Minimax and minimal distance martingale measures and their relationship to portfolio optimization

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Abstract

In this paper we give a characterization of minimal distance martingale measures with respect to f-divergence distances in a general semimartingale market model. We provide necessary and sufficient conditions for minimal distance martingale measures and determine them explicitly for exponential Lévy processes with respect to several classical distances. It is shown that the minimal distance martingale measures are equivalent to minimax martingale measures with respect to related utility functions and that optimal portfolios can be characterized by them. Related results in the context of continuous-time diffusion models were first obtained by He and Pearson (1991b) and Karatzas et al. (1991) and in a general semimartingale setting by Kramkov and Schachermayer (1999). Finally parts of the results are extended to utility-based hedging.

Key words: f-divergences, derivative pricing, utility maximization, hedging

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1 Introduction

A common approach to derivative pricing in incomplete markets is to base the prices on a minimal distance martingale measure with respect to certain distances like L^2-distance (Schweizer (1996), Delbaen and Schachermayer (1996)), Hellinger distance (Keller (1997)), entropy distance (Frittelli (2000)) and others. In this paper we consider the class of all f-divergence distances defined by strictly

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convex, differentiable functions f which includes all the distances above (see Liese and Vajda (1987)) and obtain some necessary and some sufficient conditions for projections of the underlying measure on the set of martingale measures in general semimartingale market models.

Minimal distance martingale measures are related to minimax martingale measures. These were introduced and studied in various forms by He and Pearson (1991 a, b), Karatzas et al. (1991), Bellini and Frittelli (1997, 1998) and Kallsen (1998). As a consequence the characterization of minimal distance martingale measures is closely related to the determination of optimal portfolios. The crucial property is an integral representation of a transform of the minimal distance martingale measure, for which we give a short proof. In general the minimal distance martingale measure may not exist. It is shown in Kramkov and Schachermayer (1999) that the optimal portfolio is characterized by the solution of a dual variational problem which is related to the problem of finding a minimal distance martingale measure.

In comparison to Kramkov and Schachermayer (1999) our approach to this kind of results is different. Our main focus is on a general characterization of minimal distance martingale measures for general f-divergence distances. This characterization yields new sufficient criteria for projections on the set of martingale measures and allows to determine in a unified way explicitly projections. In particular we get a complete discussion in the important class of exponential Lévy processes. We also determine the projection with respect to the reverse entropy for non-continuous semimartingales. For continuous semimartingales Schweizer (1999) had proved that the projection is given by the minimal martingale measure. Based on the characterization of minimal distance martingale measures we obtain directly a duality result for optimal portfolios. We do not insist on the most generality concerning the duality theorem as in the paper of Kramkov and Schachermayer (1999) but we assume the existence of projections. (The approach in this paper could be extended to obtain a more general duality result by allowing finitely additive measures.) From our characterization of minimal distance martingale measures we finally obtain directly the equivalence to minimax martingale measures. This implies an existence result for minimax measures from a well-known existence result for f-projections.

The paper is organized as follows. In Section 2 we recall a theorem of Rüschendorf (1984) about f-projections on moment families, which is our main tool to characterize minimal distance martingale measures. Moreover we give an existence result for minimal distance measures and a general result on the equivalence of f-projections. In Section 3 we show how these results can be applied to minimal distance martingale measures. Some necessary and some sufficient conditions for minimal distance martingale measures are derived. In Section 4 we introduce our notion of a minimax measure with respect to concave utility functions and convex sets of probability measures, which is weaker than the notions of minimax measures of He and Pearson (1991 a, b) and Bellini and Frittelli (1997, 1998). It is shown that our notion of minimax measures is equivalent to minimal distance measures with respect to f-divergence distances induced by the convex conjugate of the utility function. In Section 5 we show that under weak conditions the different notions of a minimax measure coincide. It is pointed out how the results on minimal distance martingale measures are related to optimal portfolios maximizing the expected utility of terminal wealth. In Section 6 the sufficient conditions on minimal distance martingale measures allow us to calculate some examples explicitly. For exponential Lévy processes the minimal distance martingale measures are determined with respect to several classical distances. Finally in Section 7 we extend our duality result to utility based hedging of claims as introduced in Föllmer and Leukert (2000).

2 f-divergences and minimal distance measures

In the following we define f-divergence distances and recall some relevant results about f-projections. For general reference we refer to Liese and Vajda (1987) or Vajda (1989). Let $(\Omega,\, \mathcal{F},\, P)$ be a probability space.

Definition 2.1 Let $Q \ll P$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then the f-divergence between Q and P is defined as

$$f(Q||P) := \begin{cases} \int f(\frac{dQ}{dP})dP & \text{, if the integral exists} \\ \infty & \text{, else} \end{cases}$$

where $f(0) = \lim\limits_{x \downarrow 0} f(x)$.

Examples of f-divergence distances are the Kullback-Leibler or entropy distance for $f(x) = x \log x$, the total variation distance for $f(x) = |x-1|$, the Hellinger distance for $f(x) = -\sqrt{x}$, the reverse relative entropy for $f(x) = -\log(x)$ and many others (see Liese and Vajda (1987)).

In the following we assume that f is a continuous, strictly convex and differentiable function. $\mathcal{K}$ denotes a convex set of probability measures on $(\Omega, \mathcal{F})$ dominated by P. A measure $Q^* \in \mathcal{K}$ is called f-projection of P on $\mathcal{K}$ if $f(Q^*||P) = \inf\limits_{Q \in \mathcal{K}} f(Q||P) =: f(\mathcal{K}||P)$.

Remarks. (1) If f is strictly convex and $f(\mathcal{K}||P) < \infty$, then there exists at most one f-projection of P on $\mathcal{K}$ (Liese and Vajda (1987), Proposition 8.2).
(2) If $\mathcal{K}$ is closed in the variational distance topology and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \infty$, then there exists a f-projection of P on $\mathcal{K}$ (see Liese and Vajda (1987), Proposition 8.5).

For $\widehat{Q} \ll P$ and a vector subspace $F \subset L^1(\widehat{Q})$ with $1 \in F$ define the generalized moment family induced by F and $\widehat{Q}$,

$$\mathcal{K}_F := \{Q \ll P : F \subset L^1(Q) \text{ and } E_Q f = E_{\widehat{Q}} f \text{ for all } f \in F\}.$$

In classical moment families some moments of the distributions are fixed. In our application to mathematical finance the moment family $\mathcal{K}_F$ typically represents some class of martingale measures. The following result was given in Rüschendorf (1984).

Theorem 2.2 *(i) Let $Q^* \in \mathcal{K}$ satisfy $f(Q^*||P) < \infty$. Then Q^* is the f-projection of P on $\mathcal{K}$ if and only if*

$$\int f'(\frac{dQ^*}{dP})(dQ^* - dQ) \leq 0 \text{ for all } Q \in \mathcal{K} \text{ with } f(Q||P) < \infty.$$

(ii) Let $Q^* \in \mathcal{K}_F$ satisfy $f(Q^*||P) < \infty$ and $f'(\frac{dQ^*}{dP}) \in L^1(Q^*)$. If Q^* is the f-projection on $\mathcal{K}_F$, then

$$f'(\frac{dQ^*}{dP}) \in L^1(F, Q^*), \text{ the closure of } F \text{ in } L^1(Q^*).$$

(iii) Let $Q^* \in \mathcal{K}_F$ satisfy $f(Q^*||P) < \infty$. If $f'(\frac{dQ^*}{dP}) \in F$, then Q^* is the f-projection on $\mathcal{K}_F$.

Remarks. (1) In Theorem 5 in Rüschendorf (1984) the assumption $f'(\frac{dQ^*}{dP}) \in L^1(Q^*)$ was also stated for part (i) but was not used for the proof of this part.
(2) For every $Q \in \mathcal{K}$ with $f(Q||P) < \infty$ $\int f'(\frac{dQ^*}{dP})(dQ - dQ^*)$ coincides with the directional derivative of the function $f(\cdot||P)$ (see Vajda (1989), Lemma 9.31.i)). Hence the condition in Theorem 2.2 (i) can be understood as a condition on the directional derivative in Q^*.
(3) Under some additional growth conditions on f (see Liese and Vajda (1987)) one gets the following equivalence for a measure Q^* with $f(Q^*||P) < \infty$:
Q^* is the f-projection of P on $\mathcal{K}$ if and only if $f'(\frac{dQ^*}{dP}) \in L^1(Q)$ and $E_{Q^*} f'(\frac{dQ^*}{dP}) \leq E_Q f'(\frac{dQ^*}{dP})$ for all $Q \in \mathcal{K}_f$, where $\mathcal{K}_f := \{Q \in \mathcal{K} : f(Q||P) < \infty\}$.
(4) Theorem 2.2 (ii) is a generalization of a theorem of Csiszár (1975) on the entropy distance. This result was applied in recent papers on mathematical finance for the characterization of minimal relative entropy martingale measures by Frittelli (2000), Grandits and Rheinländer (1999) and Rheinländer (1999).

Frittelli (2000) studied the minimal entropy martingale measure, which corresponds to $f(x) = x \log(x)$. He showed that in this case the f-projection of P on $\mathcal{K}$ is necessarily equivalent to P if there is a measure $Q \in \mathcal{K}$ with $Q \sim P$ and $f(Q||P) < \infty$. Based on Theorem 2.2 (i) this result can be extended to general f-projections.

Corollary 2.3 *Let $f'(0) = -\infty$. Assume the existence of a measure $Q \in \mathcal{K}$ such that $Q \sim P$ and $f(Q||P) < \infty$. If Q^* is the f-projection of P, then $Q^* \sim P$.*

Proof. Suppose Q^* is not equivalent to P, i.e., $P(\frac{dQ^*}{dP} = 0) > 0$. Because $Q \sim P$ this implies $Q(\frac{dQ^*}{dP} = 0) > 0$. Since $f'(0) = -\infty$ this leads to a contradiction to the necessary condition on a f-projection of Theorem 2.2 (i). $\square$

3 Characterization of the minimal distance martingale measure

In the following we apply Theorem 2.2 to characterize f-projections on the set of martingale measures. Our mathematical framework is as follows. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ is a filtered probability space in the sense of Jacod and Shiryaev (1987), Definition I.1.2, where $\mathcal{F} = \mathcal{F}_T$. Securities $0, \dots, d$ are modeled by their price process $S := (S^0, \dots, S^d)$. We assume that S is a $\mathbb{R}^{d+1}$-valued semimartingale with deterministic S_0. Security 0 serves as a numeraire and hence we may assume without loss of generality $S^0 \equiv 1$. Vector stochastic integrals are written as $\int_0^t \varphi_s dS_s = \varphi \cdot S_t$. (For the definition of a vector stochastic integral, see Jacod (1980).)
Let $\mathcal{M}$ ($\mathcal{M}_{loc}$) be the set of P-absolutely continuous (local) martingale measures and $\mathcal{M}^e$ ($\mathcal{M}^e_{loc}$) the subset of $\mathcal{M}$ ($\mathcal{M}_{loc}$) consisting of probability measures which are equivalent to P. If in some context of the paper we consider $\mathcal{M}$ as well as $\mathcal{M}_{loc}$, then we use the notation $\mathcal{M}_{(loc)}$.
We define

$$\begin{aligned} G: &= \{\varphi\cdot S_T : \varphi^i = H^i 1_{]s_i,t_i]}, s_i < t_i, H^i \text{ bounded } \mathcal{F}_{s_i}\text{-measurable}\} \\ &\quad \cup\{1_B : P(B) = 0\} \end{aligned}$$

and

$$\begin{aligned} G_{loc}: &= \{\varphi\cdot S_T : \varphi^i = H^i 1_{]s_i,t_i]} 1_{[0,\widehat{T}^i]}, s_i < t_i, \\ &\qquad H^i \text{ bounded } \mathcal{F}_{s_i}\text{-measurable}, \widehat{T}^i \in \gamma^i\} \cup \{1_B : P(B) = 0\} \\ \text{with } \gamma^i: &= \{\widehat{T}^i \text{ stopping time}; (S^i)^{\widehat{T}^i} \text{ is bounded}\}. \end{aligned}$$

For a $\mathbb{R}^d$-valued local martingale N Jacod (1980) defined the class $L^1_{loc}(N)$ of predictable integrands. For $Q \in \mathcal{M}_{loc}$ we denote by $L^1_{loc}(S, Q)$ the class $L^1_{loc}(S)$ with respect to Q.

Theorem 3.1 *Let $Q^* \in \mathcal{M}$ satisfy $f(Q^*||P) < \infty$ and $f'(\frac{dQ^*}{dP}) \in L^1(Q^*)$. If Q^* is the f-projection of P on $\mathcal{M}$, then*

$$f'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dS_s \quad Q^*\text{-a.s.} \tag{3.1}$$

for some process $\varphi \in L^1_{loc}(S, Q^*)$ such that $\int_0^\cdot \varphi_s dS_s$ is a martingale under Q^*.

Proof. We have the following characterization of $\mathcal{M}$ as a moment family

$$\mathcal{M} = \{Q \text{ prob. measure on } (\Omega, \mathcal{F}) : G \subset L^1(Q) \text{ and } E_Q g = 0 \quad \forall g \in G\}.$$

Let F be the vector space generated by 1 and G. Theorem 2.2 (ii) yields:

$$f'(\frac{dQ^*}{dP}) \quad = \quad \xi \quad Q^*\text{-a.s.}$$

for some ξ in $L^1(F, Q^*)$, the $L^1(Q^*)$-closure of F.
By a theorem of Yor (1978, Corollary 2.5.2) (for a multidimensional version see Delbaen and Schachermayer (1999, Theorem 1.6)) on the closedness of stochastic integrals $L^1(G, Q^*)$ is contained in $\{\varphi \cdot S_T : \varphi \in L^1_{loc}(S, Q^*),\ \text{such that } \varphi \cdot S \text{ is a } Q^*\text{-martingale}\}$. According to Jacod (1979), Proposition 1.1, this result is valid without the assumption of a complete filtration. Since F is generated by 1 and G we get the characterization (3.1) (see for example Schaefer (1971, Proposition 3.3)). $\square$

The following theorem is a variant of Theorem 3.1. It shows that the necessary condition in Theorem 3.1 is also valid for the set of local martingale measures under the additional assumption that the price process is locally bounded. For the case of the relative entropy it was independently shown by Grandits and Rheinländer (1999).

Theorem 3.2 *Let S be locally bounded. Let $Q^* \in \mathcal{M}_{loc}$ satisfy $f(Q^*||P) < \infty$ and $f'(\frac{dQ^*}{dP}) \in L^1(Q^*)$. If Q^* is the f-projection of P on $\mathcal{M}_{loc}$, then*

$$f'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dS_s \quad Q^*\text{-a.s.} \tag{3.2}$$

for some process $\varphi \in L^1_{loc}(S, Q^*)$ such that $\int_0^\cdot \varphi_s dS_s$ is a martingale under Q^*.

Proof. We have the following characterization of $\mathcal{M}_{loc}$ as a moment family

$$\mathcal{M}_{loc} = \{Q \text{ prob. measure on } (\Omega, \mathcal{F}) : G_{loc} \subset L^1(Q) \text{ and } E_Q g = 0 \quad \forall g \in G_{loc}\}.$$

Therefore, we can follow the proof of Theorem 3.1. $\square$

Remarks. (1) If $Q^* \sim P$ and additionally $-f'(\frac{dQ^*}{dP})$ is bounded from below then the attainability of $f'(\frac{dQ^*}{dP})$ as a stochastic integral may also be derived from Theorem 2.2 (i) and a martingale representation result of Jacka (1992), Theorem 3.4, and Ansel and Stricker (1994), Theorem 3.2.
(2) If S is (locally) bounded then $\mathcal{M}_{(loc)}$ is closed with respect to the variational distance. This can be verified by the characterization of $\mathcal{M}_{(loc)}$ with the help of $G_{(loc)}$. If moreover $\lim\limits_{x \to \infty} \frac{f(x)}{x} = \infty$ then there exists a f-projection of P on $\mathcal{M}_{(loc)}$ (see Liese and Vajda (1987), Proposition 8.5). This condition on f is in particular fulfilled in the case of the relative entropy.

In the following we give sufficient conditions for a f-projection of P on $\mathcal{M}_{(loc)}$. The first one is also considered in the case of the relative entropy in Frittelli (2000) and Rheinländer (1999).

Proposition 3.3 *Let $Q^* \in \mathcal{M}_{(loc)}$ such that $f(Q^*||P) < \infty$ and*

$$f'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dS_s \quad P\text{-a.s.} \tag{3.3}$$

for $\int_0^T \varphi_s dS_s \in G_{(loc)}$. Then Q^* is the f-projection of P on $\mathcal{M}_{(loc)}$.

Proof. See Theorem 2.2 (iii). $\square$

The condition in Proposition 3.3 is not satisfactory. Usually the transform of the density $f'(\frac{dQ^*}{dP})$ cannot be represented as elementary stochastic integral (see also the discussion in Rheinländer (1999)).

From Theorem 2.2 (i) one can derive a more general sufficient condition for f-projections of P. This extension allows us to determine minimal distance martingale measures with respect to several classical distances explicitly in Section 6. We denote by $L(S)$ the set of predictable, S-integrable processes with respect to P (see Jacod (1980)).

Theorem 3.4 *Let $Q^* \in \mathcal{M}_{(loc)}$ with $f(Q^*||P) < \infty$ such that for a predictable process $\varphi \in L(S)$*

$$f'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dS_s \quad P\text{-a.s.},$$

$$-\int_0^\cdot \varphi_s dS_s \text{ is bounded from below } P\text{-a.s.},$$

$$E_{Q^*}(\int_0^T \varphi_s dS_s) = 0.$$

Then Q^* is the f-projection of P on $\mathcal{M}_{(loc)}$.

Proof. From Jacod (1979), Proposition 7.26.b, one gets that φ is also S-integrable with respect to any $Q \in \mathcal{M}_{loc}$ in dimension $d = 1$. Using Proposition 3 in Jacod (1980) the proof of this result can be extended to dimension $d \geq 1$. Hence by Ansel and Stricker (1994), Corollaire 3.5, $-\varphi \cdot S$ is a Q-local martingale and hence a Q-supermartingale for any $Q \in \mathcal{M}_{loc}$. Therefore,

$$\begin{aligned} E_Q f'(\frac{dQ^*}{dP}) &= c + E_Q(\varphi \cdot S_T)) \\ &\geq E_{Q^*} f'(\frac{dQ^*}{dP}). \end{aligned}$$

Now the result follows from Theorem 2.2 (i). $\square$

If the set of martingale measures is restricted a priori to the class $\mathcal{M}_{H^q} := \{Q \in \mathcal{M} : S^i \in H^q(Q) \text{ for } i \in \{1, \ldots, d\}\}$ for some $q \in [1, \infty)$, where $H^q(Q)$ consists of all martingales M such that $[M, M]_\infty^{\frac{1}{2}} \in L^q(Q)$ (for further informations about the H^q-spaces see for example Jacod (1979)), then the following sufficient condition for a f-projection of P on $\mathcal{M}_{H^q}$ allows a larger class of integrands for a sufficient criterion in comparison to Proposition 3.3.

Theorem 3.5 *Let $Q^* \in \mathcal{M}_{H^q}$ with $f(Q^*||P) < \infty$ such that for a bounded and predictable process φ*

$$f'(\frac{dQ^*}{dP}) = c + \int_0^T \varphi_s dS_s \quad P\text{-a.s.}.$$

Then Q^* is the f-projection of P on $\mathcal{M}_{H^q}$.

Proof. Let $Q \in \mathcal{M}_{H^q}$. Since $S^i \in H^q(Q)$ we have that $\varphi^i \in L^q(S^i)$ and hence (see Jacod (1979) , Theorem 4.60) $\varphi \cdot S \in H^q(Q)$. Now the result follows from Theorem 2.2 (i). □

4 Minimal distance and minimax measures

Again $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{K}$ denotes a convex set of probability measures on $(\Omega, \mathcal{F})$ dominated by P. $\mathcal{K}$ may be thought of as a subclass of the class of all absolutely continuous local martingale measures.
A utility function $u: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is assumed to be strictly increasing, strictly concave, continuously differentiable in $dom(u) := \{x \in \mathbb{R} \mid u(x) > -\infty\}$ and to satisfy

$$u'(\infty) = \lim_{x \to \infty} u'(x) = 0, \tag{4.1}$$
$$u'(\bar{x}) = \lim_{x \downarrow \bar{x}} u'(x) = \infty \tag{4.2}$$

for $\bar{x} := \inf\{x \in \mathbb{R} \mid u(x) > -\infty\}$. This implies either $dom(u) = (\bar{x}, \infty)$ or $dom(u) = [\bar{x}, \infty)$.
We denote by $I := (u')^{-1}$. Because of assumption (4.1) we have $I(0) = \infty$. The convex conjugate function of u, $u^* : \mathbb{R}_+ \to \mathbb{R}$, is defined by $u^*(y) := \sup_{x \in \mathbb{R}}\{u(x) - xy\} = u(I(y)) - yI(y)$.
In the following we introduce minimax measures. In the case where $\mathcal{K}$ is the set of equivalent martingale measures they were first introduced in a stronger form in He and Pearson (1991 a, b). Recently they were studied in another modified form by Bellini and Frittelli (1997, 1998) and in a finite market setting by Kallsen (1998). The minimax martingale measure has an economic interpretation. It produces prices which are least favourable for an investor with a given utility profile, i.e., the maximal expected utility with respect to prices based on a martingale measure is minimal. For a brief discussion of the economical significance of the minimax martingale measure see He and Pearson (1991b).
For $Q \in \mathcal{K}$ and $x > \bar{x}$ define

$$U_Q(x) := \sup\{Eu(Y) : Y \in L^1(Q), E_Q Y \leq x, Eu(Y)^- < \infty\}. \tag{4.3}$$

The following lemma gives a well-known representation of $U_Q(x)$.

Lemma 4.1 *Let $Q \in \mathcal{K}$ and $E_Q(I(\lambda \frac{dQ}{dP})) < \infty \quad \forall \lambda > 0$. Then*

(i) $U_Q(x) = \inf_{\lambda > 0}\{E(u^*(\lambda \frac{dQ}{dP})) + \lambda x\}$.

(ii) There is a unique solution for λ in the equation $E_Q(I(\lambda \frac{dQ}{dP})) = x$, denoted as $\lambda_Q(x) \in (0, \infty)$, and $U_Q(x) = E[u(I(\lambda_Q(x) \frac{dQ}{dP}))]$.

Proof. Let $Y \in L^1(Q)$ with $E_Q Y \le x$ and $Eu(Y)^- < \infty$. Then we have for $\lambda > 0$:

$$\begin{aligned} Eu(Y) &\le Eu(Y) + \lambda(x - E_Q Y) \\ &\le Eu^*(\lambda \frac{dQ}{dP}) + \lambda x \\ &= Eu(I(\lambda \frac{dQ}{dP})) + \lambda(x - E_Q I(\lambda \frac{dQ}{dP})). \end{aligned}$$

The inequalities hold as equalities if and only if Y is given as $I(\lambda_Q(x)\frac{dQ}{dP})$. Since we have $E_Q(I(\lambda \frac{dQ}{dP})) < \infty$ for all $\lambda > 0$, one can conclude that $E_Q(I(\lambda \frac{dQ}{dP}))$ is a continuous, monotonically decreasing function of λ with values in $(\bar{x}, \infty)$. This guarantees the existence of $\lambda_Q(x)$. Finally one has to check that $E[u(I(\lambda_Q(x)\frac{dQ}{dP}))]^- < \infty$. From the inequality $u(x) - xy \le u(I(y)) - yI(y)$ one gets that $E[u(I(\lambda \frac{dQ}{dP})) - \lambda \frac{dQ}{dP} I(\lambda \frac{dQ}{dP})]^- < \infty$. The inequality

$$\left[u(I(\lambda \frac{dQ}{dP}))\right]^- \le \left[u(I(\lambda \frac{dQ}{dP})) - \lambda \frac{dQ}{dP} I(\lambda \frac{dQ}{dP})\right]^- + \left[\lambda \frac{dQ}{dP} I(\lambda \frac{dQ}{dP})\right]^-$$

implies that the condition $E[u(I(\lambda_Q(x)\frac{dQ}{dP}))]^- < \infty$ is fulfilled. □

Remarks. (1) $I(\lambda_Q(x)\frac{dQ}{dP})$ can be interpreted as optimal claim which is financeable under the pricing measure Q.
(2) Notice that if for $Q \in \mathcal{K}$ there exists $\lambda > 0$ with $Eu^*(\lambda \frac{dQ}{dP}) < \infty$, then $U_Q(x) < \infty$ for all $x > \bar{x}$. Moreover if for $Q \in \mathcal{K}$ with $U_Q(x) < \infty$ the assumption of Lemma 4.1 is fulfilled, then $Eu^*(\lambda_Q(x)\frac{dQ}{dP}) < \infty$.
(3) For $\log x$, $\frac{x^p}{p}$, $1 - e^{-x}$ the corresponding convex conjugate functions are $-\log x - 1$, $-\frac{p-1}{p} x^{\frac{p}{p-1}}$, $1 - x + x \log x$. Hence for $u(x) = 1 - e^{-x}$ the u^*-divergence distance is the relative entropy, for $u(x) = \log x$ the reverse relative entropy and for $u(x) = -x^{-1}$ the Hellinger distance.

Definition 4.2 A measure $Q^* = Q^*(x) \in \mathcal{K}$ is called minimax measure for x and $\mathcal{K}$ if it minimizes $Q \mapsto U_Q(x)$ over all $Q \in \mathcal{K}$, i.e.,

$$U_{Q^*}(x) = U(x) := \inf_{Q \in \mathcal{K}} U_Q(x).$$

Remark. In general the minimax measure Q^* will depend on x. Fortunately for the standard utility functions like $u(x) = \frac{x^p}{p}$ ($p \in (-\infty, 1) \setminus \{0\}$), $u(x) = \log x$ and $u(x) = 1 - e^{-px}$ ($p \in (0, \infty)$) the minimax measure is independent of x.

Our weak notion of minimax measures allows to formulate a complete equivalence to minimal distance measures with respect to related f-divergence distances. This is the reason why we did not use the stronger forms of this notion in He and Pearson (1991b) or Bellini and Frittelli (1998). Later on we will see that under weak conditions the weak notion of a minimax measure coincides

with the stronger notion in He and Pearson (1991b) and also with that of Bellini and Frittelli (1998).
We assume throughout this section that

$$\exists x > \bar{x} \text{ with } U(x) < \infty, \tag{4.4}$$

$$E_Q I(\lambda \frac{dQ}{dP}) < \infty \quad \forall \lambda > 0 \quad \forall Q \in \mathcal{K}. \tag{4.5}$$

Remarks. (1) Assumption (4.5) is fulfilled for $u(x) = \log x$. If for every $Q \in \mathcal{K}$ $u^*(Q||P) < \infty$, then assumption (4.5) is also fulfilled for $u(x) = \frac{x^p}{p}$ $(p \in (-\infty, 1) \setminus \{0\})$ and $u(x) = 1 - e^{-px}$ $(p \in (0, \infty))$. In these cases one could substitute the set $\mathcal{K}$ by the convex subset $\{Q \in \mathcal{K} : u^*(Q||P) < \infty\}$.
(2) Assumption (4.5) implies according Remark 2 after Lemma 4.1 that

$$\{Q \in \mathcal{K} : U_Q(x) < \infty\} = \{Q \in \mathcal{K} : \forall \lambda > 0 : u^*_\lambda(Q||P) < \infty\}.$$

As usual we denote by $\partial U(x)$ the subdifferential of the function U at x. If $f(x) = u^*(\lambda_0 x)$, we denote the corresponding f-divergence by $u^*_{\lambda_0}(\cdot||\cdot)$.

Proposition 4.3 *Let $x > \bar{x}$, $\lambda_0 \in \partial U(x)$, $\lambda_0 > 0$. Then*

(i) $U(x) = u^*_{\lambda_0}(\mathcal{K}||P) + \lambda_0 x$.

(ii) If $Q^* \in \mathcal{K}$ is an $u^*_{\lambda_0}$-projection of P on $\mathcal{K}$, then Q^* is a minimax measure and $\lambda_0 = \lambda_{Q^*}(x)$.

(iii) If $Q^* \in \mathcal{K}$ is a minimax measure, then Q^* is an $u^*_{\lambda_{Q^*}(x)}$-projection of P on $\mathcal{K}$, $\lambda_{Q^*}(x) \in \partial U(x)$ and the following equation holds

$$U_{Q^*}(x) \quad = \quad \inf_{Q \in \mathcal{K}} U_Q(x) = \sup\{Eu(Y) : \sup_{Q \in \mathcal{K}(x)} E_Q Y \le x\},$$

where $\mathcal{K}(x) := \{Q \in \mathcal{K} : U_Q(x) < \infty\}$.

Proof. (i) From Lemma 4.1 we obtain

$$\begin{aligned} U(x) &= \inf_{Q \in \mathcal{K}} \inf_{\lambda > 0} \{Eu^*(\lambda \frac{dQ}{dP}) + \lambda x\} \\ &= \inf_{\lambda > 0} \{u^*_\lambda(\mathcal{K}||P) + \lambda x\}. \end{aligned}$$

Define $H : (0, \infty) \to \mathbb{R} \cup \{\infty\}$ as $H(\lambda) := u^*_\lambda(\mathcal{K}||P)$. According to Remark 2 after Lemma 4.1 the assumptions (4.4), (4.5) guarantee, that there is a $\lambda > 0$ with $H(\lambda) < \infty$. This implies that $U(x) < \infty$ for every $x \in dom(u) = dom(U)$. Hence we get $H(\lambda) < \infty$ for every $\lambda > 0$. In the following it is shown that H is a convex function. Let $\varepsilon > 0$ and $Q_1, Q_2 \in \mathcal{K}$, such that $H(\lambda_1) + \varepsilon \geq Eu^*(\lambda_1 \frac{dQ_1}{dP})$ and $H(\lambda_2) + \varepsilon \geq Eu^*(\lambda_2 \frac{dQ_2}{dP})$. Then we have

$$\begin{aligned} \gamma H(\lambda_1) + (1-\gamma)H(\lambda_2) + 2\varepsilon &\geq \gamma Eu^*(\lambda_1 \frac{dQ_1}{dP}) + (1-\gamma)Eu^*(\lambda_2 \frac{dQ_2}{dP}) \\ &\geq Eu^*(\gamma\lambda_1 \frac{dQ_1}{dP} + (1-\gamma)\lambda_2 \frac{dQ_2}{dP}) \\ &\geq \inf_{Q\in\mathcal{K}} Eu^*((\gamma\lambda_1 + (1-\gamma)\lambda_2)\frac{dQ}{dP}) \\ &= H(\gamma\lambda_1 + (1-\gamma)\lambda_2). \end{aligned}$$

The second inequality holds because u^* is convex and the last inequality holds because $\frac{\gamma\lambda_1}{\gamma\lambda_1+(1-\gamma)\lambda_2}\frac{dQ_1}{dP} + \frac{(1-\gamma)\lambda_2}{\gamma\lambda_1+(1-\gamma)\lambda_2}\frac{dQ_2}{dP} \in \mathcal{K}$. By Rockafellar (1970), Theorem 23.5, $\inf_{\lambda>0}\{H(\lambda)+\lambda x\}$ achieves its infimum in $\lambda = \lambda_0$ if and only if $-x \in \partial H(\lambda_0)$. By Rockafellar (1970), Theorem 7.4 and Corollary 23.5.1, this is equivalent to $\lambda_0 \in \partial U(x)$.
(ii) This follows from Lemma 4.1.
(iii) The first statement follows from Lemma 4.1. According to Remark 2 after assumption (4.5) we have

$$\{Q \in \mathcal{K} : U_Q(x) < \infty\} \quad = \quad \{Q \in \mathcal{K} : u^*_{\lambda_{Q^*(x)}}(Q||P) < \infty\}$$

and hence the equation follows from Theorem 2.2 (i) and Lemma 4.1. □

Corollary 4.4 *Assume that the hypotheses of Proposition 4.3 hold and moreover U is differentiable in x. Then we have the following equivalence: Q^* is a minimax measure if and only if Q^* is the $u^*_{\lambda_0}$-projection, where $\lambda_0 = \nabla U(x)$.*

Since $U(x)$ typically is not known explicitly it is of interest to be able to determine λ_0. In Proposition 4.7 we will address the question $\lambda_0 \in \partial U(x)$. Notice that this problem vanishes for the standard utility functions like $u(x) = \frac{x^p}{p}$ ($p \in (-\infty, 1) \setminus \{0\}$), $u(x) = \log x$ and $u(x) = 1 - e^{-px}$ ($p \in (0,\infty)$). In these cases the minimax measure does not depend on x respectively the u^*_λ-projection does not depend on λ.

Proposition 4.5 *Assume that $\bar{x} = 0$ and u is bounded from above. Then U is differentiable in every $x > 0$.*

Proof. According to Rockafellar (1970), Theorem 26.3, it is sufficient to prove that the function $H(\lambda) = u^*_\lambda(\mathcal{K}||P)$ is strictly convex.
Define $\bar{\mathcal{K}}$ as the closure of $\mathcal{K}$ with respect to $\sigma(ba, L^\infty)$. For any $\lambda > 0$ there is due to Lemma 3.3 of Kramkov and Schachermayer (1999) and the convexity of $\mathcal{K}$ a minimizing sequence $\{Q_n\}$ in $\mathcal{K}$ such that $\frac{dQ_n}{dP}$ converges almost surely. Since according to Alaoglu's Theorem $\bar{\mathcal{K}}$ is weak-star compact the sequence $\{Q_n\}$ has a cluster point $\bar{Q} \in \bar{\mathcal{K}}$ and hence $\frac{dQ_n}{dP} \to \frac{d\bar{Q}^r}{dP}$, where $\bar{Q}^r$ denotes the regular part for $\bar{Q} \in \bar{\mathcal{K}}$. By Lemma 3.4 of Kramkov and Schachermayer (1999) it follows that $\lim_{n\to\infty} Eu^*(\lambda\frac{dQ_n}{dP})^- = Eu^*(\lambda\frac{d\bar{Q}^r}{dP})^-$. Since $\bar{x} = 0$ and u is

bounded from above, it follows that u^* is bounded from above. The theorem of dominated convergence implies that $\lim_{n\to\infty} Eu^*(\lambda \frac{dQ_n}{dP}) = Eu^*(\lambda \frac{d\bar{Q}^r}{dP})$ and hence $\inf_{Q\in\mathcal{K}} Eu^*(\lambda \frac{dQ}{dP}) = Eu^*(\lambda \frac{d\bar{Q}^r}{dP})$.
Let $\lambda_1,\ \lambda_2 \in \mathbb{R}_+,\ \gamma \in (0,1)$. Due to the consideration above there are $\bar{Q}_1$, $\bar{Q}_2 \in \bar{\mathcal{K}}$ with $H(\lambda_1) = Eu^*(\lambda_1 \frac{d\bar{Q}_1^r}{dP})$ and $H(\lambda_2) = Eu^*(\lambda_2 \frac{d\bar{Q}_2^r}{dP})$. Therefore,

$$\begin{aligned}
\gamma H(\lambda_1) + (1-\gamma)H(\lambda_2) &= \gamma Eu^*(\lambda_1 \frac{d\bar{Q}_1^r}{dP}) + (1-\gamma)Eu^*(\lambda_2 \frac{d\bar{Q}_2^r}{dP}) \\
&> Eu^*(\gamma\lambda_1 \frac{d\bar{Q}_1^r}{dP} + (1-\gamma)\lambda_2 \frac{d\bar{Q}_2^r}{dP}) \\
&\geq \inf_{Q\in\mathcal{K}} Eu^*((\gamma\lambda_1 + (1-\gamma)\lambda_2)\frac{dQ}{dP}) \\
&= H(\gamma\lambda_1 + (1-\gamma)\lambda_2).
\end{aligned}$$

The strict inequality holds, since u^* is strictly convex and the inequality because of the convexity of the set $\mathcal{K}$. Hence H is strictly convex and we are done. □

The differentiability condition in Corollary 4.4 is fulfilled if for every $\lambda > 0$ there is a u^*_λ-projection. From Proposition 8.5 in Liese and Vajda (1987) one gets the following result which was already obtained - using different methods - by Bellini and Frittelli (1998) (see also Schachermayer (1999)).

Proposition 4.6 *Assume that $\mathcal{K}$ is closed in the variational distance topology and dom$(u) = (-\infty,\infty)$. Then for every $\lambda > 0$ there is a u^*_λ-projection of P on $\mathcal{K}$.*

Proof. According to Liese and Vajda (1987), Proposition 8.5, it is sufficient to check whether $\lim_{x\to\infty} \frac{u^*(\lambda x)}{x} = \infty$. Since $u^*(\lambda x) \geq u(-\frac{n}{\lambda}) + nx$ with $u(-\frac{n}{\lambda}) > -\infty$ it follows that $\lim_{x\to\infty} \frac{u^*(\lambda x)}{x} \geq n$ for every $n \in \mathbb{N}$. □

Using the sufficient conditions for projections in section 3 we now provide a way to determine the parameter $\lambda_0 \in \partial U(x)$ and, therefore, the f-divergence distance related to a minimax measure.

Proposition 4.7 *Let $Q^* \in \mathcal{M}$ ($\mathcal{M}_{loc}$, $\mathcal{M}_{H^q}$), $\lambda > 0$ with $u^*_\lambda(Q^*||P) < \infty$ such that for a S-integrable process φ*

$$I(\lambda \frac{dQ^*}{dP}) \quad = \quad x + \int_0^T \varphi_s dS_s \quad P\text{-a.s.}.$$

Assume moreover that any of the sufficient conditions on a u^*_λ-projection of Proposition 3.3 or the Theorems 3.4 or 3.5 hold. Then Q^* is the minimax measure for x and $\lambda \in \partial U(x)$.

Remark. Notice that the conditions in Proposition 3.3 and in the Theorems 3.4, 3.5 are formulated for $(u^*_\lambda)'(x) = -\lambda I(\lambda x)$.

Proof. In the following $\mathcal{K}$ stands for either $\mathcal{M}$, $\mathcal{M}_{loc}$ or $\mathcal{M}_{H^q}$. Since $E_{Q^*}I(\lambda\frac{dQ^*}{dP}) = x$ one gets from Lemma 4.1 that $\lambda = \lambda_{Q^*}(x)$. As Q^* is the u^*_λ-projection of P on $\mathcal{K}$ the condition of Theorem 2.2 is fulfilled. Hence for all measures $Q \in \mathcal{K}$ satisfying $u_\lambda(Q||P) < \infty$ one gets $E_Q I(\lambda\frac{dQ^*}{dP}) \le x$ and one can conclude that $U_{Q^*}(x) = Eu((I(\lambda\frac{dQ^*}{dP})) \le U_Q(x)$. From assumption (4.5) one has $\{Q \in \mathcal{K} : u^*_\lambda(Q||P) < \infty\} = \{Q \in \mathcal{K} : U_Q(x) < \infty\}$ and it follows that Q^* is a minimax measure for x and $\mathcal{K}$. By Proposition 4.3 one can conclude that $\lambda = \lambda_{Q^*}(x) \in \partial U(x)$. $\square$

5 Relationship to portfolio optimization

In this section we point out how the results of Section 3 and 4 are related to portfolio optimization. We assume that assumptions (4.4) and (4.5) hold for $\mathcal{K} = \mathcal{M}$ ($\mathcal{M}_{loc}$) and that $\bar{x} > -\infty$.
We call a predictable S-integrable $\mathbb{R}^{d+1}$-valued process φ an admissible strategy if $\sum_{i=0}^d \varphi^i_t S^i_t = x + \int_0^t \varphi dS$ for any $t \in [0,T]$ and $\int_0^\cdot \varphi dS$ is bounded from below. The set of admissible strategies is denoted by $\mathcal{A}$. We say that $\widehat{\varphi} \in \mathcal{A}$ is an *optimal portfolio* if it maximizes $\varphi \mapsto Eu(x + \int_0^T \varphi dS)$ over all $\varphi \in \mathcal{A}$. Notice that for $\bar{x} = -\infty$ the optimal portfolio typically is not bounded from below and hence not admissible. Therefore, in this case one needs to consider a larger class of strategies (see Schachermayer (1999) and Kallsen (2000)).
With the results of the previous section one gets the following theorem.

Theorem 5.1 *Let $Q^* \in \mathcal{M}^e_{(loc)}$ such that $u^*_{\lambda_0}(Q^*||P) < \infty$ and $I(\lambda_0\frac{dQ^*}{dP}) \in L^1(Q^*)$, ($S$ locally bounded), and let $\lambda_0 \in \partial U(x)$. Then*

(i) The following statements are equivalent:

(a) Q^* is a minimal distance (local) martingale measure.

(b) $E_Q I(\lambda_0\frac{dQ^*}{dP}) \le E_{Q^*} I(\lambda_0\frac{dQ^*}{dP}) \quad \forall Q \in \mathcal{M}_{(loc)}$ with $u^*_{\lambda_0}(Q||P) < \infty$.

(c) $I(\lambda_0\frac{dQ^*}{dP}) = x + \int_0^T \widehat{\varphi} dS$ and $\int_0^\cdot \widehat{\varphi} dS$ is a Q^*-martingale for some S-integrable, predictable process $\widehat{\varphi}$.

(ii) If (c) holds then $\sup_{\varphi\in\mathcal{A}} Eu(x+\int_0^T \varphi dS) = Eu(x+\int_0^T \widehat{\varphi} dS) = U_{Q^*}(x) = U(x)$ and $\widehat{\varphi}$ (with $\widehat{\varphi}^0_t := x + \int_0^t \widehat{\varphi} dS - \sum_{i=1}^d \widehat{\varphi}^i_t S_t$) is an admissible optimal portfolio-strategy.

(iii) If (a) holds then Q^* is a minimax (local) martingale measure.

Proof. Notice that $(u^*_\lambda)'(x) = -\lambda I(\lambda x)$.
(i) Due to Theorems 3.1, 3.2, 2.2 it remains to show that $(c) \Rightarrow (a)$.
Since $I : \mathbb{R} \to (\bar{x}, \infty)$ we obtain that $x + \widehat{\varphi} \cdot S_T \ge \bar{x}$. As $\widehat{\varphi} \cdot S$ is a Q^*-martingale and $Q^* \sim P$, $\widehat{\varphi} \cdot S$ is bounded from below P-a.s.. Therefore, by Theorem 3.4 Q^*

is the $u^*_{\lambda_0}$-projection of P on $\mathcal{M}_{(loc)}$.
(ii) As pointed out in 1. for a process $\widehat{\varphi}$ fulfilling condition (c) one can conclude that $\widehat{\varphi} \cdot S$ is bounded from below P-a.s.. Due to the definition of $\widehat{\varphi}^0_t$ and the assumption that S^0 equals 1 it holds that $\sum_{i=0}^d \widehat{\varphi}^i_t S^i_t = x + \int_0^t \widehat{\varphi} dS$ for any $t \in [0,T]$ and hence $\widehat{\varphi} \in \mathcal{A}$. By Ansel and Stricker (1994), Corollaire 3.5, $\varphi \cdot S$ is a Q^*-local martingale and hence a Q^*-supermartingale for any $\varphi \in \mathcal{A}$. Therefore $E_{Q*}(x + \varphi \cdot S_T) \leq x$. Analogously to Lemma 4.1 (ii) one concludes that $\sup_{\varphi \in \mathcal{A}} Eu(x + \int_0^T \varphi dS) = Eu(x + \int_0^T \widehat{\varphi} dS)$ and hence $\widehat{\varphi}$ is an optimal portfolio strategy.
(iii) This follows from Proposition 4.3. □

Theorem 5.1 together with Proposition 4.3 imply that for $\bar{x} > -\infty$ our weak form of the definition of a minimax martingale measure coincides with the strong notion of a minimax martingale measure in the sense of He and Pearson (1991 a, b) and also with that of Bellini and Frittelli (1998).

Corollary 5.2 *Let Q^* be a minimax measure for x and $\mathcal{M}_{(loc)}$.*

(i) If $u^*(\widehat{Q}||P) < \infty$ for some measure $\widehat{Q} \in \mathcal{M}^e_{(loc)}$, then $Q^* \sim P$.

(ii) If $Q^* \sim P$ (and S is locally bounded), then

$$I(\lambda_{Q^*}(x)\frac{dQ^*}{dP}) \quad = \quad x + \int_0^T \widehat{\varphi} dS,$$

where $\widehat{\varphi}$ is a optimal portfolio strategy and moreover

$$U(x) = U_{Q^*}(x) \quad = \quad \sup\{Eu(Y) : E_Q Y \leq x \text{ for all } Q \in \mathcal{M}_{loc}\}.$$

Proof. (i) Proposition 4.3 shows that Q^* is the $u^*_{\lambda_{Q^*}(x)}$-projection. Assumption (4.5) implies according to Remark 2 after Lemma 4.1 that $u^*_{\lambda_{Q^*}(x)}(\widehat{Q}||P) < \infty$. Since $I(0) = \infty$ it follows from Corollary 2.3 that $Q^* \sim P$.
(ii) By Proposition 4.3 and Theorem 5.1 one concludes that $I(\lambda_{Q^*}(x)\frac{dQ^*}{dP}) = x + \widehat{\varphi} \cdot S_T$ where $\widehat{\varphi} \in \mathcal{A}$ is a optimal portfolio strategy. This implies due to Corollaire 3.5 in Ansel and Stricker (1994) that $\widehat{\varphi} \cdot S$ is a Q-local martingale and hence a Q-supermartingale for any $Q \in \mathcal{M}_{loc}$. Therefore $E_Q(x + \widehat{\varphi} \cdot S_T) \leq x$ and Lemma 4.1 implies that

$$U(x) = U_{Q^*}(x) \quad = \quad sup\{Eu(Y) : E_Q Y \leq x \text{ for all } Q \in \mathcal{M}_{loc}\}.$$

□

Remarks. (1) Theorem 5.1 is an extension of Theorem 9.4 in Karatzas et al. (1991) and Theorem 2 in He and Pearson (1991b) from continuous-time diffusion models to general incomplete semimartingale models.
(2) Recently Kramkov and Schachermayer (1999) obtained a characterization of

the optimal portfolio in general semimartingale models by a solution of a dual problem which is related to the problem of finding a minimal distance martingale measure. They considered a variational problem with respect to a properly defined set of supermartingale measures, which contains the set of absolutely continuous martingale respectively local martingale measures. Kramkov and Schachermayer (1999) give an example (Example 5.1 bis in that paper) where the optimal portfolio cannot be characterized by a probability density $\frac{dQ^*}{dP}$ as in Theorem 5.1 (i), but only by a measure with mass strictly less than 1.
(3) Combining Theorem 2.2 (ii), (iv) in Kramkov and Schachermayer (1999) and Theorem 2.2 (iv) in Schachermayer (1999) one could also derive a version of Theorem 3.2 for u_λ^*-projections.
(4) Theorem 5.1 shows that if derivative prices are computed by a minimax respectively minimal distance martingale measure Q^* then the optimal claim, i.e., the solution of problem (4.3) for $Q = Q^*$ can be duplicated by a portfolio strategy $\widehat{\varphi}$. Hence no derivative trade increases the maximal expected utility in comparison to an optimal portfolio if derivative prices are computed by Q^*. We have

$$E_P u(x + \widehat{\varphi} \cdot S_T) \quad \geq \quad E_P u(Y)$$

for all claims Y such that $E_{Q^*} Y \leq x$.

Davis (1997) proposes as reasonable derivative price, the price such that an infinitesimal long- or short-position of the derivative does not increase the expected utility of terminal wealth in comparison to an optimal portfolio. Under certain assumptions he gets that the fair price of a contingent claim H according to an initial endowment x is given by

$$p(H) = \frac{E\left(u'\left(x + \widehat{\varphi} \cdot S_T\right) H\right)}{const}, \tag{5.1}$$

where $\widehat{\varphi}$ is an optimal portfolio strategy. Thus according to the characterization of Theorem 5.1 (i) the minimal distance martingale measure yields the fair derivative price suggested by Davis (1997) by taking the expectation of the derivative H under this measure. Hence not only infinitesimally but even general trading of the derivative does not increase the maximal expected utility.

Corollary 5.3 *Assume that the hypotheses of Theorem 5.1 hold. If Q^* is the minimal distance martingale measure, then Davis' fair derivative price is given by*

$$p(H) = E_{Q^*} H. \tag{5.2}$$

6 Examples

6.1 Minimizing relative entropy

The distance corresponding to the utility-function $u(x) = 1 - e^{-px}$ is the relative entropy with $f(x) = x \log x$. Necessary and sufficient conditions for the minimal

entropy martingale measure have been given in Frittelli (2000) and Grandits and Rheinländer (1999). In the setting of an exponential Lévy process the minimal entropy martingale measure has been determined by Miyahara (1999) and Chan (1999).
Theorems 3.1-3.5 gives as necessary respectively sufficient condition for minimal distance martingale measures a representation of the density of the form

$$\frac{dQ^*}{dP} = \frac{1}{\lambda_0} \exp(-p(x + \varphi \cdot S_T)). \tag{6.1}$$

The condition $\varphi \cdot S_T \in L^1(G_{(loc)}, Q^*)$ is necessary, (see the Theorems 3.1, 3.2 and also Grandits and Rheinländer (1999)), the sufficient condition $\varphi \cdot S_T \in G_{(loc)}$ has been given in Frittelli (2000). Grandits and Rheinländer (1999) prove that $\frac{dP}{dQ^*} \in L^\varepsilon(P)$ for an $\varepsilon > 0$ and $\varphi \cdot S \in BMO(Q^*)$ is a sufficient criterion. Theorem 3.4 gives a further quite general sufficient condition which can be checked in our subsequent examples. If the stochastic integral in (6.1) is bounded from below P-a.s. and is a Q^*-martingale then Q^* is the minimal entropy (local) martingale measure and φ is an optimal portfolio strategy. In general the question, whether the process $\varphi \cdot S$ is bounded from below is a delicate point. For finite state markets this problem vanishes.
Suppose that the positive price process $S = (S^1, \ldots, S^d)$ is of the form

$$S^i = S_0^i \, \mathscr{E}(X^i), \tag{6.2}$$

where $X = (X^1, \ldots, X^d)$ is a $\mathbb{R}^d$-valued Lévy process and $\mathscr{E}$ is the stochastic exponential. By Lemma A.8 in Goll and Kallsen (2000), these processes coincide with those of the form $S^i = S_0^i \exp(\tilde{X}^i)$ for $\mathbb{R}^d$-valued Lévy processes $\tilde{X}$. In the last couple of years exponential Lévy processes have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein and Keller (1995), Eberlein et al. (1998), Madan and Senata (1990), Barndorff-Nielsen (1998)). In this setting one can derive a candidate for the local martingale measure minimizing the relative entropy from results in Kallsen (2000), who studies the corresponding portfolio optimization problem.
Assume (b, c, F) to be the characteristic triplet of X relative to some truncation function $h : \mathbb{R}^d \mapsto \mathbb{R}^d$ in the sense of Jacod and Shiryaev (1987). Assume that there exists some $\gamma \in \mathbb{R}^d$ with the following properties:

1. $\int |xe^{-\gamma^\top x} - h(x))| F(dx) < \infty$,
2. $$b - c\gamma + \int \left(xe^{-\gamma^\top x} - h(x) \right) F(dx) = 0. \tag{6.3}$$

Let

$$\varphi_t^i := \frac{\gamma^i}{S_{t_-}^i} \text{ for } i = 1, \ldots, d, \quad \varphi_t^0 := x + \int_0^t \varphi_s dS_s - \sum_{i=1}^d \varphi_t^i S_t^i$$

for $t \in (0, T]$.

Define $Z_t = \mathscr{E}\left(-\gamma^\top X_s^c + (e^{-\gamma^\top x} - 1) * (\mu^X - \nu)_s \right)_t$.

Corollary 6.1 *The measure Q^* defined by $\frac{dQ^*}{dP} = Z_T$ is an equivalent local martingale measure. If $\gamma \cdot X$ is bounded from below, then Q^* minimizes the relative entropy between P and $\mathcal{M}_{loc}$.*

Proof. Theorem 3.3 in Kallsen (2000) shows that Z as defined above is a martingale such that $S^i Z$ is a local martingale with respect to P for $i \in \{1, \dots, d\}$. Moreover the density $Z_T = \frac{dQ^*}{dP}$ of Q^* with respect to P has a representation as in (6.1) with $\varphi_T^i := \frac{\gamma^i}{S_{t_-}^i}$. Furthermore, we have $E_{Q^*}(\varphi \cdot S_T) = 0$, which implies that the relative entropy between Q^* and P is finite, i.e., $E_{Q^*} \log(\frac{dQ^*}{dP}) < \infty$. If the process $\varphi \cdot S = \gamma \cdot X$ is bounded from below, then one can conclude by Theorem 3.4 that Q^* minimizes the relative entropy between P and $\mathcal{M}_{loc}$. $\square$

Remarks. (1) Under the measure Q^* as defined above X is again a Lévy process (see Kallsen (2000)).
(2) Equation (6.3) is also part of the condition given in Chan (1999) and Miyahara (1999).
(3) The condition that $\gamma \cdot X = \varphi \cdot S$ is bounded from below is not fulfilled in general. In connection with portfolio optimization this question is discussed in Schachermayer (1999) and Kallsen (2000).
(4) Corollary 6.1 extends to exponential Lévy processes of the form $S^i = S_0^i \exp(X^i)$ using Lemma A.8 in Goll and Kallsen (2000), which shows that these processes coincide with those of the form (6.2).

6.2 Minimizing the reverse relative entropy

In this section we consider the reverse relative entropy, i.e., the f-divergence distance for $f(x) = -\log x$, which corresponds to the logarithmic utility function $u(x) = \log x$.
Assume that the characteristics (B, C, ν) of the $\mathbb{R}^d$-valued semimartingale $(S^1, \dots, S^d)$ relative to some fixed truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ (in the sense of Jacod (1979), Jacod and Shiryaev (1987)) are given in the form

$$B = \int_0^\cdot b_t dA_t, \quad C = \int_0^\cdot c_t dA_t, \quad \nu = A \otimes F, \tag{6.4}$$

where $A \in \mathscr{A}_{\mathrm{loc}}^+$ is a predictable process, b is a predictable $\mathbb{R}^d$-valued process, c is a predictable $\mathbb{R}^{d \times d}$-valued process whose values are non-negative, symmetric matrices, and F is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$.
Assume that there exists a $\mathbb{R}^d$-valued, S-integrable process H with the following properties:

1. $1 + H_t^\top x > 0$ for $(A \otimes F)$-almost all $(t, x) \in [0, T] \times \mathbb{R}^d$,

2. $\int |\frac{x}{1+H_t^\top x} - h(x))|F_t(dx) < \infty$ $(P \otimes A)$-almost everywhere on $\Omega \times [0,T]$,

3.
$$b_t - c_t H_t + \int \left(\frac{x}{1+H_t^\top x} - h(x) \right) F_t(dx) = 0 \qquad (6.5)$$

$(P \otimes A)$-almost everywhere on $\Omega \times [0,T]$.

Let

$$\varphi_t^i := x H_t^i \, \mathscr{E}\left(\int_0^{\cdot} H_s dS_s \right)_{t-} \quad \text{for } i = 1, \ldots, d, \quad \varphi_t^0 := x + \int_0^t \varphi_s dS_s - \sum_{i=1}^d \varphi_t^i S_t^i$$

for $t \in (0,T]$.
Goll and Kallsen (2000) show that φ as defined above is an optimal portfolio strategy for the logarithmic utility maximization problem. Based on this paper we get from Theorem 3.4 a characterization of the local martingale measure minimizing the reverse relative entropy.

Corollary 6.2 *If* $Z_t := \mathscr{E}\left(-H \cdot S_s^c + (\frac{1}{1+H^\top x} - 1) * (\mu^S - \nu)_s \right)_t$ *is a martingale, then the corresponding measure* Q^* *is an equivalent local martingale measure and it minimizes the reverse relative entropy.*

Proof. Theorem 3.1 in Goll and Kallsen (2000) shows that Z_t is a positive local martingale, such that $S^i Z$ is a local martingale for $i \in \{1, \ldots, d\}$ and $\frac{x}{Z_T} = x + \varphi \cdot S_T$, where φ is an admissible portfolio-strategy. If Z is even a martingale, then Z_T is the density of an equivalent local martingale measure Q^*. For Q^* we have

$$E(-\log(\frac{dQ^*}{dP})) = E \log(\frac{1}{Z_T}) = E(\log(x + \varphi \cdot S_T)) - \log(x).$$

From Lemma 4.1 it follows that

$$E \log(x + \varphi \cdot S_T) \quad \leq \quad E(-\log(\frac{dQ}{dP})) + x$$

for all measures $Q \in \mathcal{M}_{loc}$. The last inequality shows that if Q^* has infinite reverse relative entropy, then also all other measures $Q \in \mathcal{M}_{loc}$. Hence it follows due to $E_{Q^*}(x + \varphi S_T) = x$ from Theorem 3.4 that Q^* minimizes the reverse relative entropy. □

Remark. In the case of a continuous price process S equation (6.5) simplifies considerably and the minimal distance martingale measure Q^* is given by the minimal martingale measure. In this case the result is due to Schweizer (1999).

6.3 Derivative pricing by Esscher-transforms

Assume that the price process $S = (S_t)_{t \leq T}$ is generated by some Lévy process $X = (X_t)_{t \leq T}$ with $X_0 = 0$, in the sense that $S_t = e^{X_t}$. Let M be the moment generating function of X with $M(u,t) = M(u)^t = Ee^{uX_t}$. M is assumed to exist for $|u| < C$ for some constant $C > 0$. By means of Esscher transforms one defines a set of measures $\{Q^\theta : |\theta| < C\}$ by $\frac{dQ^\theta}{dP} = \frac{e^{\theta X_T}}{M(\theta)^T}$. If $\widehat{\theta}$ is a solution of

$$0 = \log \frac{M(\theta+1)}{M(\theta)}, \tag{6.6}$$

then $Q^{\widehat{\theta}}$ is an equivalent martingale measure (see Eberlein and Keller (1995) and Shiryaev (1999)).
Let u be the utility-function given by $u(x) = \frac{x^p}{p}$, $p \in (-\infty, 1) \setminus \{0\}$. Then condition (c) of Theorem 5.1 (i) becomes

$$\frac{dQ^*}{dP} = \frac{(x + \widehat{\varphi} \cdot S_T)^{p-1}}{\lambda_0} \quad \text{and} \quad \widehat{\varphi} \cdot S \text{ is a } Q^*\text{-martingale.}$$

Hence $Q^{\widehat{\theta}}$ fulfills condition (c) of Theorem 5.1 (i) respectively the assumption of Theorem 3.3 for $\widehat{p} = \widehat{\theta} + 1$ and $\widehat{\varphi} = const$. Thus we obtain the following result.

Corollary 6.3 *In the model of an exponential Lévy process the Esscher transform $Q^{\widehat{\theta}}$ is a minimal distance martingale measure for $f(x) = -\frac{\widehat{p}-1}{\widehat{p}} x^{\frac{\widehat{p}}{\widehat{p}-1}}$, if $\widehat{\theta}$ solves (6.6) and $\widehat{p} = \widehat{\theta} + 1 < 1$. Moreover $Q^{\widehat{\theta}}$ is a minimax martingale measure for the power utility function $\frac{x^{\widehat{p}}}{\widehat{p}}$.*

Derivative pricing by the Esscher transform for exponential Lévy processes was proposed and studied in Eberlein and Keller (1995). Corollary 6.3 shows that the martingale measure obtained by the Esscher transform corresponds to a specific power utility function $u(x) = \frac{x^{\widehat{p}}}{\widehat{p}}$, where the parameter $\widehat{p}$ is determined in such a way that $const \cdot e^{X_T}$ is the value of the optimal portfolio at time T. Thus the optimal portfolio strategy constantly invests the whole wealth into the risky asset (see Section 6.4 for the solution for general power utility functions).

Remarks. (1) Gerber and Shiu (1994) noted that the derivative price computed by the Esscher transform corresponds to the derivative price suggested by Davis (1997) (see (5.1)) for the power utility function as specified in Corollary 6.3.
(2) Chan (1999) studies a generalized Esscher transform for geometric Lévy processes $dS_t = \sigma_t S_{t-} dX_t + b_t S_{t-} dt$. He shows that for this model the martingale measure constructed via the generalized Esscher transform minimizes the relative entropy. This connection can be seen by the following consideration. If we have constant coefficients σ, b, then $\frac{1}{\sigma S_{t_-}} dS_t - \frac{b}{\sigma} dt = dX_t$ and the Esscher transform $\frac{dQ^\theta}{dP} = \frac{e^{\theta X_T}}{M(\theta)^T}$ can be written as $\frac{dQ^\theta}{dP} = \frac{e^{-\varphi \cdot S}}{E(e^{-\varphi \cdot S})}$, where $\varphi_t = -\frac{\theta}{\sigma S_{t_-}}$. Hence the density of the Esscher transform has a representation as in equation (6.1) which corresponds to the measure minimizing the relative entropy.

6.4 Distance minimization for power utility functions

In the following we determine the local martingale measure minimizing the f-divergence distance for $f(x) = -\frac{p-1}{p}x^{\frac{p}{p-1}}$ ($p \in (-\infty, 1) \setminus \{0\}$), if the discounted price process $S = (S^1, \dots, S^d)$ is of the form

$$S^i = S_0^i \, \mathscr{E}(X^i) \tag{6.7}$$

for a $\mathbb{R}^d$-valued Lévy-process $X = (X^1, \dots, X^d)$. This problem corresponds according to Theorem 5.1 to the problem of portfolio optimization with respect to $u(x) = \frac{x^p}{p}$ ($u^* = f$).
Assume (b, c, F) to be the characteristic triplet of X relative to some truncation function $h : \mathbb{R}^d \mapsto \mathbb{R}^d$. Assume that there exists some $\gamma \in \mathbb{R}^d$ with the following properties:

1. $F(\{x \in \mathbb{R}^d : 1 + \gamma^\top x \leq 0\}) = 0$,
2. $\int |\frac{x}{(1+\gamma^\top x)^{1-p}} - h(x)|F(dx) < \infty$,
3. $$b + (p-1)c\gamma + \int \left(\frac{x}{(1+\gamma^\top x)^{1-p}} - h(x) \right) F(dx) = 0. \tag{6.8}$$

Let

$$\varphi_t^i := \frac{\gamma^i}{S_{t_-}^i} V_{t_-} \text{ for } i = 1, \dots, d, \quad \varphi_t^0 := x + \int_0^t \varphi_s dS_s - \sum_{i=1}^d \varphi_t^i S_t^i$$

for $t \in (0, T]$, where V is the wealth process with respect to φ.
Kallsen (2000) shows that φ as defined above is an optimal portfolio strategy for the utility maximization problem with respect to $u(x) = \frac{x^p}{p}$. Based on this paper we get from Theorem 3.4 a characterization of the local martingale measure minimizing the f-divergence distance for $f(x) = -\frac{p-1}{p}x^{\frac{p}{p-1}}$.
Define $Z_t = \mathscr{E}\left((p-1)\gamma^\top X_s^c + ((1+\gamma^\top x)^{p-1} - 1) * (\mu^X - \nu)_s \right)_t$.

Corollary 6.4 *The measure Q^* defined by $\frac{dQ^*}{dP} = Z_T$ is an equivalent local martingale measure and it minimizes the f-divergence distance for $f(x) = -\frac{p-1}{p}x^{\frac{p}{p-1}}$.*

Proof. Theorem 3.2 in Kallsen (2000) shows that Z is a positive martingale, such that $S^i Z$ is a local martingale with respect to P for $i \in \{1, \dots, d\}$. Moreover the density $Z_T = \frac{dQ^*}{dP}$ of Q^* with respect to P has the representation $Z_T = \frac{(x+\varphi \cdot S_T)^{p-1}}{E(x+\varphi \cdot S_T)^{p-1}}$ with $\varphi_T^i := \frac{\gamma^i}{S_{t_-}^i} V_{t_-}$. Furthermore we have $E_{Q^*}(\varphi \cdot S_T) = 0$, which implies that $f(Q^*||P) < \infty$. Since it turns out that the process $\varphi \cdot S$ is bounded from below the result follows from Theorem 3.4. $\square$

Remarks. (1) Under the measure Q^* as defined above X is again a Lévy process (see Kallsen (2000)).
(2) For $p = -1$ the measure Q^* minimizes the Hellinger distance. This result has also been obtained independently in Grandits (1999).
(3) Some related results for the f-divergence distances for $f(x) = -\frac{p-1}{p} x^{\frac{p}{p-1}}$ with $p < 0$ have been obtained independently in Xia and Yan (2000).
(4) Corollary 6.4 extends to exponential Lévy processes of the form $S^i = S_0^i \exp(X^i)$ using Lemma A.8 in Goll and Kallsen (2000), which shows that these processes coincide with those of the form (6.7).

7 Utility-based hedging

In an incomplete market an investor may apply superhedging to eliminate the financial risk of a contingent claim. But this is often quite expensive. To superhedge a European call option an investor may be forced to buy the underlying asset in t=0 (see Eberlein and Jacod (1997)).
Therefore, it is reasonable to ask for hedging strategies which require less capital than superhedging strategies. Recently Föllmer and Leukert (1999) proposed hedging strategies maximizing the probability that the hedge is successful. Föllmer and Leukert (2000) also studied further hedging criteria like minimizing the shortfall risk, which is defined as the expected shortfall weighted by some loss function or maximizing expected utility with respect to a state-dependent utility function. In the following we show how our approach to portfolio optimization can be extended to utility based hedging, i.e., we study the following problem

$$\sup_{\varphi \in \mathcal{A}} Eu(x + \int_0^T \varphi dS - H), \tag{7.1}$$

where H is a non-negative $\mathcal{F}_T$-measurable random variable modeling the contingent claim in question. In problem (7.1) risk-aversion of the investor is described by the utility function u.
If $\bar{x} := \inf\{x \in \mathbb{R} \mid u(x) > -\infty\} \geq 0$ then criterion (7.1) only allows superhedging strategies. Since we want to allow more general strategies we assume

$$-\infty < \bar{x} < 0 \qquad \text{and} \qquad x - \bar{x} > \sup_{Q \in \mathcal{M}_{loc}} E_Q H. \tag{7.2}$$

It is known that $\sup_{Q \in \mathcal{M}_{loc}^e} E_Q H$ corresponds to the minimal cost for a superhedging strategy (Föllmer and Kabanov (1998)).
We define

$$U^H(x) \quad := \quad \inf_{Q \in \mathcal{M}_{loc}} \sup_{E_Q Y \leq x} Eu(Y - H).$$

In the following we assume that $U^H(x) < \infty$ and that assumption (4.5) is fulfilled for $\mathcal{M}_{loc}$. Moreover we assume that S is locally bounded.

Instead of problem (7.1) we consider the following dual problem for $\lambda_0 \in \partial U^H(x)$:

$$\inf_{Q\in\mathcal{M}_{loc}} E(u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP}H). \qquad (D)$$

Then one gets the following duality result for utility based hedging.

Theorem 7.1 *Let $\lambda_0 \in \partial U^H(x)$ and let $Q^* \in \mathcal{M}^e_{loc}$, such that $u^*_{\lambda_0}(Q^*||P) < \infty$ and $I(\lambda_0 \frac{dQ^*}{dP}) + H \in L^1(Q^*)$. Then*

(i) The following statements are equivalent:

   (a) Q^* solves problem (D).

   (b) $E_Q(I(\lambda_0 \frac{dQ^*}{dP}) + H) \leq E_{Q^*}(I(\lambda_0 \frac{dQ^*}{dP}) + H) \quad \forall Q \in \mathcal{M}_{loc}$ with $E(u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP}H) < \infty$.

   (c) $I(\lambda_0 \frac{dQ^*}{dP}) + H = x + \int_0^T \widehat{\varphi} dS$ and $\int_0^\cdot \widehat{\varphi} dS$ is a Q^*-martingale for some S-integrable, predictable process $\widehat{\varphi}$.

(ii) If (c) holds then $\widehat{\varphi}$ (with $\widehat{\varphi}^0_t := x + \int_0^t \widehat{\varphi} dS - \sum_{i=1}^d \widehat{\varphi}^i_t S_t$) is an optimal hedging strategy.

For the proof of Theorem 7.1 one needs the analogous result to Theorem 2.2 (i).

Proposition 7.2 *Let $Q^* \in \mathcal{M}_{loc}$ satisfy $u^*_{\lambda_0}(Q^*||P) < \infty$ and $I(\lambda_0 \frac{dQ^*}{dP}) \in L^1(Q^*)$. Then Q^* solves (D) if and only if*

$$E_Q(I(\lambda_0 \frac{dQ^*}{dP}) + H) \quad \leq \quad E_{Q^*}(I(\lambda_0 \frac{dQ^*}{dP}) + H) \quad \forall Q \in \mathcal{M}_{loc}$$

with $E(u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP}H) < \infty$.

Proof. For $Q \in \mathcal{M}_{loc}$ with $E(u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP}H) < \infty$ and $\alpha \in [0, 1]$ define

$$\begin{aligned} h_\alpha \quad &:= \quad \frac{1}{\alpha - 1}(u^*(\lambda_0(\alpha \frac{dQ^*}{dP} + (1-\alpha)\frac{dQ}{dP})) - \lambda_0 H(\alpha \frac{dQ^*}{dP} + (1-\alpha)\frac{dQ}{dP}) \\ &\quad\quad -u^*(\lambda_0 \frac{dQ^*}{dP}) + \lambda_0 \frac{dQ^*}{dP}H)) \\ &= \quad -\lambda_0 H(\frac{dQ^*}{dP} - \frac{dQ}{dP}) + \frac{1}{\alpha - 1}(u^*(\lambda_0(\alpha \frac{dQ^*}{dP} + (1-\alpha)\frac{dQ}{dP})) \\ &\quad\quad -u^*(\lambda_0 \frac{dQ^*}{dP})). \end{aligned}$$

For $\alpha \uparrow 1$, h_α increases to $\lambda_0 H(\frac{dQ}{dP} - \frac{dQ^*}{dP}) + \lambda_0 I(\lambda_0 \frac{dQ^*}{dP})(\frac{dQ}{dP} - \frac{dQ^*}{dP})$ and, therefore, by the monotone convergence theorem, using

$$h_\alpha \quad \geq \quad (u^*(\lambda_0 \frac{dQ^*}{dP}) - \lambda_0 \frac{dQ^*}{dP}H) - (u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP}H),$$

$\int h_\alpha dP$ increases to $\int(\lambda_0 H + \lambda_0 I(\lambda_0 \frac{dQ^*}{dP}))(dQ - dQ^*)$.
If Q^* solves (D), then the left hand side is ≤ 0 for each α, which implies, that

also the limit on the right hand side is ≤ 0. If conversely the right hand side is ≤ 0, then by the nondecreasing property of h_α we have that

$$\begin{aligned}\int h_0 dP &= \int (u^*(\lambda_0 \frac{dQ^*}{dP}) - \lambda_0 \frac{dQ^*}{dP} H - u^*(\lambda_0 \frac{dQ}{dP}) - \lambda_0 \frac{dQ}{dP} H) dP \\ &\leq \int (\lim_{\alpha \uparrow 1} h_\alpha) dP \\ &\leq 0.\end{aligned}$$

□

Proof of Theorem 7.1. (i) Due to Proposition 7.2 it is sufficient to show $(b) \Leftrightarrow (c)$.
$(b) \Rightarrow (c)$: By Rüschendorf (1984), Proposition 1, we know that $I(\lambda_0 \frac{dQ^*}{dP}) + H \in L^1(F, Q^*)$, the closure of F in $L^1(Q^*)$. The representation of this closure in Theorem 3.2 yields that $I(\lambda_0 \frac{dQ^*}{dP}) + H = c + \widehat{\varphi} \cdot S_T$ for a S-integrable predictable process $\widehat{\varphi}$. Analogously to Proposition 4.3 one can show that if $Q^* \in \mathcal{M}_{loc}$ solves (D) then

$$\begin{aligned}U^H(x) &= \sup\{Eu(Y - H) : E_{Q^*} Y \leq x, Eu(Y - H)^- < \infty\} \\ &= Eu(I(\lambda_0 \frac{dQ^*}{dP}))\end{aligned}$$

and $E_{Q^*}(I(\lambda_0 \frac{dQ^*}{dP}) + H) = x$. Since $\widehat{\varphi} \cdot S$ is a Q^*-martingale it follows that $c = x$ and $I(\lambda_0 \frac{dQ^*}{dP}) + H = x + \widehat{\varphi} \cdot S_T$.
$(c) \Rightarrow (b)$: Since $I : \mathbb{R} \to (\bar{x}, \infty)$ we obtain that $x + \widehat{\varphi} \cdot S_T \geq \bar{x}$. As $\widehat{\varphi} \cdot S$ is a Q^*-martingale and $Q^* \sim P$, $\widehat{\varphi} \cdot S$ is bounded from below P-a.s.. By Ansel and Stricker (1994), Corollaire 3.5, $\widehat{\varphi} \cdot S$ is a Q-local martingale and hence a Q-supermartingale for any $Q \in \mathcal{M}_{loc}$. Therefore

$$\begin{aligned}E_Q(I(\frac{dQ^*}{dP}) + H) &= x + E_Q(\widehat{\varphi} \cdot S_T) \\ &\leq x = E_{Q^*}(I(\frac{dQ^*}{dP}) + H).\end{aligned}$$

(ii) As pointed out in (i) for a process $\widehat{\varphi}$ fulfilling condition (c) one can conclude that $\widehat{\varphi} \cdot S$ is bounded from below P-a.s. and $\widehat{\varphi} \in \mathcal{A}$. Let $\varphi \in \mathcal{A}$ be a admissible strategy, then

$$\begin{aligned}E(u(x + \varphi \cdot S_T - H)) &\leq E(u(x + \widehat{\varphi} \cdot S_T - H) \\ &\quad + u'(x + \widehat{\varphi} \cdot S_T - H)(\varphi \cdot S_T - \widehat{\varphi} \cdot S_T)) \\ &= E(u(x + \widehat{\varphi} \cdot S_T - H) + \lambda_0 \frac{dQ^*}{dP}(\varphi \cdot S_T - \widehat{\varphi} \cdot S_T)) \\ &\leq E(u(x + \widehat{\varphi} \cdot S_T - H).\end{aligned}$$

The first inequality holds since u is concave, the equality holds because $u'(x + \int_0^T \widehat{\varphi} dS - H) = \lambda_0 \frac{dQ^*}{dP}$ and the second inequality holds since $\varphi \cdot S$ is bounded from below and therefore $E_{Q^*}(\varphi \cdot S_T - \widehat{\varphi} \cdot S_T) \leq 0$ by Ansel and

Stricker (1994), Corollaire 3.5. □

Remarks. (1) The results of this section can be also obtained with $\mathcal{M}$ or $\mathcal{M}_{H^q}$ instead of $\mathcal{M}_{loc}$ without the assumption of local boundedness of S.
(2) In a recent paper (which we got to know only after finishing this paper) Cvitanić et al. (1999) study the problem of utility maximization in incomplete markets with random endowment for utility functions defined on $\mathbb{R}_+$. This includes utility-based hedging for $\bar{x} > -\infty$ as a special case. They obtain a characterization of the optimal portfolio strategy by the solution of a dual problem which is similar to our dual problem (D).
(3) Delbaen et al. (2000) study the case of exponential hedging, i.e., hedging with respect to a exponential utility function. Considering special classes of hedging strategies they prove a duality relation between the problem of utility-based hedging and problem (D) for the case of the exponential utility function.
(4) After essentially having finished the paper and during the process of revision we got copies of the papers of Cvitanić et al. (1999), Delbaen et al. (2000), Grandits (1999), Grandits and Rheinländer (1999), Rheinländer (1999), Schachermayer (1999) and Xia and Yan (2000) where some related results were obtained independently.

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