.10)

$$T_n^\varepsilon = \min\{T_{n,1}^\varepsilon, T_{n,2}^\varepsilon\} \tag{2.33}$$

with

$$T_{n,1}^{\varepsilon} = \inf\Big\{i : (i > n - [n\varepsilon]; X_{\widehat{m}_n} > c_n, X_i \geq w_{n-i})\Big\}$$

$$T_{n,2}^{\varepsilon} = \inf\Big\{i : \sqrt{n} < i \leq n - [n\varepsilon], X_{\widehat{m}_n} \leq c_n, \text{ and } \Big[\widehat{Y}_i > \widehat{Y}_{i-2}, \text{ and } X_i - a_n \geq E\Big((\widehat{Y}_i - a_n + \dot{Z}_1^1 + b\dot{Z}_2^1) \vee u\Big(\frac{i}{n}\Big)\Big)\Big] \text{ or } \Big[\widehat{Y}_i \leq \widehat{Y}_{i-2}, \text{ and } X_i - a_n \geq u\Big(\frac{i}{n}\Big)\Big]\Big\}.$$

Then we obtain the following analog to Theorem 2.2.

Theorem 2.4 *Let F, (X_i), (c_n) be as in (2.31), (2.32) then*

$$EX_{T_n} - a_n \to a_*$$

and (T_n^{ε}) is an asymptotically optimal double sequence of stopping times, i.e.

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}(EX_{T_n^{\varepsilon}} - a_n) = a_*.$$

Proof: For the proof we can repeat the steps of the proof of Theorem 2.2. It remains to establish that the estimates $\widehat{Y}_i$ uniformly approximate the Y_i, i.e.

$$\sup_{\sqrt{n}<i\leq n}\{|Y_i - \widehat{Y}_i|\} \xrightarrow{P} 0. \tag{2.34}$$

To that purpose we obtain by induction that the Y_i satisfy

$$Y_i = (0\;1)\mathcal{M}^j \begin{pmatrix} Y_{i-j-1} \\ Y_{i-j-2} \end{pmatrix} + \sum_{\ell=i-j+1}^{i} X_\ell\, a_{i-\ell+1}, \quad 1 \leq j \leq n.$$

In particular

$$Y_i = (0\;1)\mathcal{M}^{i-\widehat{m}_n-1} \begin{pmatrix} Y_{\widehat{m}_n} \\ Y_{\widehat{m}_n-1} \end{pmatrix} + \sum_{\ell=\widehat{m}_n}^{i} X_\ell\, a_{i-\ell+1}. \tag{2.35}$$

The minimal polynomial m of $\mathcal{M}$ is given by

$$m(x) = \text{Det}\left(\begin{pmatrix} 0 & -1 \\ 1 & -b \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right) = x^2 + bx + 1,$$

with roots $x_0^{1,2} = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - 1}$, both of norm 1. Therefore, $\{\mathcal{M}^j : j \in \mathbb{N}\}$ is bounded. From the definition of c_n and $\widehat{m}_n$ we conclude that

$Y_{\widehat{m}_n} \vee Y_{\widehat{m}_n-1} \xrightarrow{P} 0$. This implies as $n \to \infty$

$$\sup_{\sqrt{n}<i\leq n}\left\{(0\ 1)\mathcal{M}^{i-\widehat{m}_n-1}\begin{pmatrix}Y_{\widehat{m}_n}\\ Y_{\widehat{m}_n-1}\end{pmatrix}\right\}\xrightarrow{P}0.$$

From (2.35) and the definition of $\widehat{Y}_i$ we obtain (2.34). □

The construction of similar uniformly consistent predictions should be feasible for much more general sequences.

3 Deterministic infinite cluster case

For infinite moving average processes with polynomial tails the following point process result was proved in Resnick (1987). Let (Y_i) be iid with df F, let $\alpha > 1$, $c_j \in \mathbb{R}$, $j \in \mathbb{N}$, where some $c_j \neq 0$. Let $c_1 \neq 0$ and $\sum_{j=1}^{\infty} |c_j|^{\delta} < \infty$ for some $0 < \delta < \alpha \wedge 1$. We introduce the following conditions:

A1) $F \in D(\Phi_\alpha)$ and all $c_j \geq 0$

A2) $F(0-\cdot) \in D(\Phi_\alpha)$ and all $c_j \leq 0$

A3) $P(|Y_1| > x) \in RV_{-\alpha}$, i.e. $P(|Y_1| > x)$ is of regular variation of order $-\alpha$. Further the limit $\lim_{x\to\infty} \frac{P(Y_1>x)}{P(|Y_1|>x)} =: p$ exists, where some $c_j > 0$ in case $p = 1$ and some $c_j < 0$ in case $p = 0$.

In this section we consider the infinite moving average process

$$X_i := \sum_{j=1}^{\infty} c_j Y_{i-j+1}, \quad i \in \mathbb{N}. \tag{3.1}$$

Then for the imbedded point processes convergence in distribution holds on $[0,1]\times(0,\infty]$ to a deterministic infinite cluster process N:

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)} \;\to\; N = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \varepsilon_{(\tau_k, c_i y_k)}. \tag{3.2}$$

Here $N' = \sum_{i=1}^{\infty} \varepsilon_{(\tau_i,y_i)}$ is an underlying Poisson process with intensity $\mu = \lambda_{[0,1]} \otimes \nu$, $\nu([x,\infty]) = x^{-\alpha}$ and (a_n) is the normalization of the limit law of the maxima $M_n = Y_1 \vee \cdots \vee Y_n$ (see Resnick (1987, chap. 4.5)).

In comparison to section 2 we have infinite deterministic clusters. For the construction of optimal stopping curves the idea is to wait after appearance of the first point of a large cluster until the point with the biggest coefficient is observed and then to compare it with the stopping curve. No additional estimation step is necessary compared to the random cluster case. Denote by $u = u_\alpha^\Phi$ the asymptotically optimal stopping curve for the iid sequence $Y_1, \dots, Y_n$ resp. the exact optimal curve for the Poisson process N' above given by

$$u_\alpha^\Phi(t) = \left(\frac{\alpha}{\alpha^2-1}\right)^{\frac{1}{\alpha}} \left(1-t^{1+\alpha}\right)^{\frac{1}{\alpha}} \tag{3.3}$$

(see KR (2000b, Theorem 4.4)). T_n denotes as in the first part the optimal stopping time for the stopping of $X_1, \dots, X_n$.

Theorem 3.1 *Consider the moving average process* $X_i = \sum_{j=1}^\infty c_j Y_{i-j+1}$ *as in (3.1):*
a) Under conditions A1) or A2) and assuming w.l.g. $\sup_i |c_i| = 1$ *we define*

$$m := \inf\{i:\ c_i = \sup_j\{|c_j|\}\}, \quad w := \sup\{i:\ |c_i| \geq |c_1|\}$$

and the stopping time

$$\begin{aligned} T_n' \quad := \quad \inf\Bigg\{ & i \geq w+1 : X_i \geq a_n \frac{c_1}{c_m} u_\alpha^\Phi\left(\frac{i}{n}\right) \text{ and} \\ & X_{i-1} \vee \dots \vee X_{i-w} < \frac{1}{2} a_n \frac{c_1}{c_m} u_\alpha^\Phi\left(\frac{i}{n}\right)\Bigg\} + m - 1. \end{aligned}$$

b) Under condition A3) we define

$$\begin{aligned} & m_s := \inf\{i:\ c_i = \sup_j\{c_j\}\}, \qquad m_\ell := \inf\{i:\ c_i = \inf_j\{c_j\}\}, \text{ and} \\ & i^+ := \inf\{i:\ c_i > 0\}, \qquad i^- := \inf\{i:\ c_i < 0\}, \qquad c_\infty := 0. \end{aligned}$$

We assume w.l.g. $p(c_{m_s})_+^{\frac{1}{2}} + (1-p)(-c_{m_\ell})_+^{\frac{1}{2}} = 1$ *and we define the stopping time*

$$T_n' = \min\{T_n^1, T_n^2\} \tag{3.4}$$

where

$$\begin{aligned} T_n^1 \ := \ \inf\Bigg\{ & i : X_i \geq a_n \frac{(c_i)_+}{(c_{m_s})_+} u_\alpha^\Phi\left(\frac{i}{n}\right) \text{ and } X_{i-1} \vee \dots \vee X_{i-m_s} \\ & < \frac{1}{2} a_n \frac{(c_i)_+}{(c_{m_s})_+} u_\alpha^\Phi\left(\frac{i}{n}\right)\Bigg\} + m_s - i^+ \end{aligned}$$

and

$$\begin{aligned} T_n^2 \ := \ \inf\Bigg\{ & i : -X_i \geq a_n \frac{(c_i)_-}{(c_{m_\ell})_-} u_\alpha^\Phi\left(\frac{i}{n}\right) \text{ and } |X_{i-1} \vee \dots \vee X_{i-m_\ell}| \\ & < \ -\frac{1}{2} a_n \frac{(c_i)_-}{(c_{m_\ell})_-} u_\alpha^\Phi\left(\frac{i}{n}\right)\Bigg\} + m_\ell - i^-. \end{aligned}$$

Then in both cases a), b) holds

$$\frac{EX_{T_n}}{a_n} \to \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}}, \tag{3.5}$$

and T_n' *is an asymptotically optimal sequence of threshold stopping times.*

Furthermore in case a) holds:

$$P\left(\left\{\frac{X_{T_n}}{a_n} \leq x\right\}\right) \to \begin{cases} 1 - x^{-\alpha}\frac{1}{2-\frac{1}{\alpha}}, & x \geq \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ \frac{\alpha}{2\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ 0, & x \leq 0. \end{cases} \tag{3.6}$$

Remark 3.2 Theorem 3.1 states that the asymptotic optimal stopping value under the normalizations on c_i for the moving average case is identical to that of stopping the corresponding iid sequence $Y_1, \ldots, Y_n$. Further convergence of optimal stopping times holds. The idea of the definition of the stopping time is similar to that in Theorem 2.2. As soon as the first point of a *large* cluster is approximatively reached, then one stops after reaching the maximal point of this deterministic cluster approximatively.

Proof of Theorem 3.1: The basic idea of this proof is to compare the optimal stopping problem with the limiting cluster process by a majorization argument to a sequence of stopping problems for a Poisson process.

The imbedded point process N_n converges to the cluster process N (cp. (3.2)). As in section 2 we assume w.l.g. almost sure convergence of the points of the process. We consider first the statement under condition A1). Since $\sum_{j=1}^{\infty} |c_j| < \infty$ and $\{\frac{1}{a_n}Y_1 \vee \cdots \vee Y_n : n \in \mathbb{N}\}$ is uniformly integrable we conclude that the maxima $M_n = \max\{X_i; 1 \leq i \leq n\}$ normalized by a_n are uniformly integrable too, as

$$\max_{1\leq i\leq n}\left\{\Big|\sum_{j=1}^{\infty} Y_{i-j}c_j\Big|\right\} \leq \sum_{j=1}^{\infty} |c_j|| \max_{1\leq i\leq n}\{Y_i\}|;$$

also $\lim_{n\to\infty} \frac{EX_n}{a_n} = 0$.

For $j \in -\mathbb{N}_0$ define $c_j = 0$. Let T denote the optimal stopping time of the underlying Poisson process $N' = \sum \varepsilon_{(\tau_i, y_i)}$, i.e. $T = \inf\{\tau_i, y_i \geq u_\alpha^\Phi(\tau_i)\}$ (cp. KR (2000b)). We shall use some notations from KR (2000a, proof of Theorem 2.2).

1.) In the first step we determine as in the first step of the proof of Theorem 2.2 the asymptotics for the stopping time T_n':

$$\frac{X_{T_n'^{\geq nt}}}{a_n} \to y_{K^{T\geq t}}, \tag{3.7}$$

where again $T_n'^{\geq nt}$ denotes the restriction of the stopping time T_n' to the time domain $\geq nt$, $K^{T\geq t}$ is the first stopping index after time t and $y_{K^{T\geq t}}$ is the corresponding stopping value after time t in the limiting Poisson process N'. For the proof of (3.7) we establish convergence of the point process of points over the threshold

$$N_n^1 := \sum_{\substack{i=1,\dots,n, X_i \ge a_n \frac{c_1}{c_m} u_\alpha^\Phi(\frac{i}{n}),\\ X_{i-1}\vee\dots\vee X_{i-w} < \frac{1}{2} a_n \frac{c_1}{c_m} u_\alpha^\Phi(\frac{i}{n})}} \varepsilon_{(\frac{i}{n}, \frac{X_{i+m-1}}{a_n})}$$
$$\to \sum_i \sum_{\substack{k\in\{m\}\cup\{j+m-1: j>w, y_i c_j > \frac{c_1}{c_m} u_\alpha^\Phi(\tau_i),\\ y_i c_{j-1}\vee\dots\vee c_{j-w} < \frac{1}{2}\frac{c_1}{c_m} u_\alpha^\Phi(\tau_i)\}}} \varepsilon_{(\tau_i, y_i c_k)} =: N^1. \tag{3.8}$$

To prove (3.8) we obtain as in Resnick (1987, chap. 4.5)

$$N_n^2 := \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}, \frac{X_{i-1}\vee\dots\vee X_{i-w}}{a_n}, \frac{X_{i+m-1}}{a_n}, \frac{Y_i}{a_n}\right)}$$
$$\to \sum_i \sum_{\substack{k=1\\ c_k\neq 0}}^\infty \varepsilon_{(\tau_i, c_k y_i, y_i(c_{k-1}\vee\dots\vee c_{k-w}), y_i c_{k+m-1}, y_i(2-k)_+)} =: N^2, \tag{3.9}$$

where the convergence holds on $[0,1]\times(0,\infty)\times[-\infty,\infty]^3$.

Define

$$H := \left\{(t,x_1,x_2,x_3,x_4)\in\mathbb{R}^5 :\ t\in[0,1], x_1 > \frac{c_1}{c_m} u_\alpha^\Phi(t), x_2 < \frac{1}{2}\frac{c_1}{c_m} u_\alpha^\Phi(t)\right\}.$$

Then from (3.7) and $P(N^2(\partial H)\neq 0) = 0$ we obtain

$$N_n^1 = N_n^2(\cdot\cap H) \to N^2(\cdot\cap H) = N^1.$$

The last equality follows from the definition of H and m, w.

As in Resnick (1987, chap. 4.5, proof of 3.2 and p. 144) (see also (2.13)) we obtain from (3.8) the existence of rv's $m_{n,i}^k$ such that for $k\in\mathbb{N}$ with $c_k\neq 0$ and $k\in\{m\}\cup\{j+m-1 : j>w, y_i c_j > \frac{c_1}{c_m} u_\alpha^\Phi(\tau_i), y_i c_{j-1}\vee\dots\vee c_{j-w} < \frac{1}{2}\frac{c_1}{c_m} u_\alpha^\Phi(\tau_i)\}$ a.s. pointwise convergence holds

$$\frac{X_{m_{n,i}^k}}{a_n} \to y_i c_k, \tag{3.10}$$

where $m_{n,i}^k = m_{n,i}^1 + k - 1$. Note that $T_n' = m_{n,K^T}^1 + m - 1$. From (3.8) and (3.10) we conclude (3.7).

2.) In the second step our aim is to show that the optimal stopping values for the stopping problem of $X_1,\dots,X_n$ converge to the optimal stopping curve u of the Poisson process N'. In more detail we establish that P a.s. for all $t\in[0,1)$ and $n\to\infty$

$$E\left(\frac{\gamma_{n,[nt]}}{a_n}|\mathcal{F}_{[nt]-1}\right) \xrightarrow{P} u(t), \quad u(t) := u_\alpha^\Phi(t) = E y_{K^T\ge t}. \tag{3.11}$$

To that purpose we consider as in KR (2000a, (2.4)) a majorizing stopping problem $(X_{nk}'^i)$ with limiting problem $(X_k'^i)$ and establish

$$P^{X_{n,k}'^{i+1}|\mathcal{F}_{n,i}'} \xrightarrow{P} P^{X_k'^{i+1}}. \tag{3.12}$$

In (3.12) we use the notation $X'^{i+1}_{n,k} = \max\{X_{n,a^i_{n,k}}, \dots, X_{n,a^{i+1}_{n,k}}\}$, $a^i_{n,k} = \left[n\left(t+(1-t)\frac{i}{k}\right)\right]$ and X'^i_k are the corresponding maxima on the discretized intervals $[a^i_{n,k}, a^{i+1}_{n,k}]$ in the limiting problem for the Poisson cluster process with corresponding filtration $\mathcal{F}'_{k,i}$, i.e., we discretize the time interval into k subintervals and consider stopping of the $\max$-sequence on these subintervals. Thus we get a finite stopping problem which majorizes the original stopping problem.

For the proof of (3.12) we first obtain as in Resnick (1987, chap. 4.5, proof of (3.2)) on $[0,1] \times (0,\infty] \times (0,\infty]$ point process convergence

$$\sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}, \frac{Y_i}{a_n}\right)} \to \sum_k \sum_{\substack{i \\ c_i \neq 0}} \varepsilon_{(\tau_k, c_i y_k, y_k (2-i)_+)}, \tag{3.13}$$

and thus from the continuous mapping theorem for $0 \le s < t \le 1$ using $\sup c_i = 1$

$$\frac{Y_{[ns]} \vee \cdots \vee Y_{[nt]}}{a_n} - \frac{X_{[ns]} \vee \cdots \vee X_{[nt]}}{a_n} \xrightarrow{P} 0.$$

Using independence of (Y_i) this implies (3.12) by similar arguments as in KR (2000a, proof of (2.7)).

As consequence of (3.12) we obtain by an induction argument from the Bellman equation as in the proof of KR (2000a, (2.4)) convergence of the optimal stopping curves for $(X'^i_{n,k})$ to the optimal stopping curves u^k for (X'^i_k) in the discretized limit problem. The optimal stopping curves u^k of this majorizing discretized problem in the limit satisfy $u^k \ge u$ and by the approximation theorem KR (2000b, Theorem 3.2) we obtain

$$u^k \to u \tag{3.14}$$

using the normalization condition $\sup c_i = 1$. Thus the optimal stopping problem for the maxima in the limiting cluster process is identical to the optimal stopping of the underlying Poisson process. As consequence of the approximation theorem in KR (2000b, Theorem 3.2) we obtain from (3.14) by comparison with the majorizing stopping problem an upper bound for the optimal stopping curve $u_n(t) = E\left(\frac{\gamma_{n,[nt]}}{a_n} \mid \mathcal{F}_{[nt]-1}\right)$:

$$\lim_{n\to\infty} P\left(E\left(\frac{\gamma_{n,[nt]}}{a_n}\middle|\mathcal{F}_{[nt]-1}\right) < u(t) + \varepsilon\right) \to 1, \quad \forall \varepsilon > 0. \tag{3.15}$$

By uniform integrability and the attainment of upper bound as shown in (3.7) by application of the stopping time T'_n we conclude (3.11).

In order to investigate the distributional properties of the optimal stopping times T_n we introduce

$$\widehat{T}_n := \inf\left\{i : Y_i > a_n u^{\Phi}_{\alpha}\left(\frac{i}{n}\right)\right\}$$

and prove

$$P(T_n \in \{\widehat{T}_n + j - 1 : j \in \{1, \dots, m\}\}) \to 1. \tag{3.16}$$

From KR (2000b, Theorem 4.4) we obtain with the threshold stopping time $T := T^u$, $u = u_\alpha^\Phi$

$$\left(\frac{\widehat{T}_n}{n}, \frac{Y_{\widehat{T}_n}}{a_n}\right) \to (T, y_{K^T})\,. \tag{3.17}$$

Assuming w.l.g. a.s. convergence and following the proof in KR (2000a, Theorem 2.2) we obtain also convergence of the stopping values at random times $\widehat{T}_n + m$. Considering that after the time point $\widehat{T}_n + m$ one observes based on the clustering structure only the maximal value $y_{K^T} \vee_{i=m+1}^\infty c_i$, this leads to

$$E\left(\frac{\gamma_{n,\widehat{T}_n+m+1}}{a_n}|\mathcal{F}_{\widehat{T}_n+m}\right) \to u(T) \vee (y_{K^T} \underbrace{\vee_{i=m+1}^\infty c_i}_{<1}).$$

As in the proof in KR (2000a, Theorem 2.2) it follows that

$$P(T_n \le \widehat{T}_n + m - 1) \to 1. \tag{3.18}$$

In particular $\lim_{t\to 1}\lim_{n\to\infty} P\left(\frac{T_n}{n} \le t\right) = 0$. Further as in the proof in KR (2000a, (2.16)) using a discretization argument we obtain

$$E(\gamma_{n,g_k^n(T_n)}|\mathcal{F}_{n,T_n}) \le E(\gamma_{n,T_{n+1}}|\mathcal{F}_{n,T_n}), \tag{3.19}$$

where g_k^n maps a number in $\{1,\dots,n-1\}$ to the nearest point of the k-grid $\lceil 1 + \frac{ni}{k}\rceil$. We obtain

$$\begin{aligned}
&\left(u\left(\frac{T_n}{n}\right) - E\left(\frac{\gamma_{n,T_n+1}}{a_n}|\mathcal{F}_{T_n}\right)\right)_+ \\
&\le \underbrace{\left(u\left(\frac{T_n}{n}\right) - u\left(g_k^n\left(\frac{T_n}{n}\right)\right)\right)_+}_{\xrightarrow[k\to\infty,n\to\infty]{P} 0} + \underbrace{\left(u\left(g_k^n\left(\frac{T_n}{n}\right)\right) - E\left(\gamma_{n,g_k^n(T_n)}|\mathcal{F}_{n,T_n}\right)\right)_+}_{\xrightarrow[k\to\infty,n\to\infty]{P} 0 \text{ by } (3.11)} \\
&\quad + \underbrace{\left(E(\gamma_{n,g_k^n(T_n)}|\mathcal{F}_{n,T_n}) - E\left(\frac{\gamma_{n,T_n+1}}{a_n}|\mathcal{F}_{T_n}\right)\right)_+}_{\le 0 \text{ by } (3.19)}
\end{aligned}$$

and, therefore,

$$\left(u\left(\frac{T_n}{n}\right) - E\left(\frac{\gamma_{n,T_n+1}}{a_n}|\mathcal{F}_{T_n}\right)\right)_+ \xrightarrow{P} 0. \tag{3.20}$$

Using random variables as in (3.10) one can strengthen (3.13) and obtains that the point which converges to $(\tau_i, c_1 y_i, y_i)$ appears first in the cluster. (3.20) implies that at time T_n the stopping condition of $\widehat{T}_n$ is satisfied and thus $P(T_n \ge \widehat{T}_n) \to 1$. So with (3.18) we obtain (3.16). (3.16) implies the distributional results in (3.5), (3.6) using uniform integrability.

The proof of the other cases is similar; one has to regard the negative variables. □

Remark 3.3 a) In contrast to section 2 it has not been shown in this proof that $P(T_n = T'_n) \to 1$. In the case that for several $i \in \mathbb{N},\ c_i = \sup_j c_j$ this property is not clear at all. As a result one does not obtain several convergence properties of stopped variables and maximum before and after the optimal stopping time as in the independent case (cp. KR 2000b, Theorem 3.2)).

b) The structure of the proof in Theorem 3.1 is a variation of that of Theorem 2.2. It is relatively easy to analyse the asymptotic behaviour of the stopping time T'_n. The stopping values are approximated by those of the limiting underlying Poisson process. Then we consider majorizing stopping problems defined by maxima on a finite number of subintervals. Here one obtains convergence to the corresponding discretized problem for the limiting cluster process. Then we obtain an asymptotically upper bound of the stopping problems by letting the widths of the discretization converge to zero. The discretized limiting stopping problem for the cluster process converges by the normalization condition $\sup c_i = 1$ to the stopping problem for the underlying Poisson process. Since the upper bound is reached by the stopping time sequence (T'_n) we obtain optimality of (T'_n).

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