# From Sequential Testing to Optimal Stopping

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# I Some Examples of Sequential Testing and Detection

# Testing the Sign of the Drift of BM

$W_t$, $t \geq 0$ Brownian motion with drift $-\theta$ or $+\theta$, with $\theta > 0$ fixed.
$W_0 = 0$.

**Testing sequentially:** $H_0 : -\theta$ versus $H_1 : +\theta$

Prior: Uniform on $\{-\theta, \theta\}$

$$R(T,\delta) = \frac{1}{2}\left(P_{-\theta}\{\delta \text{ rejects } H_0\} + c\theta^2 E_{-\theta} T\right) + \frac{1}{2}\left(P_{\theta}\{\delta \text{ rejects } H_1\} + c\theta^2 E_{\theta} T\right)$$

Find $(T^*, \delta^*)$ with $\quad R(T^*, \delta^*) = \min_{(T,\delta)} R(T,\delta)$.

$\delta^*_T = 1_{\{W_T > 0\}}\,, \quad T^* = ?$

Representation of the risk:

$$R(T, \delta_T^*) = \int h(\theta|W_T|)\, dQ$$

with $h(x) = \frac{e^{-2x}}{1+e^{-2x}} + cx\,\frac{1-e^{-2x}}{1+e^{-2x}}$ and $Q = \frac{1}{2}P_\theta + \frac{1}{2}P_{-\theta}$

Note that $\left.\frac{dP_\theta}{dP_{-\theta}}\right|_{\mathcal{F}_T} = \exp(2\theta W_T)$

$h$ has a unique minimum in $a_c$ and

$$R(T, \delta_T^*) = \int h(\theta|W_T|)\, dQ \geq h(a_c)$$

Let $T^* = \min\{t > 0 \mid \theta|W_t| = a_c\}$.

Since $Q(T^* < \infty) = 1$ it follows $R(T^*, \delta_{T^*}^*) = h(a_c)$.

# The Repeated Significance Test as Bayes Test (RST)



$W_t,\ t \geq 0$ Brownian motion with drift $\theta$; $P_\theta$ underlying measure.

**Testing sequentially:** $H_0 : \theta < 0$ versus $H_1 : \theta > 0$

Prior: $G = N(0, r^{-1})$

$$R(T,\delta) = \int_{-\infty}^{0} \left(P_\theta\{\delta \text{ rejects } H_0\} + c\,\theta^2 E_\theta T\right) G(d\theta)$$

$$+ \int_{0}^{\infty} \left(P_\theta\{\delta \text{ rejects } H_1\} + c\,\theta^2 E_\theta T\right) G(d\theta)$$

Find $(T^*, \delta^*)$ with $\quad R(T^*, \delta^*) = \min_{(T,\delta)} R(T,\delta)$.

$\delta^* = \delta^*_T = 1_{\{W_T > 0\}} \qquad T^* = \ ?$

PNAS, 83 (1986)

Representation of the risk:

$$R(T,\delta_T^*) = \int g\left(\frac{W_T^2}{T+r}\right) dQ$$

with $g(x) = \Phi(-\sqrt{x}) + c\,x, \quad Q = \int P_\theta\, G(d\theta), \quad G = N\left(0, r^{-1}\right).$

$g$ is convex with unique minimum $b_c$ and

$$R(T,\delta_T^*) = \int g\left(\frac{W_T^2}{T+r}\right) dQ \geq g(b_c)$$

Let $T^* = \min\{t > 0 \mid W_t^2/(t+r) = b_c\}$.

Since $Q\{T^* < \infty\} = 1$ it follows $R(T^*, \delta_{T^*}^*) = g(b_c)$.

## The representation:

$$G = N(0, r^{-1}), \quad Q = \int P_\theta G(d\theta), \quad G_{W(T),T} = N\left(\tfrac{W(T)}{T+r}, \tfrac{1}{T+r}\right)$$

$$\begin{aligned}
\int \theta^2 E_\theta T\, G\,(d\theta) &= \int \theta^2 E_\theta (T+r)\, G\,(d\theta) - 1 \\
&= \int (T+r) \int \theta^2 G_{W(T),T}\,(d\theta)\, dQ - 1 \\
&= \int (T+r) \left( \frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r} \right) dQ - 1 \\
&= \int \frac{W(T)^2}{T+r}\, dQ
\end{aligned}$$

# The Disruption Problem

Shiryaev (1961) studied the following problem.

Observations: $W_t = B_t + \theta(t-\tau)^+$ with

$B_t,\ t \geq 0$ standard Brownian motion,

$\theta > 0$ fixed

Filtration: $\mathcal{F}_t = \sigma(W_t;\ 0 \leq s \leq t)$

Change-point: $\tau$ random time, independent of $B$

with distribution $\pi = p\delta_0 + (1-p)F$,

where $F(t) = 1 - e^{-\lambda t}$

Risk: $R(T) = P_\pi(T < \tau) + cE_\pi(T-\tau)^+$

Find $T^*$ with $R(T^*) = \min_T R(T)$.

## Theorem

Let $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$ and $T^* = \min\{t > 0 \mid \pi_t \geq p^*\}$ .

Here $p^*$ is the unique solution in $(0,1)$ of $G'(p) = 1$, where $G$ is the (finite at 0) solution of

$$\frac{\theta}{2}x^2(1-x^2)G''(x) + \lambda(1-x)G'(x) = cx.$$

Then

$$\pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t}$$

where

$$\varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t}{L_s} \lambda e^{-\lambda s}\, ds$$

with

$$L_t = \exp(\theta W_t - \theta^2 t/2).$$

$\pi_t$ is a diffusion with $d\pi_t = \lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}$ where $\overline{W_t}$ is a standard Brownian motion.

Itô's formula yields:

$$\begin{aligned} dG(\pi_t) &= G'(\pi_t)d\pi_t + \frac{1}{2}G''(\pi_t)(d\pi_t)^2 \\ &= G'(\pi_t)\left[\lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}\right] \\ &\qquad + \frac{1}{2}G''(\pi_t)\theta^2\pi_t^2(1-\pi_t)^2 dt \end{aligned}$$

If $G$ satisfies the equation

$$\frac{\theta^2}{2}x^2(1-x)^2G''(x) + \lambda(1-x)G'(x) = cx$$

and behaves well at 0, then

$$G(\pi_t) - G(\pi_0) = c\int_0^t \pi_s ds + c\int_0^t \theta\pi_s(1-\pi_s)d\overline{W_s}$$

$$\Rightarrow E_\pi\left[G(\pi_T) - G(\pi_0)\right] = c\,E_\pi\int_0^T \pi_s ds\,.$$

Then one obtains

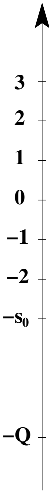
$$R(T) = P(T < \tau) + cE(T-\tau)^+$$
$$= E_\pi \left[ (1-\pi_T) + c \int_0^T \pi_s ds \right]$$
$$= \int g(\pi_T) dP - G(p)$$

with $g(x) = (1-x) + G(x)$

$g$ is convex with a unique minimum at $p^*$.

This insight opened a new direction to Bayes tests of power one for change point problems with continuous composite hypotheses.
(Disseration of M. Beibel (1994), Diploma thesis of I. Maahs (2008))

# A Parking Problem




$$S_0 = -Q$$

$$S_n = \sum_{i=1}^{n} X_i - Q$$

$X_i$ i.i.d. geometric $(p)$

$p$: probability of empty spot

Park as near as possible at ″0″!

Find a stopping time $T^*$ of $S_i, i \geq 0$ with

$$E \mid S_{T^*} \mid = \min_T E \mid S_T \mid .$$

Solution: $T^* = \min\{n \geq 1 \mid S_n > -s_0\}$

with $s_0 = \min\{s \in \mathbb{N} \mid 2(1-p)^s - 1 < 0\}$

Chow, Robbins, Siegmund (1971): Great Expectations, p. 45

# Generalized Parking Problem (GPP)

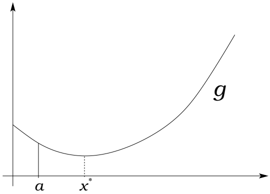
Let $g$ be a convex nonnegative function with a unique minimum at $x^* > 0$.

Assume $X_i$ i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find a stopping time $T^*$ with

$$Eg(S_{T^*}) = \min_T Eg(S_T).$$




Solution (Woodroofe–Lerche–Keener (1994)):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\}$$

# II Overshoot and Optimality in Sequential Testing

# Generalized Parking Problem (GPP)

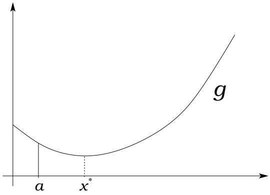
Let $g$ be a convex nonnegative function with a unique minimum at $b > 0$.

Assume $X_i$ i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find a stopping time $T^*$ with

$$Eg(S_{T^*}) = \min_T Eg(S_T).$$




Solution (Woodroofe–Lerche–Keener (1994)):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\}$$

with $a = \sup\{x \mid H^+ g(x) < g(x)\}$ where $H^+$ is the ladder-height distribution of $S_n$; $n \geq 1$ and $H^+ g(x) = \int g(x+y) H^+(dy)$.

Let $K(z) = \int_0^z \frac{1 - H^+(y)}{\nu_1}\, dy$ with $\nu_i = \int y^i\, dH^+(y)$, $i \in \mathbb{N}$.

$K$ is the asymptotic overshoot distribution function.

See Siegmund, Sequential Analysis (1985).

## Theorem 1

If $K\, g(x) < \infty$ for all $0 \leq x < \infty$, then $K\, g(x)$ is minimized at $x = a$.

**Example 1:**

If $g(x) = |x - b|$ for $x \in \mathbb{R} \Rightarrow a = b - \text{med}\, K$.

**Example 2:**

If $g(x) = (x-b)^2$ for $x \in \mathbb{R}$ and $\nu_2$ is finite $\Rightarrow a = b - \frac{\nu_2}{\nu_1}$

**Example 3:**

If $g(x) = e^{-x} + cx$ for $x \in \mathbb{R}$, with $0 < c < 1$

$\Rightarrow b = \log(\frac{1}{c})$.

If $\int x^2 H^+(dx) < \infty$ and if $\kappa := \int_0^\infty e^{-x} K(dx)$

$\Rightarrow Kg(x) = \kappa e^{-x} + c(x + \frac{\nu_2}{2\nu_1})$ and is minimized at $\log(\frac{\kappa}{c}) = b - \log(\frac{1}{\kappa})$.

# Lorden's Result on the one-sided SPRT

Let $P_0 \neq P_1$ be equivalent and let $I = E_1 \log \frac{dP_1}{dP_0}(X)$ .

Minimize $$R(T) = P_0(T < \infty) + c\, I\, E_1\, T \ .$$

Let $$\ell_n = \log \frac{dP_1^n}{dP_0^n}\ .$$

Then by Wald' identities

$$R(T) = \int e^{-\ell_T} dP_1 + c \int \ell_T dP_1 = \int g(\ell_T)\, dP_1 \quad \text{with } g(x) = e^{-x} + cx.$$

$g$ is a nonnegative convex function with a unique minimum at $\log \frac{1}{c}$.

Then $T_c^* = \min\{n \geq 1 | \ell_n \geq \log(\frac{\kappa}{c})\}$ where $\kappa = \lim_{a \to \infty} E_1 \exp(-(\ell_{\tau_a} - a))$

and $\tau_a = \min\{n \geq 1 | \ell_n \geq a\}$.

Then $$R(T_c^*) = c\left[\left(1 + \log \frac{1}{c}\right) + \log \kappa + \frac{\nu_2}{2\,\nu_1}\right].$$

Lorden (AS 1977)

# Testing the Sign of the Mean with $|\theta|$ known

$S_n$, $n \geq 0$ normal random walk ($\sigma^2 = 1$) with mean $-\theta$ or $+\theta$ and with $\theta > 0$ known, $S_0 = 0$.

**Testing sequentially:** $H_0 : \ -\theta$ versus $H_1 : \ +\theta$.

Prior: Uniform on $\{-\theta, \theta\}$

$$R(T,\delta) = \tfrac{1}{2}\left(P_{-\theta}\{\delta \text{ rejects } H_0\} + c\theta^2 E_{-\theta} T\right)$$
$$+\tfrac{1}{2}\left(P_{\theta}\{\delta \text{ rejects } H_1\} + c\theta^2 E_{\theta} T\right)$$

Let $R_c^* = \min_{(T,\delta)} R_c(T,\delta)$.

Let $\delta_T^* = 1_{\{S_T > 0\}}$.

Then

$$R_c(T, \delta^*) = \int g_c(\theta|S_T|)\, dQ$$

where $g_c(x) = \frac{e^{-2x}}{1+e^{-2x}} + c\,x\,\frac{1-e^{-2x}}{1+e^{-2x}}$ and $Q = \frac{1}{2}P_\theta + \frac{1}{2}P_{-\theta}$.

Let $a_c = \arg\min_x g_c(x)$ and

let $T_c = \min\left\{n \geq 1 \mid \theta|S_n| \geq a_c - \log\left(\frac{1}{\kappa}\right)\right\}$

with $\kappa = \lim_{a\to\infty} E_\theta e^{-(\theta S_{\tau_a} - a)}$.

Then it holds

$$R_c(T_c, \delta^*_{T_c}) - R^*_c = o(c) \qquad \text{as } c \to 0.$$

Lorden (1977), AS

# Nonlinear Parking Problem: Discrete Case

$Z_1, Z_2, \dots$ a perturbed random walk, say

$$Z_n = \widetilde{S}_n + \xi_n \quad \text{for } n = 0, 1, 2, \dots ,$$

where

$$S_n = \sum_{i=1}^{n} X_i \,, \quad n \geq 1$$

with

$$X_1, X_2, \dots \text{ i.i.d. and } 0 < EX_1 < \infty,$$

having a non-arithmetic distribution.

$\xi_n$ are slowly changing in the sense of "Woodroofe, SIAM, 1982" or Siegmund (1985).

Let $g_c$, $0 < c \leq 1$ denote convex functions. Find $T_c^*$ with

$$Eg_c\left(Z_{T_c^*}\right) = \min_T Eg_c(Z_T).$$

## Definition

*A sequence of random variables* $(\xi_n, n \geq 1)$ *is called slowly changing if:*

i) $\lim_{\delta \downarrow 0} \sup_{n \geq 1} P\left(\max_{0 \leq k \leq n\delta} |\xi_{n+k} - \xi_n| > \varepsilon\right) = 0 \quad \forall \varepsilon > 0$

ii) $\frac{1}{n} \max(|\xi_1|, |\xi_2|, \dots, |\xi_n|) \xrightarrow{P} 0$

**Example (RST)**

$$Z_n = \frac{S_n^2}{n+r}$$

$$\widetilde{S}_n = 2\left(\theta S_n - \frac{n}{2}\theta^2\right)$$

$$\xi_n = \frac{(S_n - n\theta)^2}{r+n} - \frac{2(S_n - n\theta)r\theta}{r+n} - \frac{nr\theta^2}{r+n}$$

$\xi_n,\ n > 1$ is slowly changing.

# Overshoot for a Perturbed Random Walk



## Theorem 2

Let $Z_n = \widetilde{S}_n + \xi_n$, where $\widetilde{S}_n$ is a random walk with mean $\mu > 0$.

Let $\xi_n;\ n \geq 1$ be slowly changing.

Let $T_a = \min\{n \geq 1 \mid Z_n \geq a\}$ and $\tau_a = \min\{n \geq 1 \mid \widetilde{S}_n \geq a\}$.

Then

$$\lim_{a \to \infty} P_\mu \left( Z_{T_a} - a \leq x \right) = \lim_{a \to \infty} P_\mu \left( \widetilde{S}_{\tau_a} - a \leq x \right)$$

Lai–Siegmund (1977)

For each $0 < c \leq 1$ let $g_c$ be a convex function with a unique minimum at $b = b_c \geq 0$. Assume $\lim_{c \downarrow 0} b_c = \infty$ and there exists a convex function $h : \mathbb{R} \to \mathbb{R}$ with minimum at zero and with

$$h_c(x) := \frac{g_c(b+x) - g_c(b)}{c} \to h(x) < \infty.$$

Let $K(y) = \int_0^y \frac{1 - H(x)^+}{\gamma_1} dx$ for $\widetilde{S}_n;\ n \geq 1$ as in the GPP.

## Theorem 3 (Schwarz, Keener-L-Woodroofe)

Let $\gamma = \operatorname*{argmin}_x Kh(-x)$ and $T_{b-\gamma} = \min\{n \geq 1 \mid Z_n \geq b - \gamma\}$.

Then as $c \to 0$

1) $\inf_T Eg_c(Z_T) = g_c(b) + c \inf_T Eh(Z_T - b) + o(c)$.

2) $\inf_T Eh(Z_T - b) = Eh\left(Z_{T_{b-\gamma}}\right) + o(1)$

3) $Eh\left(Z_{T_{b-\gamma}}\right) = Kh(-\gamma) + o(1)$.

## Corollary

$$\inf_T Eg_c(Z_T) = Eg_c\left(Z_{T_{b_{-\gamma}}}\right)$$
$$= g_c(b_c) + cKh(-\gamma) + o(c)$$

# Testing the Mean sequentially

$S_n,\ n \geq 0$ normal random walk with mean $\theta$ and variance 1, $S_0 = 0$.

**Testing sequentially:** $H_0:\ \theta = 0$ versus $H_1:\ \theta \neq 0$.

Let $I(\theta) = E_\theta \log \frac{dP_\theta}{dP_0}(X_1),\ G = N\left(0, r^{-1}\right)$

$$R_c(T) = P_0(T < \infty) + c \int I(\theta) E_\theta T G(d\theta)$$

Let $P_* = \int P_\theta G(d\theta)$ and $Z_n = \log \frac{dP_*^n}{dP_0^n} = \frac{S_n^2}{2(n+r)} - \frac{1}{2} \log\left(\frac{n+r}{r}\right)$

Then

$$R_c(T) = E_* g_c(Z_T) + E_* \log\left(\frac{T+r}{r}\right)$$

with $g_c(z) = e^{-z} + cz$.

$g_c$ is convex with minimum at $b_c = \log\left(1/c\right)$.

Then by Theorem 3

$$E_* g_c(Z_T) = \int E_\theta g_c(Z_T) G(d\theta)$$
$$\geq g_c(b_c) + c \left( \int_{-\infty}^{\infty} \inf_T E_\theta h(Z_T - b_c) G(d\theta) + o(1) \right)$$

with $h(z) = z + e^{-z} - 1$.

$Z_n = \widetilde{S}_n + \xi_n$ where $\widetilde{S}_n = \theta S_n - \frac{1}{2}\theta_n^2$ and $\xi_n$ is slowly changing.

Then by Theorem 3 again

$$\inf_T E_\theta h(Z_T - b_c) \geq K^\theta h(-\gamma(\theta)) + o(1)$$

Here $K^\theta$ ist the asymptotic overshoot distribution of $\widetilde{S}_n$.

$\gamma(\theta) = \log\left({}^1/_{\kappa(\theta)}\right)$ where $\kappa(\theta) = \int e^{-x} K^\theta(dx)$.

Then

$$\inf_T E_* g(Z_T) \geq g_c(b_c) + c\left(\int_{-\infty}^{\infty} K^\theta h\left(-\gamma(\theta)\right) G(d\theta) + o(1)\right)$$

$$= g_c(b_c) + c\left(\int_{-\infty}^{\infty} \left(\kappa(\theta) + \frac{\nu_2(\theta)}{2\nu_1(\theta)} - 1\right) G(d\theta) + o(1)\right)$$

One can show:

$T_c = \inf\{n \geq 1 \mid Z_n > b_c - \Gamma_n\}$, where $\Gamma_n = E[\gamma(\theta) \mid Y_1, \ldots, Y_n]$,

assumes this lower bound.

G. Schwarz (1993)

Note: The repeated significance test does not satisfy the assumptions of Theorem 3. In his case a stabilizing $h$ does not exist.

# III A Martingale-Measure Transformation Approach to Optimal Stopping

# The Basic Idea: OS as GPP

Let $(Z_t, \mathcal{F}_t; t \geq 0)$ denote a continuous stochastic process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Find a stopping time $T^*$ with

$$E_P\left(Z_{T^*} 1_{\{T^*<\infty\}}\right) = \max_T E_P\left(Z_T 1_{\{T<\infty\}}\right).$$

**Idea:** (Beibel-Lerche (1997))

Find a process $(X_t, \mathcal{F}_t; t \geq 0)$, a nonnegative martingale $(M_t, \mathcal{F}_t; t \geq 0)$ with $E_p M_0 = 1$ and a function $f$ with unique maximum at $x^*$ such that $Z_t = f(X_t) M_t$.

Then

$$\begin{aligned} E_p Z_T 1_{\{T<\infty\}} &= E_p\left(f(X_T) M_T 1_{\{T<\infty\}}\right) \\ &\leq f(x^*) E_p M_T 1_{\{T<\infty\}} \\ &\leq f(x^*) \end{aligned}$$

With $T^* = \min\{t \geq 0 \mid X_t = x^*\}$ the inequalities become equalities, if $E_p M_{T^*} 1_{\{T^*<\infty\}} = 1$.

# Optimality of Parabolic Boundaries

Let $X_t = B_t + x_0$, $t \geq 0$ with $B_t$ standard Brownian motion. For a measurable function $g$ find a stopping time $T$ that maximizes

$$E_p\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right). \qquad \text{(Moerbeke (1974))}$$

Let $H(x) = \int_0^\infty e^{ux-u^2/2} u^{2\beta-1}\, du$ with $\beta > 0$

and assume that there exists a unique point $x^*$ with

$$\sup_{x\in\mathbb{R}} \frac{g(x)}{H(x)} = \frac{g(x*)}{H(x*)} = C^* \quad \text{and } 0 < C^* < \infty$$

$$(t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) = \int_0^\infty e^{uX_t - \frac{u^2}{2}t}\left(e^{-\frac{u^2}{2}} u^{2\beta-1}\right) du$$

is a positive martingale with starting value $H(x_0)$.

Thus $M_t = (t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) \Big/ H(x_0)$ is a positive martingale with

$E_p M_0 = 1$.

Then

$$E_p\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) = H(x_0) E_p \frac{g\left(\frac{X_T}{\sqrt{T+1}}\right)}{H\left(\frac{X_T}{\sqrt{T+1}}\right)} M_T$$

$$\leq H(x_0) C^*.$$

Let $T^* = \inf\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}$.

For $x_0 < x^*$ we have $E_p M_{T^*} = 1$. Thus

$$\sup_T E_p \left\{ (T+1)^{-\beta} g\left( \frac{X_T}{\sqrt{T+1}} \right) \right\} = E_p \left\{ (T^*+1)^{-\beta} g\left( \frac{X_{T^*}}{\sqrt{T^*+1}} \right) \right\}$$
$$= H(x_0) C^*$$

**Special case:**

$$g(x) = x, \; x_0 = 0, \; \beta = \tfrac{1}{2}$$

$$E_p(X_T/(T+1)) = \max \text{ with}$$

$$T^* = \min \left\{ t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^* \right\}$$

$x^*$ is solution of $\quad x = (1-x^2) \int_0^\infty e^{ux-u^2/2}\, du.$ Shepp (AMS 1969)

# Perpetual American Put Option

Samuelson (1965), McKean (1965)

$X_t = \sigma B_t + \mu t$ , $t \geq 0$ Brownian motion with drift $\mu$ and variance $\sigma^2$.

Find a stopping time $T^*$ which maximizes

$$E_P\, e^{-rT} (K - e^{X_T})^+ \, 1_{\{T<\infty\}}.$$

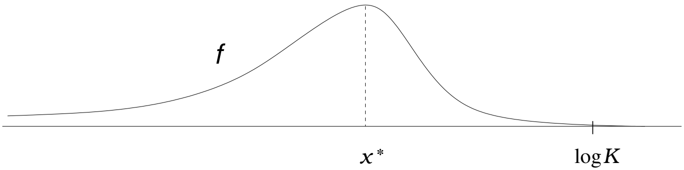
**Idea:**

Find $M$ and $f$ with $E_P\, e^{-rT} (K - e^{X_T})^+ \, 1_{\{T<\infty\}} = E_P\, f(X_T)\, M_T\, 1_{\{T<\infty\}}$,

where $f$ has a unique maximum at $x^*$.

Then

$$T^* = \min\{t \geq 0 \mid X_t = x^*\} \quad \text{if } E_P\, M_{T^*} = 1.$$



How to find $M_t$ ?

It holds for all $\alpha \in \mathbb{R}$

$$(K - e^{X_T})^+ e^{-rT} = (K - e^{X_T})^+ (e^{X_T})^{-\alpha} (e^{X_T})^{\alpha} e^{-rT}.$$

Choose $f(x) = (K - e^{x_T})^+ e^{-\alpha x}$ and $\alpha$ such that $M_t = e^{\alpha X_t} e^{-rt}$ is a martingale.

This holds when

$$\begin{aligned} M_t &= \exp\left[\alpha(\sigma B_t) + t(\alpha\mu - r)\right] \\ &= \exp\left[(\alpha\sigma)B_t - t(\alpha\sigma)^2/2\right]. \end{aligned}$$

$M_t$ is a positive martingale with $M_0 = 1$ iff $(\alpha\sigma)^2/2 + \alpha\mu - r = 0$

$\alpha^{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$ are the two possible solutions.

Thus we have two martingales $M_t^{\pm}$.

Then we have

$$E_P\, e^{-rT}\left(K - e^{X_T}\right)^+ 1_{\{T<\infty\}} = E_Q\, f(X_T)\, 1_{\{T<\infty\}}$$

with $f(x) = \dfrac{(K - e^x)^+}{e^{\alpha^- x}}$ and $\dfrac{dQ_t}{dP_t} = M_t^-$.

Let $K < 1 + (-\alpha^-)^{-1}$. Then $f$ has a unique maximum at

$x^* = \log \frac{\alpha^- K}{\alpha^- - 1} < 0$. Under $Q$ $X$ is a Brownian motion with drift

$\alpha^- \sigma^2 + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0.$

This yields $Q(T^* < \infty) = 1$ for $T^* = \inf\{t > 0 \mid X_t = x^*\}$.

Then

$$\sup_T E_P\left(e^{-rT}(K - e^{X_T})^+ 1_{\{T<\infty\}}\right) = E_Q\, f(X_{T^*}) = \frac{(K - e^{x^*})}{e^{\alpha^- x^*}}.$$

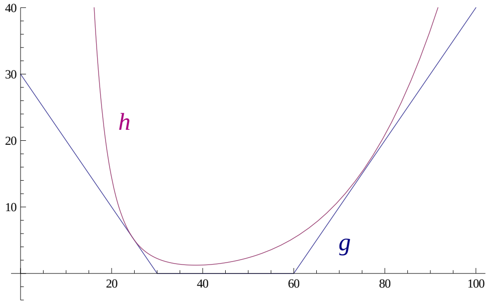
# Example: Strangle Option

$g(x) = (e^x - K)^+ \vee (L - e^x)^+$

$h(x) = p^* e^{\alpha_1 x} + (1 - p^*) e^{\alpha_2 x}$ with

$0 < p^* < 1$, $\alpha_2 < 0 < \alpha_1$

Here $h$ is convex.



# Two-Sided Boundaries

Let $g$ be measurable, $X_t = \sigma B_t + \mu t$ a Brownian motion with drift $\mu$, variance $\sigma^2$ and $X_0 = x$. Find a stopping time $T^*$ which maximizes

$$Ee^{-rT} g(X_T) 1_{\{T<\infty\}}.$$

Let $\alpha_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \quad (\alpha_2 < 0 < \alpha_1)$.

Then $M_t^{(i)} = e^{-rt} e^{\alpha_i X_t}$, $i = 1, 2$ are positive martingales.

We consider boundaries of the type

1.) $g(x) = x^2$

2.) $g(x) = \max\{(L - e^x)^+, (e^x - K)^+\}$

Let $p \in [0,1]$. Let $M_t = pM_t^{(1)} + (1-p)M_t^{(2)}$. Then

$$E_x e^{-rT} g(X_t) = E_x M_T \frac{g(X_T)}{pe^{\alpha_1 X_T} + (1-p)e^{\alpha_2 X_T}}.$$

Let $g(x)$ be nonnegative and measurable with

a) $0 < \sup_{x \geq 0}(e^{-\alpha_1 x} g(x)) < \sup_{x \leq 0}(e^{-\alpha_1 x} g(x)) < \infty$

b) $0 < \sup_{x \leq 0}(e^{-\alpha_2 x} g(x)) < \sup_{x \geq 0}(e^{-\alpha_2 x} g(x)) < \infty.$

## Lemma

*If a) and b) holds, there exists a* $p^* \in (0,1)$ *with* $\sup_{x \geq 0} G_{p^*}(x) = \sup_{x \leq 0} G_{p^*}(x)$, *where*

$$G_p(x) = \frac{g(x)}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}}.$$

## Theorem

*Let* $C^* = \sup_{x \in \mathbb{R}} G_{p^*}(x)$. *If there exists points* $x_1 > 0$ *and* $x_2 < 0$ *with* $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$, *then with* $T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}$

$$\sup_T E_x e^{-rT} g(X_T) = E_x e^{-rT^*} g(X_{T^*}) = C^*(p\, e^{\alpha_1 x} + (1-p)\, e^{\alpha_2 x})$$

*for* $x_1 \leq x \leq x_2$.

# Stopping of Diffusions with Random Exponential Discounting

$X$ a regular diffusion with $X_0 = x$ and $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$

and $B$ standard Brownian motion, state space I.

$g : \mathbb{R} \to \mathbb{R}_+$ a continuous function.

Find a stopping time $T^*$ of $X$ with

$$E_x\left(e^{-A(T)}g(X_T)1_{\{T<\infty\}}\right) = \max.$$

$A(s)$: additive continuous stochastic process adapted to $\mathcal{F}^X$

$$A(s+t) = A(s) + A(t) \circ \theta_s$$

**Example:**

$$E_x\left(\exp\left\{-r\int_0^T B_t^2\,dt\right\}\left(B_T^+\right)^\alpha 1_{\{T<\infty\}}\right) = \max$$

# The Main Idea Again

How to solve

$$V_*(x) = \sup_T E_x\left(e^{-A(T)} g(X_T)\, 1_{\{T<\infty\}}\right) \; ?$$

Let $h : I \to \mathbb{R}_+$ be such that $e^{-A(t)} h(X_t)$ is a positive local martingale and $\sup_x \frac{g}{h}(x) = C^* < \infty$. Then

$$\begin{aligned} E_x\left(e^{-A(T)} g(X_T)\, 1_{\{T<\infty\}}\right) &= E_x\left(e^{-A(T)} h(X_T) \frac{g(X_T)}{h(X_T)}\, 1_{\{T<\infty\}}\right) \\ &\leq C^* E_x\left(e^{-A(T)} h(X_T)\, 1_{\{T<\infty\}}\right) \\ &\leq C^* h(x). \end{aligned}$$

If there exists a $T^*$ with $\frac{g}{h}(X_{T^*}) = C^*$ and $E_x\left(e^{-A(T^*)} h(X_{T^*})\, 1_{\{T^*<\infty\}}\right) = h(x)$ the inequalities become equalities and the stopping problem has been solved.

# How to choose the Martingales?

$$\psi_+(x) = \begin{cases} E_x\left(e^{-A(\tau_{x_0})}1_{\{\tau_{x_0}<\infty\}}\right) & \text{for } x \le x_0 \\ \left[E_{x_0}\left(e^{-A(\tau_x)}1_{\{\tau_x<\infty\}}\right)\right]^{-1} & \text{for } x \ge x_0 \end{cases}$$

$$\psi_-(x) = \begin{cases} \left[E_{x_0}\left(e^{-A(\tau_x)}1_{\{\tau_x<\infty\}}\right)\right]^{-1} & \text{for } x \le x_0 \\ E_x\left(e^{-A(\tau_{x_0})}1_{\{\tau_{x_0}<\infty\}}\right) & \text{for } x \ge x_0. \end{cases}$$

$M_t^{(+)} = e^{-A(t)}\psi_+(X_t)$ for $b \ge x$ on $0 \le t \le \tau_b$

$M_t^{(-)} = e^{-A(t)}\psi_-(X_t)$ for $x \ge a$ on $0 \le t \le \tau_a$.

are u.i. martingales

with

$$E_x\left(M_{\tau_b}^{(+)}1_{\{\tau_b<\infty\}}\right) = \psi_+(x) \qquad \text{for } x \le b$$

$$E_x\left(M_{\tau_a}^{(-)}1_{\{\tau_a<\infty\}}\right) = \psi_-(x) \qquad \text{for } x \ge a.$$

**Note:**

If $A(t) = \int_0^t r(X_s)ds$ with $r(x) \ge 0$, then $\psi_\pm(x)$ are the solutions of $\mathcal{D}\psi = r \cdot \psi$ with

$$\mathcal{D} = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2}{\partial x^2}.$$

**Example:**

$r(x) = rx^2$ with $r > 0$, $x = 0$.

Let $\Psi(x) = e^{-x^2/2} \frac{2^{5/4}}{\Gamma(1/2)} \int_0^\infty e^{xt - t^2/2} \frac{1}{\sqrt{t}}\, dt$

Then $\psi(x) = \Psi\left(\sqrt[4]{8r}\, x\right)$ is an increasing solution of

$$\frac{1}{2}\psi''(x) = rx^2\psi(x) \qquad \text{with } \psi(0) = 1.$$

Then $\exp\left(-r\int_0^t X_s^2\, ds\right)\psi(X_t)$ is a local martingale and by the OST

$$E_x \exp\left(-r\int_0^{\tau_0} X_s^2\, ds\right) = \psi(x) \quad \text{for } x \le 0$$

and

$$E_0 \exp\left(-r\int_0^{\tau_x} X^2\, ds\right) = \psi(x)^{-1} \quad \text{for } x \ge 0.$$

Thus $\psi(x) = \psi_+(x)$.

Then $\sup_{x\in\mathbb{R}}\left[(x^+)^\alpha/\psi_+(x)\right] = \sup_{x\ge 0}\left[(x^+)^\alpha/\psi_+(x)\right] < \infty.$

Thus $E_x \exp(-A(T))(B_T^+)^\alpha \mathbb{1}_{\{T<\infty\}} = E_x \dfrac{e^{-A(T)}\psi_+(B_T)(B_T^+)^\alpha}{\psi_+(B_T)}$

$$\leq \psi_+(x)\frac{(x^*)^\alpha}{\psi_+(x^*)}$$

With $x^* = \arg\max_x \left[(x^+)^\alpha/\psi_+(x)\right] > 0$ one has $T^* = \inf\{t > 0 \mid X_t = x^*\}$.

Is $S^* = [x^*, \infty)$ the optimal stopping set?

Thus $E_x \exp(-A(T))(B_T^+)^\alpha \mathbb{1}_{\{T<\infty\}} = E_x \frac{e^{-A(T)}\psi_+(B_T)(B_T^+)^\alpha}{\psi_+(B_T)}$

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With $x^* = \arg\max_x \left[(x^+)^\alpha/\psi_+(x)\right] > 0$ one has $T^* = \inf\{t > 0 \mid X_t = x^*\}$.

Is $S^* = [x^*, \infty)$ the optimal stopping set?

What is $v(x)$?

$$\text{Thus } E_x \exp(-A(T))(B_T^+)^\alpha \mathbb{1}_{\{T<\infty\}} = E_x \frac{e^{-A(T)}\psi_+(B_T)(B_T^+)^\alpha}{\psi_+(B_T)}$$

$$\leq \psi_+(x)\frac{(x^*)^\alpha}{\psi_+(x^*)}$$

With $x^* = \arg\max_x \left[(x^+)^\alpha/\psi_+(x)\right] > 0$ one has $T^* = \inf\{t > 0 \mid X_t = x^*\}$.

Is $S^* = [x^*, \infty)$ the optimal stopping set?

What is $v(x)$?

$$v(x) = \begin{cases} \dfrac{(x^*)^\alpha}{\Psi\left(\sqrt[4]{8r}\,x^*\right)}\Psi\left(\sqrt[4]{8r}x\right) & \text{for } x \leq x^*, \\ x^\alpha, & \text{for } x \geq x^*. \end{cases}$$

# Characterization of the Stopping Set

For $r(x) = r$ a complete characterization of the stopping set has been given by Sören Christensen in his dissertation (2010).

He showed by using a Choquet-representation result for $r$-harmonic functions that the optimal stopping set $S^*$ can be characterized as

$$S^* = \left\{ x \mid \exists\, r\text{-harmonic } h \text{ with } x = \arg\max_y \frac{g(y)}{h(y)} \right\}.$$

It implies that the value function is the minimum of $r$-harmonic functions $\geq g$.

This result has been extended by Cedric Thoms to random discounting.

Let $v(x) = \sup_T E_x(e^{-A(T)} g(X_T) 1_{\{T<\infty\}})$ and let an optimal $T^*$ exist.

Let $S^* = \{x \mid v(x) = g(x)\}$, then $\tau^* = \inf\{t > 0 \mid X_t \in S^*\}$ is optimal and $P_x(\tau^* \leq T^*) = 1$ for all $x \in I$.

Define: A nonnegative function $f : I \to [0, \infty)$ is $A$-harmonic if it holds

$$E_x\left[e^{-A(\tau_{(c,d)})} f(X_{\tau_{(c,d)}})\right] = f(x)$$

for all $(c, d) \subset I$ and for all $x \in I$.

## Theorem (Christensen–Irle, Thoms)

*A point $x \in I$ is in $S^* = \{v = g\}$ if and only if an $A$-harmonic function $h$ exists (i.e. $h = \alpha\psi_+ + \beta\psi_-$, and $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$), such that*

$$x = \arg\max_y \frac{g(y)}{h(y)}.$$

## Corollary

*Let $g$, $\psi_+$ and $\psi_-$ be continuously differentiable. Let $X \in S^*$ with $\frac{g(x)}{h(x)} = 1$ and let $w(x) = \psi_+'(x)\psi_-(x) - \psi_+(x)\psi_-'(x) \neq 0$, then*

$$\alpha = \alpha(x) = \frac{g(x)\psi_+'(x) - g'(x)\psi_+(x)}{w(x)}$$

$$\text{and} \qquad \beta = \beta(x) = \frac{g'(x)\psi_-(x) - g(x)\psi_-'(x)}{w(x)}.$$

## Remark

*Using this Corollary one can show that $S^* = [x^*, \infty)$ in the example.*

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Thank you for your attention!