

A MULTIPLE-CURVE LÉVY FORWARD RATE MODEL IN A TWO-PRICE ECONOMY

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ABSTRACT. An advanced Heath-Jarrow-Morton (HJM) forward rate model driven by time-inhomogeneous Lévy processes is presented which is able to handle the recent development to multiple curves and negative interest rates. It is also able to exploit bid and ask price data. In this approach in order to model spreads between curves for different tenors, credit as well as liquidity risk is taken into account. Deterministic conditions are derived to ensure the positivity of spreads and thus the monotonicity of the curves for the various tenors. Valuation formulas for standard interest rate derivatives such as caps, floors, swaptions and digital options are established. These formulas can numerically be evaluated very fast by using Fourier based valuation methods. In order to exploit bid and ask prices we develop this approach in the context of a two-price economy. Explicit formulas for bid as well as ask prices of the derivatives are stated. A specific model framework based on Normal Inverse Gaussian and Gamma processes is proposed which allows for calibration to market data. Calibration results are presented based on multiple-curve bootstrapping and cap market quotes. We use data from September 2013 as well as September 2016. The latter is of particular interest since rates were deep in negative territory at that time.

KEYWORDS. Multiple-curve bootstrapping, multiple-curve model, HJM, time-inhomogeneous Lévy processes, monotonicity of curves, two-price theory, negative interest rates, interest rate derivatives, calibration

1. INTRODUCTION

The global financial crisis which started in early August 2007 had a lasting effect on financial markets. This concerns in particular the fixed income markets. As a consequence of a new perception of risk a number of interest rates, which until then had been roughly equivalent, drifted apart. In particular the basic rates, which are relevant for the interbank market, became tenor-dependent after market participants became aware of credit, liquidity and funding risks. These risks had been assumed to be negligible in this market segment before. In the new reality classical modelling approaches which are based on arbitrage considerations assuming tenor-independence cannot reflect the market behaviour any more. More sophisticated approaches, so-called multiple curve models, are

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needed to take the increased diversity of risks into account. In this paper we focus on the tenor-dependence of the Euribor rates. Before the crisis Euribor as well as Libor rates typically differed by a few basis points only. Starting in early August 2007 the spreads widened and reached levels up to 200 basis points (see Figure 1). More specifically, the graph shows the historical evolution of the differences between the spot quotes of the Euribor rates for maturities of one, three, six, nine and twelve months and the overnight indexed swap (OIS) rates based on EONIA with the corresponding maturities. We mention that the monotonicity of the spreads according to tenor length which is shown in this graph is generally not true for Euribor vs Euribor basis swaps (see the graph on the right side of figure 3).

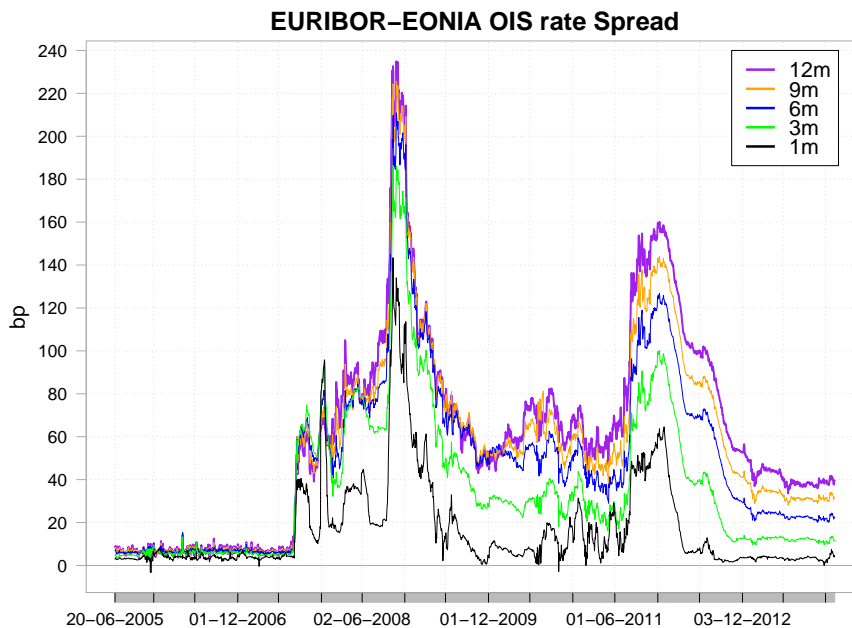


FIGURE 1. Evolution of the spreads between EURIBOR and EONIA OIS rates for different tenors.

Figure 2 presents the historical evolution of the basic discount curve and (fictitious) bond prices related to tenors of one, three, six and twelve months up to ten year maturities. These curves were bootstrapped from Bloomberg quotes of deposit rates (for short term), forward rate agreements (for mid term) and swap rates (for mid and long term). The multiple-curve bootstrapping procedure follows the guidelines of [Ametrano and Bianchetti \(2013\)](#) where we used the price data for exactly the same market instruments as mentioned before for every date considered.

The first graph on the left side of Figure 2 shows that the curves were still more or less identical in 2006. The divergence is observable in the next picture on the left side which presents the curves for September 2008. In the following graphs for 2011, 2013, 2015 and 2016 the divergence increases further. The last picture on the right side shows in particular how the interest rates entered deeply into negative territory with the consequence that the bond prices increase at the beginning.

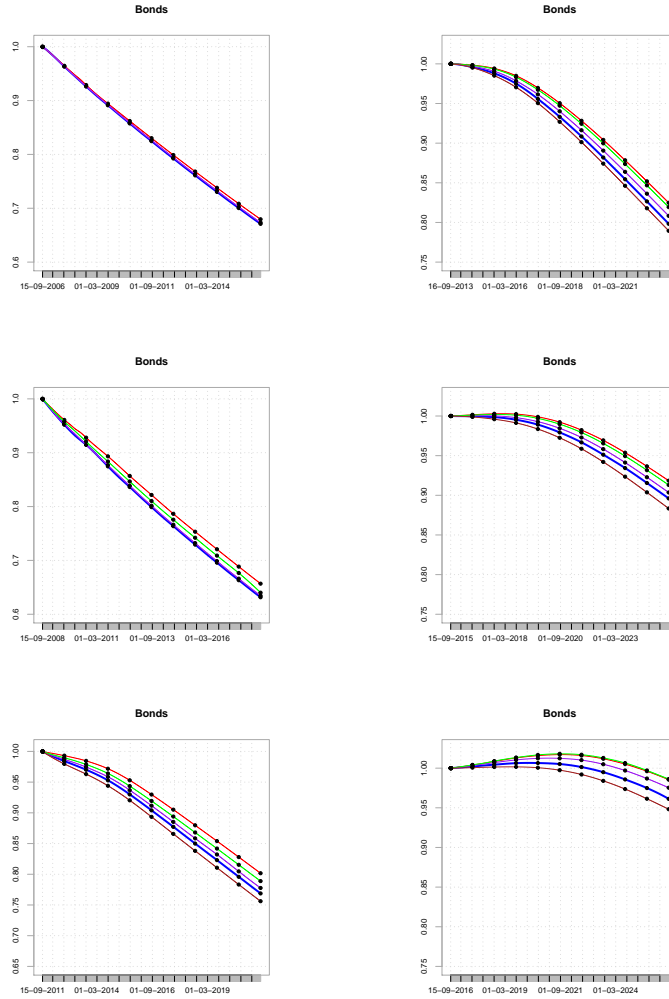


FIGURE 2. Evolution of term structure curves up to ten year maturities. Red represents the basic discount curve, green the one month, purple the three months, blue the six months and brown the twelve months curve.

There are essentially three types of interest rate model approaches, namely short rate models, forward rate or HJM-type models and market models. Hereafter we will discuss the multiple curve approach in a framework where (instantaneous) forward rates and forward spreads are taken as basic modeling quantities. The possibility that interest rates become negative in a classical (and consequently also in a Lévy driven) single curve HJM-model has often been considered to be an undesirable property. However in recent years rates for several of the major currencies, as for example for Euro and Yen, became negative. By mid-June 2016 the whole AAA Euro yield curve up to ten years was below zero. Therefore, the potential that rates become negative or even start from negative values, is an important characteristic which a realistic model has to take into account.

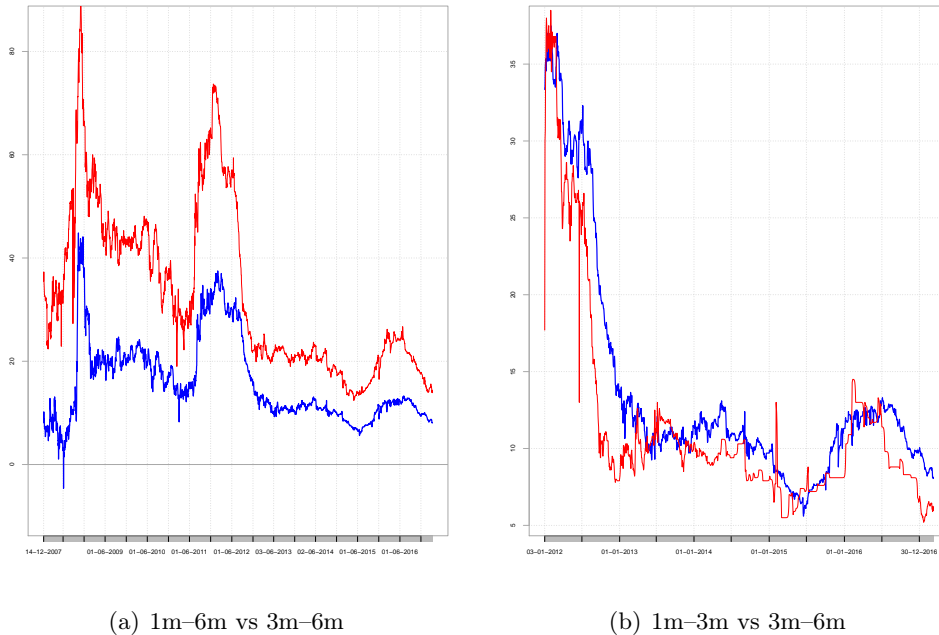


FIGURE 3. Evolution of Euribor vs Euribor basis swap spreads. The blue line represents the 3m-6m basis swap spreads. The red line shows the other mentioned spreads.

Given its importance for the financial industry, there is in the meanwhile an impressive literature on multiple curve approaches and related topics. We mention here only a few of them. [Ametrano and Bianchetti \(2009, 2013\)](#) explain which market instruments can be selected to obtain the current rates and how the bootstrapping works. A short rate approach is developed in [Kijima, Tanaka, and Wong \(2009\)](#). In an article which appeared just before the start of the crisis [Henrard \(2007\)](#) pointed to the fact that a single discount curve for all market participants is not appropriate. [Henrard \(2010\)](#) proposes a multiple curve approach. In a Gaussian framework, deterministic spreads between the curves are assumed. A diffusion driven double-curve model with regard to a foreign exchange analogy is introduced in [Bianchetti \(2010\)](#). The classical Libor market model is extended to a diffusion driven double-curve model by [Mercurio \(2010\)](#). [Moreni and Pallavicini \(2014\)](#) develop a parsimonious approach in a HJM-framework where spreads are assumed to be deterministic functions. In a very recent paper, [Cuchiero, Fontana, and Gnoatto \(2016\)](#) study a general multiple-curve approach based on multiplicative spreads with semimartingales as driving processes.

Our approach is related in spirit to the HJM-type approach in [Crépey, Grbac, and Nguyen \(2012\)](#), where a single risky rate is considered in addition to the basic rate. Another conceptual difference is that we apply multiple curves in the context of a two price market. Conic finance (see [Cherny and Madan \(2010\)](#)), as the latter is also called, is able to exploit explicitly bid and ask price data. This is highly appropriate in this context, since multiple interest rate curves emerged

from taking tenor-dependent credit and liquidity risks into account and the bid-ask spread is an important indicator for the liquidity of the corresponding market.

Our point of view in this paper is from a macro perspective. In a different strand of research a number of authors develop valuation approaches from a single deal perspective and thus take specific value adjustments into account. These adjustments are related to credit (CVA), collateral (LVA) and funding costs (FVA). For readers who are interested in this micro perspective we refer to [Brigo, Francischello, and Pallavicini \(2015\)](#) as well as [Bielecki, Cialenco, and Rutkowski \(2017\)](#) and the references therein. It is interesting to mention that these non-linear modelling approaches lead to deal-dependent pricing measures. Moreover, in the valuation formulas there is no longer a risk-free interest rate for discounting used. Instead a deal-dependent discount curve based on the cost of funding appears. Since the funding rate is usually asymmetric for the two parties involved, already this aspect leads to differential prices. Let us also mention in this context that in particular after the 2007–2009 financial crisis none of the interest rates which can be observed in the market could be classified to be risk-free in the proper sense of the word.

In section 2 we present first the class of processes which we use as drivers, namely time-inhomogeneous Lévy process. They are also called processes with independent increments and absolutely continuous characteristics (PIIAC) by [Jacod and Shiryaev \(2003\)](#). This class of processes has proved – also from the statistical point of view – to be particularly appropriate for fixed income models (see e.g. [Eberlein and Kluge \(2006\)](#)). In addition it has been shown in [Beinhofer, Eberlein, Janssen, and Polley \(2011\)](#) that the level of correlation between bond prices or interest rates for various maturities which is observed in the data can hardly be achieved in a Gaussian framework. The Lévy framework instead is flexible enough to fit the observed correlations. In section 3 the multiple-curve forward rate model is introduced in a single currency setting. We are aware that the market quotes also cross-currency basis swaps which requires to consider several economies at the same time. This more general setting is left for future research. We start with the discount curve which is the basic curve. OIS-zero coupon bond prices are typically used for the initial discount term structure. We proceed by introducing individual curves for various tenors. They are constructed by considering non-traded, fictitious bonds, which can be interpreted as risky bonds issued by a typical Euribor panel bank. The drift function in the dynamics of the underlying instantaneous forward rates is chosen such that credit and liquidity risk are explicitly represented. [Figure 1](#) as well as [figure 2](#) suggest that the spreads are usually monotone in the tenor length. A sufficient condition to achieve this goal is discussed in section 4, where we also propose a tractable model framework which guarantees the monotonicity of the curves. Variations of the processes which are used in this particular framework are possible depending on which properties one wants to secure.

In order to be able to calibrate this model, explicit valuation formulas for derivatives are needed. We consider caps, floors, swaptions as well as digital options. Fourier-based methods can be used to obtain numerically efficient versions of the pricing formulas. Since the price data consists of bid and ask prices with non-negligible spreads, the analysis is based on valuation formulas in a

two-price economy. Using distortions we derive numerically efficient formulas for bid and ask prices. For this purpose the cumulative distribution function of the stochastic component in the modeling of the underlying quantity is needed. In the last section calibration results based on specific driving processes are presented. We consider data from September 2013 as well as 2016. For the latter date we have to deal with negative rates.

2. THE DRIVING PROCESS

Let $T^* > 0$ be a finite time horizon and $\mathcal{B} := (\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]}, P)$ a stochastic basis that satisfies the usual conditions in the sense of [Jacod and Shiryaev \(2003, Definition I.1.2 and Definition I.1.3\)](#). We will consider as driving process a d -dimensional time-inhomogeneous Lévy process $L = (L^1, \dots, L^d)^\top$ on \mathcal{B} with $L^i = (L_t^i)_{t \in [0, T^*]}$ for every $i \in \{1, \dots, d\}$. This means that L is a \mathbb{F} -adapted process with independent increments and absolutely continuous characteristics (abbreviated by PIIAC) defined on \mathcal{B} (see [Eberlein, Jacod, and Raible \(2005\)](#) and [Jacod and Shiryaev \(2003\)](#)). We emphasise that L is a d -dimensional semimartingale (see [Jacod and Shiryaev \(2003, §5\)](#)).

We can assume that the paths of each component of L are càdlàg. This means that these paths are right-continuous and admit left-hand limits (almost surely). We also postulate that each component L^i starts at zero. The semimartingale characteristics of L are given by the triplet (B, C, ν) with

$$B = \int_0^\cdot b_s ds \quad C = \int_0^\cdot c_s ds \quad \nu(ds, dx) = F_s(dx)ds,$$

where $b_s = (b_s^1, \dots, b_s^d)^\top : [0, T^*] \rightarrow \mathbb{R}^d$, $c_s = (c_s^{ij})_{i, j \leq d} : [0, T^*] \rightarrow \mathbb{R}^{d \times d}$ whose values are in the set of symmetric nonnegative-definite $d \times d$ -matrices and F_s is a Lévy measure for every $s \in [0, T^*]$, i.e. a nonnegative measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that integrates $(|x|^2 \wedge 1)$ and satisfies $F_s(\{0\}) = 0$. The Euclidean scalar product on \mathbb{R}^d is denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ is the corresponding norm. The scalar product on \mathbb{R}^d is extended to complex numbers by setting $\langle w, z \rangle := \sum_{j=1}^d w_j z_j$ for every $w, z \in \mathbb{C}^d$. Thus, $\langle \cdot, \cdot \rangle$ is not the Hermitian scalar product here. We further assume that

$$\int_0^{T^*} \left[|b_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(dx) \right] ds < \infty,$$

where $\|\cdot\|$ denotes any norm on the set of $d \times d$ -matrices.

Since we shall consider exponentials of stochastic integrals with respect to the process L , we will need a priori an appropriate exponential moment condition

Assumption 2.1. (EM) There are constants $M, \epsilon > 0$, such that, for every

$$u \in [-(1 + \epsilon)M, (1 + \epsilon)M]^d,$$

we have

$$\int_0^{T^*} \int_{\{|x| > 1\}} \exp\langle u, x \rangle F_s(dx) ds < \infty.$$

Assumption (EM) is equivalent to $\mathbb{E}[\exp\langle u, L_t \rangle] < \infty$ for all $t \in [0, T^*]$ and $u \in [-(1 + \epsilon)M, (1 + \epsilon)M]^d$. A direct consequence of (EM) is that in particular the expectation of L_t is finite. In this situation one does not need a truncation function in the Lévy–Khintchine representation of its characteristic function. Consequently, the law of L_t is determined by the characteristic function in the form

$$\mathbb{E}[e^{i\langle u, L_t \rangle}] = \exp \left(\int_0^t \left[i\langle u, b_s \rangle - \frac{1}{2}\langle u, c_s u \rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \right) F_s(dx) \right] ds \right) \quad (u \in \mathbb{R}^d).$$

Likewise the canonical representation of the process L can be written in the simple form

$$L_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(ds, dx),$$

i.e. without considering an additional term for the big jumps. Here $W = (W_t)_{t \in [0, T^*]}$ is a standard d -dimensional Brownian motion (Wiener process), $\sqrt{c_s}$ is a measurable version of the square root of c_s , and μ^L is the random measure of jumps of L with compensator $\nu(ds, dx) = F_s(dx)ds$ (cf. [Jacod and Shiryaev \(2003, Corollary II.2.38\)](#)). The (extended) cumulant process associated with the process L is denoted by θ_s and given by

$$\theta_s(z) = \langle z, b_s \rangle + \frac{1}{2}\langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left(e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx)$$

for every $z \in \mathbb{C}^d$ where this function is defined. [Kallsen and Shiryaev \(2002\)](#) provide a detailed analysis of the cumulant process for semimartingales. Note that if L is a (homogeneous) Lévy process, i.e. if the increments of L are stationary, the triplet (b_s, c_s, F_s) and thus θ_s do not depend on s . In this case, we write θ for short. It then equals the cumulant (also called log moment generating function) of L_1 . Observe that the cumulant process is related to a specific measure. Which measure is meant in the following can unambiguously be seen from the notation.

3. THE MULTIPLE-CURVE LÉVY FORWARD RATE MODEL

Let us consider a complete stochastic basis $\hat{\mathcal{B}} := (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \in [0, T^*]}, \hat{P}^d)$ and a d -dimensional time-inhomogeneous Lévy process $L = (L^1, \dots, L^d)^\top$ defined on $\hat{\mathcal{B}}$. In the multiple-curve setting we will consider a discount curve which we will denote with the index \mathbf{d} (later sometimes written as 0) and m different term structures of interest rates for $m \in \mathbb{N} = \{1, 2, 3, \dots\}$. Note that this framework contains the single-curve approach by setting $m = 0$.

3.1. The Basic Curve. Let us begin with the specification of the discount curve. To this end, we consider an arbitrary discrete tenor structure $\mathcal{T}^{\mathbf{d}} := \{T_0, \dots, T_n\}$, where $T_0 < \dots < T_n$ and $T_n = T^*$. In order to avoid too heavy notation $\delta := \delta(T_{k-1}, T_k)$, the year fraction between the dates T_{k-1} and T_k , is

assumed to be independent of k and is called the tenor of \mathcal{T}^d . The time- t price of a (zero-coupon) discount bond maturing at date $T \in [0, T^*]$ is denoted by $B_t^d(T)$.

We denote by \mathcal{P} the predictable σ -field, which is the σ -field on $\hat{\Omega} \times [0, T^*]$ generated by all adapted processes with left-continuous paths (cf. [Jacod and Shiryaev \(2003, Definition I.2.1\)](#)). The optional σ -field \mathcal{O} is defined as the σ -field on $\hat{\Omega} \times [0, T^*]$ which is generated by all adapted processes with càdlàg paths (cf. [Jacod and Shiryaev \(2003, Definition I.1.20\)](#)).

The following two ingredients are needed to develop the model for the discount curve.

(D.1) The initial discount curve B_0^d defined by

$$B_0^d : \begin{cases} [0, T^*] \rightarrow (0, \infty) \\ T \mapsto B_0^d(T) \end{cases}$$

is given.

The initial discount curve can be derived from market data by using an appropriate bootstrapping technique. A general explanation of such a technique is given in [Hull \(2012\)](#) and [Ametrano and Bianchetti \(2013\)](#). One typically takes the OIS-zero-coupon bond price as an approximation of B_0^d .

(D.2) We consider a drift function α^d and a volatility structure σ^d defined by

$$\alpha^d : \begin{cases} \hat{\Omega} \times [0, T^*] \times [0, T^*] \rightarrow \mathbb{R} \\ (\hat{\omega}, s, T) \mapsto \alpha^d(\hat{\omega}, s, T) \end{cases}$$

and

$$\sigma^d : \begin{cases} \hat{\Omega} \times [0, T^*] \times [0, T^*] \rightarrow \mathbb{R}^d \\ (\hat{\omega}, s, T) \mapsto \sigma^d(\hat{\omega}, s, T) = (\sigma_1^d(\hat{\omega}, s, T), \dots, \sigma_d^d(\hat{\omega}, s, T)) \end{cases}$$

which satisfy the usual measurability and boundedness conditions (cf. [Eberlein, Jacod, and Raible \(2005\)](#))

- (a) α^d and $\sigma^d = (\sigma_1^d, \dots, \sigma_d^d)$ are measurable with respect to $\mathcal{P} \otimes \mathcal{B}([0, T^*])$.
- (b) The random functions are bounded for each $\hat{\omega} \in \hat{\Omega}$ in the sense of $\sup_{0 \leq s, T \leq T^*} (|\alpha^d(\hat{\omega}, s, T)| + |\sigma^d(\hat{\omega}, s, T)|) < \infty$.
- (c) For every $(\hat{\omega}, s, T) \in \hat{\Omega} \times [0, T^*] \times [0, T^*]$ with $T < s$, we have $\alpha^d(\hat{\omega}, s, T) = 0$ and $\sigma^d(\hat{\omega}, s, T) = (0, \dots, 0)$.

Now let us postulate that, for every fixed maturity $T \in [0, T^*]$, the dynamics of the discount instantaneous forward rate $f^d(T) = (f_t^d(T))_{t \in [0, T]}$ is given by

$$f_t^d(T) = f_0^d(T) + \int_0^t \alpha^d(s, T) ds - \int_0^t \sigma^d(s, T) dL_s. \quad (3.1)$$

The initial values $f_0^d(T)$ are assumed to be deterministic and bounded. We also require that they form a measurable mapping

$$[0, T^*] \ni T \mapsto f_0^d(T) \in \mathbb{R}.$$

It is shown by [Eberlein, Jacod, and Raible \(2005\)](#) that under these assumptions we can find a *joint-version* of all $f_t^d(T)$ such that the map $(\hat{\omega}, t, T) \mapsto f_t^d(\hat{\omega}, T)\mathbb{1}_{\{t \leq T\}}$ is $\mathcal{O} \otimes \mathcal{B}([0, T^*])$ -measurable.

By applying Fubini's Theorem (cf. [Protter \(2004, Theorem 64\)](#)) it follows from the forward rate dynamics (3.1) that the bond price $B_t^d(T)$ at time t given by

$$B_t^d(T) = \exp \left(- \int_t^T f_t^d(u) du \right)$$

can be expressed as

$$B_t^d(T) = B_0^d(T) \exp \left(\int_0^t (r_s^d - A^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s \right), \quad (3.2)$$

where the short rate r_t^d is specified by $r_t^d = f_t^d(t)$ and where we have set

$$A^d(s, T) := \int_{s \wedge T}^T \alpha^d(s, u) du \quad \text{and} \quad \Sigma^d(s, T) := \int_{s \wedge T}^T \sigma^d(s, u) du \quad (3.3)$$

(cf. [Eberlein and Kluge \(2006\)](#)). Note that the integral $\int_{s \wedge T}^T \sigma^d(s, u) du$ is understood componentwise and the initial values $f_0^d(T)$ can be obtained from the relation

$$f_0^d(T) = - \frac{\partial \ln B_0^d(T)}{\partial T}$$

if this derivative exists.

The discount factor process $\beta^d = (\beta_t^d)_{t \in [0, T^*]}$ given by

$$\beta_t^d = \exp \left(- \int_0^t r_s^d ds \right)$$

is an adapted process with continuous paths. It can obviously be written as

$$\beta_t^d = B_0^d(t) \exp \left(- \int_0^t A^d(s, t) ds + \int_0^t \Sigma^d(s, t) dL_s \right). \quad (3.4)$$

From (3.2) together with (3.4) we can easily get another representation which turns out to be useful

$$B_t^d(T) = \frac{B_0^d(T)}{B_0^d(t)} \exp \left(\int_0^t (A^d(s, t) - A^d(s, T)) ds + \int_0^t \Sigma^d(s, t, T) dL_s \right). \quad (3.5)$$

Here we have set $\Sigma^d(s, t, T) := \Sigma^d(s, T) - \Sigma^d(s, t)$. To get a tractable model and guarantee the existence of all related functions, we require in the following the model to be based on a deterministic, bounded and continuous volatility structure.

Assumption 3.1. (**DET**) The volatility structure σ^d is a deterministic and bounded function such that, for every s and T with $0 \leq s, T \leq T^*$

$$0 \leq \Sigma_j^d(s, T) \leq \hat{M} < M, \quad \text{for every } j \in \{1, \dots, d\},$$

where Σ_j^d arises from definition (3.3) (j^{th} component of Σ^d) and the constant M is from assumption (**EM**). Moreover, the mapping

$$[0, T^*] \ni s \mapsto \sigma^d(s, T) \in \mathbb{R}^d$$

is continuous for each given $T \in [0, T^*]$.

The market will be free of arbitrage if the drift function A^d is defined in such a way that the discounted bond price processes $Z^d(T) = (Z_t^d(T))_{0 \leq t \leq T}$ given by $Z_t^d(T) := \beta_t^d B_t^d(T)$ are martingales. We will repeatedly use the fact (see Eberlein and Raible (1999, Lemma 3.1)) that expectations of exponentials of stochastic integrals can be explicitly computed, namely

$$\mathbb{E}_{\hat{P}^d} \left[\exp \left(\int_0^t \Sigma^d(s, T) dL_s \right) \right] = \exp \left(\int_0^t \theta_s(\Sigma^d(s, T)) ds \right). \quad (3.6)$$

Note that the integral process in the exponent on the left side has independent increments. Using this fact, it is easy to show that the process $M = (M_t)_{t \in [0, T]}$ with

$$M_t = \frac{\exp \left(\int_0^t \Sigma^d(s, T) dL_s \right)}{E_{\hat{P}^d} \left[\exp \left(\int_0^t \Sigma^d(s, T) dL_s \right) \right]}$$

is a martingale. Combining this observation with (3.6), one sees that the drift assumption

$$A^d(s, T) = \theta_s(\Sigma^d(s, T))$$

guarantees that the discounted bond price processes $Z^d(T)$ are martingales for all $T \in [0, T^*]$. Notice that this approach for the discount curve (following the Lévy forward rate model of Eberlein and Kluge (2006)) works directly under the risk-neutral measure. This choice of the drift term is closely related to the notion of the exponential compensator (see Kallsen and Shiryaev (2002) and Jacod and Shiryaev (2003, Section II.8)). We repeat that the drift function A^d and therefore also α^d as well as the volatility structure Σ^d (and σ^d) are deterministic in this setting.

It follows that the bond price process (3.2) is given in the more specific form

$$B_t^d(T) = \frac{B_0^d(T)}{\beta_t^d} \exp \left(- \int_0^t \theta_s(\Sigma^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s \right)$$

from which we conclude that

$$\frac{Z_t^d(T)}{Z_0^d(T)} = \exp \left(- \int_0^t \theta_s(\Sigma^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s \right). \quad (3.7)$$

To end this subsection, we introduce some useful definitions. We observe that the representation (3.5) results in

$$B_t^d(T) = \frac{B_0^d(T)}{B_0^d(t)} \exp \left(\int_0^t \left[\theta_s(\Sigma^d(s, t)) - \theta_s(\Sigma^d(s, T)) \right] ds + \int_0^t \Sigma^d(s, t, T) dL_s \right). \quad (3.8)$$

For fixed $t, T \in [0, T^*]$ with $t \leq T$, the last expression is decomposed into its deterministic part

$$D^d(t, T) := \frac{B_0^d(T)}{B_0^d(t)} \exp \left(\int_0^t \left[\theta_s(\Sigma^d(s, t)) - \theta_s(\Sigma^d(s, T)) \right] ds \right)$$

and its stochastic part given as the exponential of the $\hat{\mathcal{F}}_t$ -measurable random variable

$$X^d(t, T) := \int_0^t \Sigma^d(s, t, T) dL_s.$$

Hence, we obtain the compact form

$$B_t^d(T) = D^d(t, T) \exp(X^d(t, T)). \quad (3.9)$$

3.2. Risky Tenor–Dependent Curves. Now we address the curves which take credit and liquidity risk explicitly into account. By abuse of language we call them risky curves. m different curves will be considered. Since each term structure corresponds to a discrete tenor structure, we introduce the equidistant tenor structure $\mathcal{T}^k := \{T_0^k, \dots, T_{n_k}^k\}$ for every $k \in \{1, \dots, m\}$ and $n_k \in \mathbb{N}$. We assume that $T_0^k = T_0$ and $T_{n_k}^k = T_n = T^*$ for all $k \in \{1, \dots, m\}$. The year fraction between the dates T_{j-1}^k and T_j^k is denoted by $\delta^k := \delta^k(T_{j-1}^k, T_j^k)$ for $j \in \{1, \dots, n_k\}$. δ^k is called the the tenor of \mathcal{T}^k . Moreover, for all $k, l \in \{1, \dots, m\}$ with $k \leq l$, we postulate

$$\mathcal{T}^l \subset \mathcal{T}^k \subset \mathcal{T}^d \subset [0, T^*]. \quad (3.10)$$

These nested structures will allow to take liquidity and credit risk in decreasing order of magnitude ($\delta^k < \delta^l$) into account. For every $k \in \{1, \dots, m\}$ we interpret $B_t^k(T)$ as time- t price of a fictitious risky zero-coupon bond with maturity T that corresponds to curve k .

We need two ingredients to model the multiple curves.

(MC.1) The initial multiple term structure curves B_0^1, \dots, B_0^m

$$B_0^k : \begin{cases} [0, T^*] \rightarrow (0, \infty) \\ T \mapsto B_0^k(T) \end{cases}$$

are given for every $k \in \{1, \dots, m\}$.

[Ametrano and Bianchetti \(2009, 2013\)](#) developed a bootstrapping method in the multiple-curve setting. The initial values typically satisfy

$$B_0^l(T) \leq B_0^k(T) \leq B_0^d(T)$$

for every $k, l \in \{1, \dots, m\}$ with $k \leq l$ and $T \in [0, T^*]$ (cf. Figure 2).

(MC.2) For every $k \in \{1, \dots, m\}$, we consider the drift function α^k and the volatility structure σ^k defined by

$$\alpha^k : \begin{cases} \hat{\Omega} \times [0, T^*] \times [0, T^*] \rightarrow \mathbb{R} \\ (\hat{\omega}, s, T) \mapsto \alpha^k(\hat{\omega}, s, T) \end{cases}$$

and

$$\sigma^k : \begin{cases} \hat{\Omega} \times [0, T^*] \times [0, T^*] \rightarrow \mathbb{R}^d \\ (\hat{\omega}, s, T) \mapsto \sigma^k(\hat{\omega}, s, T) = (\sigma_1^k(\hat{\omega}, s, T), \dots, \sigma_d^k(\hat{\omega}, s, T)) \end{cases}$$

which satisfy the same (measurability and boundedness) conditions as α^d and σ^d in (D.2).

For every $k \in \{1, \dots, m\}$ and $T \in [0, T^*]$, the dynamics of the instantaneous forward rates $f^k(T) = (f_t^k(T))_{t \in [0, T]}$ are postulated to be

$$f_t^k(T) = f_0^k(T) + \int_0^t \alpha^k(s, T) ds - \int_0^t \sigma^k(s, T) dL_s,$$

where the initial values $f_0^k(T)$ are assumed to be deterministic, bounded and measurable in T . Those values can be determined by the formula

$$f_0^k(T) = -\frac{\partial \ln B_0^k(T)}{\partial T} \quad (3.11)$$

if the derivative exists. In the same way as one gets representation (3.2), we obtain the form

$$B_t^k(T) = B_0^k(T) \exp \left(\int_0^t (r_s^k - A^k(s, T)) ds + \int_0^t \Sigma^k(s, T) dL_s \right) \quad (3.12)$$

from the relation

$$B_t^k(T) = \exp \left(- \int_t^T f_t^k(u) du \right)$$

for each $k \in \{1, \dots, m\}$. The rate r_t^k at t is given by $r_t^k = f_t^k(t)$ and we similarly define

$$A^k(s, T) := \int_{s \wedge T}^T \alpha^k(s, u) du \quad \text{and} \quad \Sigma^k(s, T) := \int_{s \wedge T}^T \sigma^k(s, u) du.$$

To ensure the existence of the cumulant process, we need the following

Assumption 3.2. (MC.DET) For any $k \in \{1, \dots, m\}$ and all $s, T \in [0, T^*]$ the volatility structure σ^k is deterministic and bounded in the sense of

$$0 \leq \Sigma_j^k(s, T) \leq \hat{M} < M, \quad \text{for every } j \in \{1, \dots, d\}.$$

The mapping $[0, T^*] \ni s \mapsto \sigma^k(s, T) \in \mathbb{R}^d$ is continuous. As usual, M is the constant from assumption (EM).

The constant \hat{M} in this assumption does not have to coincide with the constant \hat{M} from assumption (DET). The discounted bond price process $Z^k(T) = (Z_t^k(T))_{0 \leq t \leq T}$ corresponding to curve k is defined by $Z_t^k(T) := \beta_t^d B_t^k(T)$ for each date $T \in [0, T^*]$. One easily verifies that

$$\frac{Z_t^k(T)}{Z_0^k(T)} = \exp \left(\int_0^t [r_s^k - r_s^d - A^k(s, T)] ds + \int_0^t \Sigma^k(s, T) dL_s \right). \quad (3.13)$$

We can represent this as a stochastic exponential (see [Jacod and Shiryaev \(2003, Theorem II.8.10\)](#))

$$\frac{Z_t^k(T)}{Z_0^k(T)} = \mathcal{E}_t(\bar{Y}^k),$$

where the process \bar{Y}^k is the stochastic logarithm $\bar{Y}^k = \mathcal{L}(\exp(Y^k))$ with

$$Y_t^k = \int_0^t (r_s^k - r_s^d - A^k(s, T)) ds + \int_0^t \Sigma^k(s, T) dL_s.$$

This means that the discounted bond price process $Z^k(T)$ is the solution of

$$dZ^k(T) = Z_-^k(T) d\bar{Y}^k,$$

where the process $\bar{Y}^k = (\bar{Y}_t^k)_{0 \leq t \leq T}$ is explicitly given by

$$\begin{aligned} \bar{Y}_t^k &= \int_0^t [r_s^k - r_s^d - A^k(s, T) + \theta_s(\Sigma^k(s, T))] ds + \int_0^t \Sigma^k(s, T) \sqrt{c_s} dW_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \Sigma^k(s, T), x \rangle} - 1 \right) (\mu^L - \nu)(ds, dx). \end{aligned}$$

The last expression shows that $Z^k(T)$ is not a \hat{P}^d -(local) martingale in general. Similarly as in (3.5), we rewrite the expression (3.12) in the form

$$B_t^k(T) = \frac{B_0^k(T)}{B_0^k(t)} \exp \left(\int_0^t [A^k(s, t) - A^k(s, T)] ds + \int_0^t \Sigma^k(s, t, T) dL_s \right), \quad (3.14)$$

where we have set

$$\Sigma^k(s, t, T) := \Sigma^k(s, T) - \Sigma^k(s, t).$$

To simplify the notation, for fixed $t, T \in [0, T^*]$ with $t \leq T$, we define the factor

$$D^k(t, T) := \frac{B_0^k(T)}{B_0^k(t)} \exp \left(\int_0^t [A^k(s, t) - A^k(s, T)] ds \right)$$

and the $\hat{\mathcal{F}}_t$ -measurable random variable

$$X^k(t, T) := \int_0^t \Sigma^k(s, t, T) dL_s.$$

Then, we obtain

$$B_t^k(T) = D^k(t, T) \exp(X^k(t, T)),$$

where at this stage $D^k(t, T)$ could be random.

Now, we will specify the drift function A^k in such a manner that credit and liquidity risk are taken into account. We follow the line of thought of [Crépey, Grbac, and Nguyen \(2012, Section 2.3.2\)](#). Their idea is based on no-arbitrage requirements in defaultable HJM-models with an additional liquidity component which lead to the required drift condition.

Let us temporarily assume that defaultable bonds with respect to each curve can be traded in the market. The time- t price of such a bond maturing at T is denoted by $\bar{B}_t^k(T)$. Keep in mind that such bonds are actually not traded in the market but can be considered as bonds that are issued by an average Libor or Euribor panel member. In fact, they are rather mathematical concepts which represent the credit risk of the panel bank members but are not defaultable in the classical sense.

Hereinafter, we specify the intensity-based credit risk model and construct default times τ^1, \dots, τ^m . To this end, we need to enlarge the initial stochastic basis $\hat{\mathcal{B}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \in [0, T^*]}, \hat{P}^d)$ as follows. Let $\Gamma^1, \dots, \Gamma^m$ be real-valued, $\hat{\mathbb{F}}$ -adapted, continuous and increasing stochastic processes defined on $\hat{\mathcal{B}}$. It is assumed that $\Gamma_0^k = 0$ and $\Gamma_\infty^k := \lim_{t \uparrow \infty} \Gamma_t^k = \infty$ for every $k \in \{1, \dots, m\}$. We additionally consider an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ endowed with a family of independent random variables ξ_1, \dots, ξ_m that are uniformly distributed on the interval $[0, 1]$. We state the product space

$$(\Omega, \mathcal{G}, P^d) := (\hat{\Omega} \times \tilde{\Omega}, \hat{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \hat{P}^d \otimes \tilde{P})$$

and denote by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]}$ the trivial extension of $\hat{\mathbb{F}}$ to the enlarged probability space $(\Omega, \mathcal{G}, P^d)$. This means that each $A \in \mathcal{F}_t$ is of the form $\hat{A} \times \tilde{\Omega}$ for some $\hat{A} \in \hat{\mathcal{F}}_t$. Observe that \mathbb{F} is right-continuous and denotes the reference filtration here. All the random variables (functions) and stochastic processes defined on $\hat{\mathcal{B}}$ or $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ are extended to the enlarged filtered probability space $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P^d)$ in the usual canonical way. We retain their names when we consider them on this complete stochastic basis to avoid unnecessary and confusing notation (cf. [Eberlein and Özkan \(2003\)](#) and [Kluge \(2005, section 4.2\)](#)). Observe that each $(\hat{\mathbb{F}}, \hat{P}^d)$ -(local) martingale is also a (\mathbb{F}, P^d) -(local) martingale.

For every $k \in \{1, \dots, m\}$, let us define a random time $\tau^k : \Omega \rightarrow \mathbb{R}_+$ on $(\Omega, \mathcal{G}, P^d)$ by setting

$$\tau^k := \inf\{t \in \mathbb{R}_+ \mid e^{-\Gamma_t^k} \leq \xi_k\} = \inf\{t \in \mathbb{R}_+ \mid \Gamma_t^k \geq \eta_k\},$$

where the random variable $\eta_k := -\ln \xi_k$ is exponentially distributed with mean one under P^d . Obviously, η_1, \dots, η_m is a family of independent random variables. For every $k \in \{1, \dots, m\}$, we denote by $\mathbb{H}^k = (\mathcal{H}_t^k)_{t \in \mathbb{R}_+}$ the right-continuous filtration generated by the default process $H^k = (H_t^k)_{t \in \mathbb{R}_+}$ that is defined by $H_t^k = \mathbb{1}_{\{\tau^k \leq t\}}$. More precisely we have $\mathcal{H}_t^k = \sigma(H_u^k : 0 \leq u \leq t) = \sigma(\{\tau^k \leq u\} : 0 \leq u \leq t)$. We postulate that each random time τ^k possesses a \mathbb{F} -intensity γ^k , i.e. $\Gamma_t^k = \int_0^t \gamma_s^k ds$ for any $t \geq 0$ and some non-negative,

\mathbb{F} -progressively measurable stochastic process γ^k with integrable sample paths (see [Bielecki and Rutkowski \(2002, chapter 5 and 8\)](#)).

Further, we assume that the defaultable bonds pay a certain recovery upon default. This recovery payment is specified by the terminal recovery process $R^k = (R_t^k)_{t \in [0, T^*]}$ for every curve $k \in \{1, \dots, m\}$. The process R^k is \mathbb{F} -adapted and (locally) bounded on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P}^d)$ (cf. [Bielecki and Rutkowski \(2002, Section 13.1.9\)](#)). In financial interpretation, the amount $R_{\tau^k}^k$ is the recovery payment made at maturity T if the default of the bond issuer occurs at time $\tau^k \leq T$. More specifically, the value of the defaultable bond price $\bar{B}_T^k(T)$ at time T is given by

$$\begin{aligned} \bar{B}_T^k(T) &= \mathbb{1}_{\{\tau^k > T\}} B_T^k(T) + B_T^d(T) R_{\tau^k}^k \mathbb{1}_{\{\tau^k \leq T\}} \\ &= \mathbb{1}_{\{\tau^k > T\}} + R_{\tau^k}^k \mathbb{1}_{\{\tau^k \leq T\}}. \end{aligned}$$

Then, the time- t price results in

$$\bar{B}_t^k(T) = \mathbb{1}_{\{\tau^k > t\}} B_t^k(T) + B_t^d(T) R_{\tau^k}^k \mathbb{1}_{\{\tau^k \leq t\}}$$

and we obtain its discounted value $\bar{Z}_t^k(T) := \beta_t^d \bar{B}_t^k(T)$ as

$$\bar{Z}_t^k(T) = \mathbb{1}_{\{\tau^k > t\}} Z_t^k(T) + R_{\tau^k}^k \mathbb{1}_{\{\tau^k \leq t\}} Z_t^d(T).$$

Consequently, the time- t bond price $B_t^k(T)$ is interpreted as the pre-default price of the associated defaultable zero-coupon bond.

As mentioned above, we require that the process $\bar{Z}^k(T) = (\bar{Z}_t^k(T))_{t \in [0, T]}$ to be a local martingale. Note that, in general, the default times τ^1, \dots, τ^m are not stopping times with respect to the reference filtration \mathbb{F} . Therefore, we need to enlarge \mathbb{F} . Unfortunately, the filtration $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{t \in [0, T^*]}$ induced by $\tilde{\mathcal{G}}_t := \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^m := \sigma(\mathcal{F}_t, \mathcal{H}_t^1, \dots, \mathcal{H}_t^m)$ does not have to be right-continuous (cf. [Song \(2013\)](#)). Therefore, we endow the probability space $(\Omega, \mathcal{G}, P^d)$ with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T^*]}$ given by

$$\mathcal{G}_t := \bigcap_{s > t} \tilde{\mathcal{G}}_s, \quad \text{for any } t \in [0, T^*].$$

This filtration trivially satisfies the right-continuity. Since the stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, P^d)$ is complete (see also [Jacod and Shiryaev \(2003, §1a. 1.4\)](#)) it easily follows that the enlarged stochastic basis $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \in [0, T^*]}, P^d)$ is also complete. We conclude that \mathbb{G} is specified as the smallest enlargement of \mathbb{F} containing $\tilde{\mathbb{G}}$. It can be shown that the martingale invariance property (see [Brémaud and Yor \(1978\)](#), [Dellacherie and Meyer \(1978\)](#) and [Bielecki and Rutkowski \(2002\)](#)) is satisfied in this framework.

The following theorem states the conditions which ensure the absence of arbitrage. To be precise, we derive conditions such that for every $k \in \{1, \dots, m\}$, the discounted defaultable bond price processes $\bar{Z}^k(T)$ are (\mathbb{G}, P^d) -local martingales for each $T \in [0, T^*]$.

Theorem 3.3. *Assume that, for each $k \in \{1, \dots, m\}$ and $T \in [0, T^*]$, the condition*

$$Z_{t-}^k(T) [\lambda_t^{k,d} - A^k(t, T) + \theta_t(\Sigma^k(t, T))] = (Z_{t-}^k(T) - R_t^k Z_t^d(T)) \gamma_t^k \quad (3.15)$$

is satisfied for all $t \in [0, T]$, where we set $\lambda_t^{k,d} := r_t^k - r_t^d$. Then, for each $k \in \{1, \dots, m\}$ and $T \in [0, T^*]$, the process $\bar{Z}^k(T)$ is a (\mathbb{G}, P^d) -local martingale.

Proof. For every $k \in \{1, \dots, m\}$ and $T \in [0, T^*]$, we obviously have that $H^k = \mathbb{1}_{\{\tau^k \leq \cdot\}}$ as well as $1 - H^k = \mathbb{1}_{\{\tau^k > \cdot\}}$ are finite variation processes and $Z^k(T)$ is a semimartingale. Let us define by $\hat{H}_t^k := R_{\tau^k}^k H_t^k$ a \mathbb{G} -adapted finite variation process $\hat{H}^k = (\hat{H}_t^k)_{t \in [0, T^*]}$. Then, by [Jacod and Shiryaev \(2003, Proposition I.4.49\)](#), we obtain that

$$\begin{aligned} \bar{Z}_t^k(T) &= \int_0^t (1 - H_{s-}^k) dZ_s^k(T) + \int_0^t Z_s^k(T) d(1 - H_s^k) + \bar{Z}_0^k(T) \\ &\quad + \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t Z_s^d(T) d\hat{H}_s^k \\ &= \int_0^t (1 - H_{s-}^k) dZ_s^k(T) - \int_0^t Z_s^k(T) dH_s^k + \bar{Z}_0^k(T) \\ &\quad + \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t R_s^k Z_s^d(T) dH_s^k. \end{aligned}$$

One gets

$$\begin{aligned} \bar{Z}_t^k(T) &= \int_0^t (1 - H_{s-}^k) Z_{s-}^k(T) [\lambda_s^{k,d} - A^k(s, T) + \theta_s(\Sigma^k(s, T))] ds + \bar{Z}_0^k(T) \\ &\quad + \int_0^t (1 - H_{s-}^k) Z_{s-}^k(T) \Sigma^k(s, T) \sqrt{c_s} dW_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (1 - H_{s-}^k) Z_{s-}^k(T) \left(e^{\langle \Sigma^k(s, T), x \rangle} - 1 \right) (\mu^L - \nu)(ds, dx) \\ &\quad + \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t (R_s^k Z_s^d(T) - Z_s^k(T)) dH_s^k. \end{aligned}$$

Since condition [\(3.15\)](#) is assumed to be satisfied, we have

$$\begin{aligned} &\int_0^t (1 - H_s^k) Z_{s-}^k(T) [\lambda_s^{k,d} - A^k(s, T) + \theta_s(\Sigma^k(s, T))] ds \\ &= \int_0^t (1 - H_s^k) (Z_{s-}^k(T) - R_s^k Z_s^d(T)) \gamma_s^k ds \\ &= \int_0^t (1 - H_s^k) (Z_s^k(T) - R_s^k Z_s^d(T)) \gamma_s^k ds \end{aligned}$$

and it follows that

$$\begin{aligned}
\bar{Z}_t^k(T) &= \bar{Z}_0^k(T) + \int_0^t (1 - H_{s-}^k) Z_{s-}^k(T) \Sigma^k(s, T) \sqrt{c_s} dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}^d} (1 - H_{s-}^k) Z_{s-}^k(T) \left(e^{\langle \Sigma^k(s, T), x \rangle} - 1 \right) (\mu^L - \nu)(ds, dx) \\
&\quad + \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t (R_s^k Z_s^d(T) - Z_s^k(T)) dM_s^k, \tag{3.16}
\end{aligned}$$

where the (\mathbb{G}, P^d) -martingale $M^k = (M_t^k)_{t \in [0, T^*]}$ is defined by

$$M_t^k = H_t^k - \int_0^t (1 - H_s^k) \gamma_s^k ds.$$

By taking into account the valid martingale invariance property, we observe that all the considered stochastic integrals in equation (3.16) have (\mathbb{G}, P^d) -local martingales as integrators. Hence, we conclude that $\bar{Z}^k(T)$ is a (\mathbb{G}, P^d) -local martingale. \square

Let us assume that the terminal recovery process is of the form

$$R_t^k = R^k B_{t-}^k(T) B_t^d(T)^{-1},$$

where $R^k \in [0, 1)$. Note that this choice corresponds to the fractional recovery of market value (see [Bielecki and Rutkowski \(2002, section 1.1.1\)](#)). By easy computations, we obtain the following convenient form of condition (3.15). For each $k \in \{1, \dots, m\}$ and $T \in [0, T^*]$

$$\lambda_t^{k,d} - A^k(t, T) + \theta_t(\Sigma^k(t, T)) = (1 - R^k) \gamma_t^k \tag{3.17}$$

for every $t \in [0, T]$. One verifies that condition (3.17) can equivalently be formulated as

$$\begin{aligned}
\lambda_t^{k,d} &= (1 - R^k) \gamma_t^k \\
A^k(t, T) &= \theta_t(\Sigma^k(t, T)). \tag{3.18}
\end{aligned}$$

Hence, the credit risk component of the model is given by equation (3.18).

Since the crisis was caused by a mixture of credit and liquidity risk (cf. [Filipović and Trolle \(2013\)](#) and [Eberlein \(2015\)](#)), we add an additional liquidity component to the pure credit risk factor $\theta_t(\Sigma^k(t, T))$ in (3.18). For this reason, we need another ingredient in the model.

(MC.3) For every $k \in \{1, \dots, m\}$, we consider the liquidity component l^k defined by

$$l^k : \begin{cases} [0, T^*] \times [0, T^*] \rightarrow \mathbb{R} \\ (t, T) \mapsto l^k(t, T) \end{cases}$$

which is assumed to be a bounded function.

Finally, we consider the drift function

$$A^k(t, T) = \theta_t(\Sigma^k(t, T)) + l^k(t, T) \quad (3.19)$$

for every $k \in \{1, \dots, m\}$. We stress that the drift function A^k (and α^k) and the volatility function Σ^k (and σ^k) are deterministic functions. Therefore, the factor $D^k(t, T)$ is deterministic.

4. MONOTONICITY OF THE CURVES

We want to secure monotonicity of the curves according to the ordering given in (3.10). In other words bonds that are related to a riskier curve should have a lower price than bonds that correspond to a curve associated with less credit and liquidity risk. This approach is motivated by Figure 2.

4.1. The Monotonicity Condition. Recall that we usually have for every $k, l \in \{1, \dots, m\}$ with $k \leq l$ and all $T \in [0, T^*]$ monotonicity of the initial curves

$$B_0^l(T) \leq B_0^k(T) \leq B_0^d(T).$$

We want to design the model such that monotonicity holds at any time. This means that we want to achieve

$$B_t^l(T) \leq B_t^k(T) \leq B_t^d(T) \quad (4.1)$$

for every $t, T \in [0, T^*]$ satisfying $t \leq T$. This reflects the fact that the higher the risk is, the lower the price of the bond should be. The monotonicity will be guaranteed by additional restrictions on the model parameters. The inequalities (4.1) can obviously be achieved if

$$f_t^d(T) \leq f_t^k(T) \leq f_t^l(T). \quad (4.2)$$

For every $k, j \in \{d, 1, \dots, m\}$ and $T \in [0, T^*]$, we define the additive (forward) spread between the curves k and j by

$$s_t^{k,j}(T) := f_t^k(T) - f_t^j(T). \quad (4.3)$$

One sees that the dynamics $s^{k,j}(T) = (s_t^{k,j}(T))_{t \in [0, T]}$ are given by

$$s_t^{k,j}(T) = s_0^{k,j}(T) + \int_0^t \alpha^{k,j}(s, T) ds - \int_0^t \sigma^{k,j}(s, T) dL_s, \quad (4.4)$$

where we set $\alpha^{k,j}(s, T) := \alpha^k(s, T) - \alpha^j(s, T)$ and $\sigma^{k,j}(s, T) := \sigma^k(s, T) - \sigma^j(s, T)$. Note that $s_t^{k,k}(T) = 0$ for every $k \in \{d, 1, \dots, m\}$ and we deduce from (4.2) that

$$0 \leq s_0^{k,d}(T) \leq s_0^{l,d}(T) \quad \text{and} \quad 0 \leq s_0^{l,k}(T)$$

for all $k, l \in \{1, \dots, m\}$ with $k \leq l$ and $T \in [0, T^*]$. For every $k, j \in \{d, 1, \dots, m\}$, the short term spread between k and j is defined by $\lambda_t^{k,j} := r_t^k - r_t^j$ and we set

$$A^{k,j}(s, T) := \int_{s \wedge T}^T \alpha^{k,j}(s, u) du \quad \text{and} \quad \Sigma^{k,j}(s, T) := \int_{s \wedge T}^T \sigma^{k,j}(s, u) du.$$

Then, we clearly have the relations

$$A_t^{k,j}(s, T) = A^k(s, T) - A^j(s, T) \quad \text{and} \quad \Sigma^{k,j}(s, T) = \Sigma^k(s, T) - \Sigma^j(s, T).$$

For any $k \in \{1, \dots, m\}$, we obviously have

$$s_t^{k,d}(T) = \sum_{j=1}^k s_t^{j,j-1}(T), \quad (4.5)$$

where we define $s_t^{1,0}(T) := s_t^{1,d}(T)$. Therefore, we obtain

$$f_t^k(T) = f_t^d(T) + \sum_{j=1}^k s_t^{j,j-1}(T).$$

Note that, for every $k \in \{1, \dots, m\}$, the short term spread $\lambda^{k,d}$ is given by

$$\lambda_t^{k,d} = \sum_{j=1}^k \lambda_t^{j,j-1} = \sum_{j=1}^k s_t^{j,j-1}(t), \quad (4.6)$$

where for notational convenience we will write sometimes 0 instead of d , i.e. $\lambda_t^{1,0} := \lambda_t^{1,d}$. Moreover, one easily sees that the drift function and the volatility structure related to curve k can be represented by

$$\alpha^k(s, T) = \alpha^d(s, T) + \sum_{j=1}^k \alpha^{j,j-1}(s, T)$$

and

$$\sigma^k(s, T) = \sigma^d(s, T) + \sum_{j=1}^k \sigma^{j,j-1}(s, T),$$

where we denote as mentioned above $\alpha^{1,0}(s, T) := \alpha^{1,d}(s, T)$ and $\sigma^{1,0}(s, T) := \sigma^{1,d}(s, T)$. It follows that we have

$$A^k(s, T) = A^d(s, T) + \sum_{j=1}^k A^{j,j-1}(s, T)$$

and

$$\Sigma^k(s, T) = \Sigma^d(s, T) + \sum_{j=1}^k \Sigma^{j,j-1}(s, T), \quad (4.7)$$

where we define $A^{1,0}(s, T) := A^{1,d}(s, T)$ and $\Sigma^{1,0}(s, T) := \Sigma^{1,d}(s, T)$. Consequently, due to relation (4.3), we can specify the dynamics of the quantity $f^k(T)$ by modelling the forward spreads $s^{k,j}(T)$ and the forward rates $f^j(T)$.

It is evident that the relation (4.2) is equivalent to the condition

$$0 \leq s_t^{k,d}(T) \leq s_t^{l,d}(T) \quad (4.8)$$

for every $k, l \in \{1, \dots, m\}$ with $k \leq l$ and $t, T \in [0, T^*]$ satisfying $t \leq T$. Then, we conclude from representation (4.5) that condition (4.8) is valid if, for all $j \in \{1, \dots, m\}$ and $t, T \in [0, T^*]$ with $t \leq T$, we have

$$0 \leq s_t^{j,j-1}(T). \quad (4.9)$$

To sum up, this approach results in the non-negative specification of the forward spreads between two subsequent curves (4.9). Then, condition (4.8) is automatically satisfied. This fact implicitly guarantees the relations (4.2) and we therefore ensure the required monotonicity (4.1).

4.2. A Tractable Model Framework. Below, we present a tractable model which ensures the non-negativity of the consecutive forward spreads. Through the analysis made in the previous subsection, it follows that the monotonicity of the curves (4.1) is then guaranteed. This framework is motivated by Crépey, Grbac, and Nguyen (2012). We emphasise that we specify the model quantities corresponding to each curve $k \in \{1, \dots, m\}$ by modelling the relevant spreads and the discount curve.

Let $d, m, l \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ with $l + m \leq d$ and a d -dimensional driving process $L = (L^1, \dots, L^d)^\top$ be given on the enlarged stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, P^d)$. Its components are divided into l real-valued Lévy processes and $d - l$ Lévy processes with negative values. More precisely, we specify the d -dimensional (time-homogeneous) Lévy process L as follows

- (1) $Y^1 := (L^1, \dots, L^l)^\top$ is an \mathbb{R}^l -valued Lévy process.
- (2) $Y^2 := (L^{l+1}, \dots, L^d)^\top = (-Z^{l+1}, \dots, -Z^d)^\top$, where $Z := (Z^{l+1}, \dots, Z^d)^\top = -Y^2$ is an \mathbb{R}_+^{d-l} -valued Lévy process whose components are subordinators (see Sato (1999, Definition 21.4.)). The cumulant process of Z is of the form

$$\theta^Z(z) = \langle z, b \rangle + \int_{\mathbb{R}_+^{d-l}} (e^{\langle z, x \rangle} - 1) F(dx),$$

where $z \in \mathbb{C}^{d-l}$ such that $\operatorname{Re}(z) \in [-(1 + \epsilon)M, (1 + \epsilon)M]^{d-l}$. The drift term b satisfies $b^j \geq 0$ for any $j \in \{1, \dots, d - l\}$ and the Lévy measure F has its support on \mathbb{R}_+^{d-l} .

We make the following

Assumption 4.1. ($\forall L$) For every $k \in \{1, \dots, m\}$, the non-negative volatility functions Σ^d and $\Sigma^{k,k-1}$ as well as the liquidity function l^k are deterministic, differentiable and stationary in the following sense: For every $k \in \{1, \dots, m\}$, $j \in \{1, \dots, d\}$ and s, T with $0 \leq s \leq T \leq T^*$, the functions are of the form

$$\begin{aligned} \Sigma_j^d(s, T) &= G_j(T - s) \\ \Sigma_j^{k,k-1}(s, T) &= G_j^k(T - s) \\ l^k(s, T) &= G_l^k(T - s), \end{aligned}$$

where $G_j : [0, T^*] \rightarrow \mathbb{R}_+$ and $G_j^k : [0, T^*] \rightarrow \mathbb{R}_+$ are differentiable functions satisfying $G_j(0) = G_j^k(0) = 0$ that are bounded in the sense of

$$G_j(s) + \sum_{k=1}^m G_j^k(s) \leq \hat{M} < M$$

for all $s \in [0, T^*]$. The function $G_l^k : [0, T^*] \rightarrow \mathbb{R}$ is differentiable and bounded satisfying $G_l^k(0) = 0$.

It follows that the conditions (DET) and (MC.DET) are fulfilled under assumption ($\forall L$).

Proposition 4.2. *The forward spread $s_t^{1,d}(T)$ can be written as*

$$\begin{aligned} s_t^{1,d}(T) &= s_0^{1,d}(T) - \theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^{1,d}(0, T)) \\ &\quad - l^1(t, T) + l^1(0, T) + \theta(\Sigma^d(t, T)) - \theta(\Sigma^d(0, T)) \\ &\quad - \int_0^t \sigma^{1,d}(s, T) dL_s \end{aligned} \quad (4.10)$$

and, in the case where $m \geq 2$, the forward spread $s_t^{j,j-1}(T)$ is given by

$$\begin{aligned} s_t^{j,j-1}(T) &= s_0^{j,j-1}(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) + \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,i-1}(0, T)) \\ &\quad - l^{j,j-1}(t, T) + l^{j,j-1}(0, T) + \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)) \\ &\quad - \theta(\Sigma^d(0, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, T)) - \int_0^t \sigma^{j,j-1}(s, T) dL_s \end{aligned} \quad (4.11)$$

for any $j \in \{2, \dots, m\}$ where we have set $l^{j,j-1}(t, T) = l^j(t, T) - l^{j-1}(t, T)$. The corresponding short term spreads result in

$$\lambda_t^{1,d} = s_0^{1,d}(t) + \theta(\Sigma^d(0, t) + \Sigma^{1,d}(0, t)) + l^1(0, t) - \theta(\Sigma^d(0, t)) - \int_0^t \sigma^{1,d}(s, t) dL_s$$

and

$$\begin{aligned} \lambda_t^{j,j-1} &= s_0^{j,j-1}(t) + \theta(\Sigma^d(0, t) + \sum_{i=1}^j \Sigma^{i,i-1}(0, t)) + l^{j,j-1}(0, t) \\ &\quad - \theta(\Sigma^d(0, t) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, t)) - \int_0^t \sigma^{j,j-1}(s, t) dL_s. \end{aligned}$$

Proof. We observe that

$$\begin{aligned} \frac{\partial}{\partial T} G_j(T-s) &= -\frac{\partial}{\partial s} G_j(T-s) \\ \frac{\partial}{\partial T} G_j^k(T-s) &= -\frac{\partial}{\partial s} G_j^k(T-s) \\ \frac{\partial}{\partial T} G_l^k(T-s) &= -\frac{\partial}{\partial s} G_l^k(T-s). \end{aligned}$$

This implies

$$\begin{aligned} &\frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) + \frac{\partial}{\partial T} l^1(s, T) - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T)) \\ &= -\frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) - \frac{\partial}{\partial s} l^1(s, T) + \frac{\partial}{\partial s} \theta(\Sigma^d(s, T)) \end{aligned}$$

from which we get

$$\begin{aligned}
& \int_0^t \left[\frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) + \frac{\partial}{\partial T} l^1(s, T) - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T)) \right] ds \\
&= - \int_0^t \left[\frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) + \frac{\partial}{\partial s} l^1(s, T) - \frac{\partial}{\partial s} \theta(\Sigma^d(s, T)) \right] ds \\
&= -\theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^{1,d}(0, T)) - l^1(t, T) + l^1(0, T) \\
&\quad + \theta(\Sigma^d(t, T)) - \theta(\Sigma^d(0, T)).
\end{aligned}$$

Equation (4.10) follows now if we combine the representation of $s_t^{1,d}(T)$ in (4.4) with the choice of the drift coefficient according to (3.19). Analogously, one shows that

$$\begin{aligned}
& \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial T} l^{j,j-1}(s, T) \\
&\quad - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)) \\
&= - \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) - \frac{\partial}{\partial s} l^{j,j-1}(s, T) \\
&\quad + \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
& \int_0^t \left[\frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial T} l^{j,j-1}(s, T) \right. \\
&\quad \left. - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)) \right] ds \\
&= - \int_0^t \left[\frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial s} l^{j,j-1}(s, T) \right. \\
&\quad \left. - \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)) \right] ds \\
&= -\theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) + \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,i-1}(0, T)) - l^{j,j-1}(t, T) \\
&\quad + l^{j,j-1}(0, T) + \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)) - \theta(\Sigma^d(0, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, T)).
\end{aligned}$$

The last expressions immediately lead to the form (4.11). The representations of the short term spreads follow from (4.6) \square

Next, we derive necessary and sufficient deterministic conditions for the non-negativity of the (consecutive) forward spreads. Let $m \geq 2$. We shall exploit the representations obtained in the previous Proposition by considering the following deterministic terms

$$\begin{aligned}\mu^{1,d}(t, T) &:= s_t^{1,d}(T) + \int_0^t \sigma^{1,d}(s, T) dL_s \\ &= s_0^{1,d}(T) - \theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^{1,d}(0, T)) \\ &\quad - l^1(t, T) + l^1(0, T) + \theta(\Sigma^d(t, T)) - \theta(\Sigma^d(0, T)) \\ \mu^{1,d}(t) &:= \lambda_t^{1,d} + \int_0^t \sigma^{1,d}(s, t) dL_s \\ &= s_0^{1,d}(t) + \theta(\Sigma^d(0, t) + \Sigma^{1,d}(0, t)) + l^1(0, t) - \theta(\Sigma^d(0, t))\end{aligned}$$

and

$$\begin{aligned}\mu^{j,j-1}(t, T) &:= s_t^{j,j-1}(T) + \int_0^t \sigma^{j,j-1}(s, T) dL_s \\ &= s_0^{j,j-1}(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) \\ &\quad + \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,i-1}(0, T)) - l^{j,j-1}(t, T) + l^{j,j-1}(0, T) \\ &\quad + \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)) - \theta(\Sigma^d(0, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, T)) \\ \mu^{j,j-1}(t) &:= \lambda_t^{j,j-1} + \int_0^t \sigma^{j,j-1}(s, t) dL_s \\ &= s_0^{j,j-1}(t) + \theta(\Sigma^d(0, t) + \sum_{i=1}^j \Sigma^{i,i-1}(0, t)) + l^{j,j-1}(0, t) \\ &\quad - \theta(\Sigma^d(0, t) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, t)).\end{aligned}$$

Proposition 4.3. *Let $T \in [0, T^*]$. We assume that the following is satisfied*

(1) *For any $t \in [0, T]$ and each $k \in \{1, \dots, l\}$, we have*

$$\sigma_k^{1,d}(t, T) = 0$$

and, if $m \geq 2$, we have for every $j \in \{2, \dots, m\}$

$$\sigma_k^{j,j-1}(t, T) = 0$$

(2) *For all $t \in [0, T]$, we have*

$$0 \leq \mu^{1,d}(t, T) = \mu^{1,d}(T) - \theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) - l^1(t, T) + \theta(\Sigma^d(t, T)) \quad (4.12)$$

and, if $m \geq 2$, we have for every $j \in \{2, \dots, m\}$

$$\begin{aligned}
0 \leq \mu^{j,j-1}(t, T) &= \mu^{j,j-1}(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) - l^{j,j-1}(t, T) \\
&+ \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)).
\end{aligned} \tag{4.13}$$

Then the forward spreads satisfy

$$0 \leq s_t^{1,d}(T) \quad \text{and} \quad 0 \leq s_t^{j,j-1}(T)$$

for every $t, T \in [0, T^*]$ with $t \leq T$ and each $j \in \{2, \dots, m\}$.

Proof. It follows from the definition of $\mu^{1,d}(t, T)$ that the non-negativity of the term $-\int_0^t \sigma^{1,d}(s, T) dL_s$, for any $t \in [0, T]$, together with assumption (4.12) imply $s_t^{1,d}(T) \geq 0$ for all $t \in [0, T]$. Similar arguments lead to the non-negativity of the spread $s_t^{j,j-1}(T)$ for any $t \in [0, T]$ with $t \leq T$ and every $j \in \{2, \dots, m\}$ if $m \geq 2$. \square

Note that the conditions (4.12) and (4.13) result in additional restrictions on the considered distribution parameters of the driving process as well as the parameters of the volatility and liquidity functions. We mention that with some additional effort we can also derive a model framework where Lévy processes are replaced by time-inhomogeneous Lévy processes.

5. VALUATION FORMULAS IN A TWO-PRICE ECONOMY

5.1. Preface. We begin with some preliminary remarks related to the valuation approach. Let $m \in \mathbb{N}$ be the number of the risky curves and $\mathcal{T}^k = \{T_0^k, \dots, T_{n_k}^k\}$ be the discrete tenor structure with tenor δ^k . To simplify the notation, we omit the superscripts in the symbols for the dates and tenors related to the discount curve.

We conclude from the equations (3.8) and (3.14) that the time- T_{j-1}^k price of the bond maturing at date T_j^k can be represented in the case $k = 0 = d$ as

$$\begin{aligned}
B_{T_{j-1}}^d(T_j) &= \frac{B_0^d(T_j)}{B_0^d(T_{j-1})} \exp \left(\int_0^{T_{j-1}} \Sigma^d(s, T_{j-1}, T_j) dL_s \right. \\
&\quad \left. + \int_0^{T_{j-1}} [\theta_s(\Sigma^d(s, T_{j-1})) - \theta_s(\Sigma^d(s, T_j))] ds \right)
\end{aligned}$$

for every $j \in \{1, \dots, n\}$ and

$$B_{T_{j-1}^k}^k(T_j^k) = \frac{B_0^k(T_j^k)}{B_0^k(T_{j-1}^k)} \exp \left(\int_0^{T_{j-1}^k} \Sigma^k(s, T_{j-1}^k, T_j^k) dL_s \right. \\ \left. + \int_0^{T_{j-1}^k} [\theta_s(\Sigma^k(s, T_{j-1}^k)) + l^k(s, T_{j-1}^k) - \theta_s(\Sigma^k(s, T_j^k)) - l^k(s, T_j^k)] ds \right)$$

for every $j \in \{1, \dots, n_k\}$ with $k \in \{1, \dots, m\}$. These expressions can be written in a compact form as

$$B_{T_{j-1}^k}^k(T_j^k) = D_j^k \exp(X_j^k), \quad (5.1)$$

where we set the deterministic part as

$$D_j^k := D^k(T_{j-1}^k, T_j^k) = \frac{B_0^k(T_j^k)}{B_0^k(T_{j-1}^k)} \exp \left(\int_0^{T_{j-1}^k} [\theta_s(\Sigma^k(s, T_{j-1}^k)) + l^k(s, T_{j-1}^k) - \theta_s(\Sigma^k(s, T_j^k)) - l^k(s, T_j^k)] ds \right)$$

and the stochastic term as

$$X_j^k := X^k(T_{j-1}^k, T_j^k) = \int_0^{T_{j-1}^k} \Sigma^k(s, T_{j-1}^k, T_j^k) dL_s$$

for every $k \in \{\mathbf{d}, 1, \dots, m\}$. Furthermore, the discount factor process $\beta^{\mathbf{d}}$ at date $T \in [0, T^*]$ is calculated as

$$\beta_T^{\mathbf{d}} = B_0^{\mathbf{d}}(T) \exp \left(- \int_0^T \theta_s(\Sigma^{\mathbf{d}}(s, T)) ds + \int_0^T \Sigma^{\mathbf{d}}(s, T) dL_s \right). \quad (5.2)$$

The forward martingale measure for the date $T \in [0, T^*]$, denoted by $P_T^{\mathbf{d}}$, is defined by the Radon-Nikodym derivative

$$\frac{dP_T^{\mathbf{d}}|_{\mathcal{G}_T}}{dP^{\mathbf{d}}|_{\mathcal{G}_T}} = \frac{\beta_T^{\mathbf{d}}}{B_0^{\mathbf{d}}(T)}. \quad (5.3)$$

For any $t \leq T$, the restriction to the σ -field \mathcal{G}_t is the martingale

$$\frac{dP_T^{\mathbf{d}}|_{\mathcal{G}_t}}{dP^{\mathbf{d}}|_{\mathcal{G}_t}} = \mathbb{E}_{P^{\mathbf{d}}} \left[\frac{\beta_T^{\mathbf{d}}}{B_0^{\mathbf{d}}(T)} \middle| \mathcal{G}_t \right] = \frac{\beta_t^{\mathbf{d}} B_t^{\mathbf{d}}(T)}{B_0^{\mathbf{d}}(T)}.$$

By using [Eberlein and Kluge \(2006\)](#), we observe that the characteristic function of X_j^k under P_T^d can be determined as

$$\varphi_{X_j^k}^T(u) = \exp \left(\int_0^{T_{j-1}^k} \theta_s^T \left(iu \Sigma^k(s, T_{j-1}^k, T_j^k) \right) ds \right),$$

where θ_s^T denotes the cumulant with respect to P_T^d and $T \in \mathcal{T}^d$.

By changing to the spot martingale measure P^d with the use of [\(5.3\)](#) together with equation [\(5.2\)](#), we get the useful representation

$$\varphi_{X_j^k}^{T_j^k}(u) = \exp \left(\int_0^{T_{j-1}^k} \left[\theta_s \left(\Sigma^d(s, T_j^k) + iu \Sigma^k(s, T_{j-1}^k, T_j^k) \right) - \theta_s \left(\Sigma^d(s, T_j^k) \right) \right] ds \right). \quad (5.4)$$

Note that this expression can be extended to complex numbers by using assumption [\(EM\)](#) (cf. [Sato \(1999, Theorem 25.17\)](#)).

The payoffs of the derivatives that we will consider are functions of the spot reference rates $L_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k)$ which are related to the risky bond prices by

$$1 + \delta^k L_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) = \frac{1}{B_{T_{j-1}^k}^k(T_j^k)}. \quad (5.5)$$

The corresponding forward rates are derived from this by setting

$$L_t^k(T_{j-1}^k, T_j^k) = \mathbb{E}_{P_{T_j^k}^d} \left[L_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) | \mathcal{F}_t \right]. \quad (5.6)$$

Note that in the last formula we used the smaller filtration \mathbb{F} which is justified since due to the martingale invariance property between \mathbb{F} and \mathbb{G} we can replace \mathcal{G}_t by \mathcal{F}_t in all risk-neutral pricing formulas.

Hereafter, we consider $k \in \{d, 1, \dots, m\}$. The related valuation formulas for the single-curve setting are well-known. To simplify the notation, we choose the notional amount always to be equal to one.

5.2. Caps and Floors. For $t \leq T_0$ the time- t value of a cap with strike rate K and maturity T^* results in

$$\begin{aligned} \text{Cap}_t(\mathcal{T}^k, \delta^k, K) &= \sum_{j=1}^{n_k} (\beta_t^d)^{-1} \mathbb{E}_{P^d} \left[\beta_{T_j^k}^d \delta^k (L_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) - K)^+ | \mathcal{F}_t \right] \\ &= \sum_{j=1}^{n_k} \mathbb{E}_{P_{T_j^k}^d} \left[B_t^d(T_j^k) (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ | \mathcal{F}_t \right], \end{aligned}$$

where we have defined $\tilde{K}^k := 1 + \delta^k K$ and $\eta_j^k := \frac{1}{D_j^k}$. Hence, the time-0 price of a caplet with strike rate K is given by

$$\text{Cpl}_0(T_{j-1}^k, T_j^k, K) = \mathbb{E}_{P^d} \left[B_{T_j^k}^d (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ \right] \quad (5.7)$$

$$= \mathbb{E}_{P_{T_j^k}^d} \left[B_0^d(T_j^k) (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ \right]. \quad (5.8)$$

To simplify the notation, we set $Y_j^k := -X_j^k$. The extended characteristic function of Y_j^k relative to the pricing measure $P_{T_j^k}^d$ can be calculated as

$$\begin{aligned} \varphi_{Y_j^k}^{T_j^k}(z) = \exp \left(\int_0^{T_{j-1}^k} [\theta_s(\Sigma^d(s, T_j^k) - iz\Sigma^k(s, T_{j-1}^k, T_j^k)) \right. \\ \left. - \theta_s(\Sigma^d(s, T_j^k))] ds \right) \end{aligned} \quad (5.9)$$

for every $z \in \mathbb{C}$ where this function exists.

To evaluate the price of the caplet, we use the Fourier based valuation method (see [Eberlein, Glau, and Papapantoleon \(2010\)](#)).

Proposition 5.1. *The risk-neutral price of the caplet at time 0 can be written as*

$$\text{Cpl}_0(T_{j-1}^k, T_j^k, K) = B_0^d(T_j^k) \frac{e^{-R\xi_j^k}}{\pi} \int_0^\infty \text{Re} \left(\frac{e^{-iu\xi_j^k} \varphi_{Y_j^k}^{T_j^k}(u - iR) (\tilde{K}^k)^{1-R-iu}}{(-R - iu)(1 - R - iu)} \right) du \quad (5.10)$$

for any $R \in \left(1, \frac{M - \hat{M}}{\hat{M}}\right]$, where we set $\xi_j^k := -\ln \eta_j^k$ and \hat{M} is assumed to be chosen such that $\hat{M} < \frac{M}{2}$.

Proof. Clearly, we have

$$\begin{aligned} \varphi_{Y_j^k}^{T_j^k}(u - iR) = \exp \left(\int_0^{T_{j-1}^k} [\theta_s(\Sigma^d(s, T_j^k) - (iu + R)\Sigma^k(s, T_{j-1}^k, T_j^k)) \right. \\ \left. - \theta_s(\Sigma^d(s, T_j^k))] ds \right) \end{aligned}$$

and

$$\begin{aligned} |\text{Re}(\Sigma_l^d(s, T_j^k) - (iu + R)\Sigma_l^k(s, T_{j-1}^k, T_j^k))| &= |\Sigma_l^d(s, T_j^k) - R\Sigma_l^k(s, T_{j-1}^k, T_j^k)| \\ &\leq \hat{M} + |R|\hat{M} \leq \hat{M} + \frac{M - \hat{M}}{\hat{M}}\hat{M} \\ &= M \end{aligned}$$

for every $l \in \{1, \dots, d\}$. Then the result follows from [Eberlein, Glau, and Papapantoleon \(2010, Theorem 2.2.\)](#) and an obvious symmetry property of the integrand. \square

Note that the price of a floorlet can be obtained in an analogous way. From the pricing formula for caplets (floorlets) we immediately deduce the valuation formula of the cap (floor).

5.3. Swaptions. We consider a swap where a fixed rate R is exchanged for a floating rate (Euribor) with tenor δ^k . Let $\mathcal{T}^k = \{T_0^k, \dots, T_{n_k}^k\}$ be the corresponding tenor structure. We assume that the swap is initiated at $T_0 = T_0^k$ and the rates are exchanged at time points $T_1^k < \dots < T_{n_k}^k = T_n$. For $t \leq T_0$ the

time- t value of a payer swap - where we set again the notional amount $N = 1$ - is then

$$\begin{aligned} \text{Sw}(t, T_0, T_n, R) &= \sum_{j=1}^{n_k} B_t^d(T_j^k) \delta^k \mathbb{E}_{P_{T_j^k}^d} [L_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) - R | \mathcal{F}_t] \\ &= \sum_{j=1}^{n_k} B_t^d(T_j^k) \delta^k \left(L_t^k(T_{j-1}^k, T_j^k) - R \right). \end{aligned} \quad (5.11)$$

Consequently the swap rate, i.e. the rate which makes this value equal to zero, is given by

$$R(t, T_0, T_n) = \frac{\sum_{j=1}^{n_k} B_t^d(T_j^k) \delta^k L_t^k(T_{j-1}^k, T_j^k)}{\sum_{j=1}^{n_k} B_t^d(T_j^k) \delta^k}.$$

Now consider a swaption, i.e. an option to enter this swap at time point T_0 . Its value at $t \leq T_0$ is given by

$$\begin{aligned} \text{Swpt}(t, T_0, T_n, R) &= B_t^d(T_0) \mathbb{E}_{P_{T_0}^d} [(\text{Sw}(T_0, T_0, T_n, R))^+ | \mathcal{F}_t] \\ &= B_t^d(T_0) \mathbb{E}_{P_{T_0}^d} \left[\sum_{j=1}^{n_k} B_{T_0}^d(T_j^k) \delta^k (R(T_0, T_0, T_n) - R)^+ | \mathcal{F}_t \right]. \end{aligned} \quad (5.12)$$

This valuation formula can be written in a more elegant way with the help of the swap measure (see e.g. section 1.4.7 in [Grbac and Runggaldier \(2015\)](#)). If we define the process $A_t = \sum_{j=1}^{n_k} B_t^d(T_j^k) \delta^k$ ($0 \leq t \leq T_1^k$) then $\left(\frac{A_t}{B_t^d(T_0)} \right)_{0 \leq t \leq T_0}$ is a $P_{T_0}^d$ -martingale and thus allows to define a density process

$$\frac{dP_{\text{Swap}}^d |_{\mathcal{F}_t}}{dP_{T_0}^d |_{\mathcal{F}_t}} = \frac{A_t}{B_t^d(T_0)} \frac{B_0^d(T_0)}{A_0} \quad (0 \leq t \leq T_0).$$

Using the Bayes formula for conditional expectations we get

$$\text{Swpt}(t, T_0, T_n, R) = A_t \mathbb{E}_{P_{\text{Swap}}^d} [(R(T_0, T_0, T_n) - R)^+ | \mathcal{F}_t].$$

In order to make the valuation formula for swaptions numerically easier accessible we assume as usual that the volatilities factorise in the following way

$$\Sigma^d(s, T) = \Sigma_1(s) \Sigma_2^d(T)$$

and

$$\Sigma^k(s, T) = \Sigma_1(s) \Sigma_2^k(T)$$

where Σ_1 is \mathbb{R}^d -valued, whereas Σ_2^d and Σ_2^k are \mathbb{R} -valued. At the price of considering two-dimensional distributions instead of a one-dimensional one, the factor Σ_1 could be chosen to differ for Σ^d and Σ^k .

Using (3.9) we get

$$B_{T_0}^d(T_j^k) = D^d(T_0, T_j^k) \exp \left(\Sigma_2^d(T_0, T_j^k) \int_0^{T_0} \Sigma_1(s) dL_s \right) \quad (5.13)$$

where we wrote for short $\Sigma_2^d(T_0, T_j^k) := \Sigma_2^d(T_j^k) - \Sigma_2^d(T_0)$. In the same way we use (5.6), (5.5) and (5.1) to express

$$\begin{aligned}
L_{T_0}^k(T_{j-1}^k, T_j^k) &= \mathbb{E}_{P_{T_j^k}^d} \left[\frac{1}{\delta^k} \left(\frac{1}{B_{T_{j-1}^k}^k(T_j^k)} - 1 \right) \middle| \mathcal{F}_{T_0} \right] \\
&= \frac{1}{\delta^k} \left(\mathbb{E}_{P_{T_j^k}^d} \left[(D_j^k)^{-1} \exp(-X_j^k) \middle| \mathcal{F}_{T_0} \right] - 1 \right) \\
&= \frac{1}{\delta^k} \left(\eta_j^k \exp \left(\int_{T_0}^{T_{j-1}^k} \theta_s^{T_j^k} (\Sigma^k(s, T_j^k, T_{j-1}^k)) ds + \Sigma_2^k(T_j^k, T_{j-1}^k) \int_0^{T_0} \Sigma_1(s) dL_s \right) - 1 \right)
\end{aligned} \tag{5.14}$$

where we have set $\Sigma_2^k(T_j^k, T_{j-1}^k) := \Sigma_2^k(T_{j-1}^k) - \Sigma_2^k(T_j^k)$. Writing $X_{T_0} := \int_0^{T_0} \Sigma_1(s) dL_s$ for the random term in (5.13) and (5.14) and introducing these formulas in (5.12) with the help of (5.11) we see that the value of the swaption at time $t = 0$ can be represented as

$$\begin{aligned}
\text{Swpt}(0, T_0, T_n, R) &= B_0^d(T_0) \mathbb{E}_{P_{T_0}^d} \left[\left(\sum_{j=1}^{n_k} D^d(T_0, T_j^k) \exp \left(\Sigma_2^d(T_0, T_j^k) X_{T_0} \right) \right. \right. \\
&\quad \left. \left. \left(\eta_j^k \exp \left(\int_{T_0}^{T_{j-1}^k} \theta_s^{T_j^k} (\Sigma^k(s, T_j^k, T_{j-1}^k)) ds + \Sigma_2^k(T_j^k, T_{j-1}^k) X_{T_0} \right) - (1 + \delta^k R) \right) \right)^+ \right].
\end{aligned}$$

If we define the function $f^k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned}
f^k(x) &= \left(\sum_{j=1}^{n_k} D^d(T_0, T_j^k) \exp \left(\Sigma_2^d(T_0, T_j^k) x \right) \right. \\
&\quad \left. \left(\eta_j^k \exp \left(\int_{T_0}^{T_{j-1}^k} \theta_s^{T_j^k} (\Sigma^k(s, T_j^k, T_{j-1}^k)) ds + \Sigma_2^k(T_j^k, T_{j-1}^k) x \right) - (1 + \delta^k R) \right) \right)^+
\end{aligned}$$

the value of the swaption can be written as

$$\text{Swpt}(0, T_0, T_n, R) = B_0^d(T_0) \mathbb{E}_{P_{T_0}^d} [f^k(X_{T_0})]$$

which is numerically accessible with the Fourier based approach.

5.4. Interest Rate Digital Options. A standard interest rate digital call (put) with strike rate B is an option that pays an amount of one currency unit to its owner if and only if the reference rate for the period $[T_{j-1}^k, T_j^k]$ lies above (below) B at maturity T_{j-1}^k of the option. An interest rate digital option is called delayed if the payment date T differs from the maturity date T_{j-1}^k , where $T > T_{j-1}^k$. Thus the payoff at time T can be given in the form

$$\text{DD}_T^k(T_{j-1}^k, T_j^k, T, B, w) = \mathbb{1}_{\{wL_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) > wB\}},$$

where we assume that $B > -\frac{1}{\delta^k}$ and

$$w = \begin{cases} 1, & \text{for a delayed digital call} \\ -1, & \text{for a delayed digital put.} \end{cases}$$

Then, the time- t price for $t \leq T_{j-1}^k$ can be represented by

$$\begin{aligned} \text{DD}_t^k(T_{j-1}^k, T_j^k, T, B, w) &= (\beta_t^d)^{-1} \mathbb{E}_{P^d} \left[\beta_T^d \mathbb{1}_{\{wL_{T_{j-1}^k}^k(T_{j-1}^k, T_j^k) > wB\}} \mid \mathcal{F}_t \right] \\ &= B_t^d(T) \mathbb{E}_{P_T^d} \left[\mathbb{1}_{\{w(1+\delta^k B)^{-1} > wB_{T_{j-1}^k}^k(T_j^k)\}} \mid \mathcal{F}_t \right]. \end{aligned}$$

This formula can further be written as

$$\text{DD}_t^k(T_{j-1}^k, T_j^k, T, B, w) = B_t^d(T) \mathbb{E}_{P_T^d} \left[\mathbb{1}_{\{w(1+\delta^k B)^{-1} > wB_t^k(T_j^k)B_t^k(T_{j-1}^k)^{-1}H^k(t, T_{j-1}^k)\}} \mid \mathcal{F}_t \right]$$

with

$$H^k(t, T_{j-1}^k) := \exp \left(- \int_t^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds + X_{t,j}^k \right),$$

where we set

$$A^k(s, T_{j-1}^k, T_j^k) := A^k(s, T_j^k) - A^k(s, T_{j-1}^k)$$

and

$$X_{t,j}^k := \int_t^{T_{j-1}^k} \Sigma^k(s, T_{j-1}^k, T_j^k) dL_s.$$

By the independence of $X_{t,j}^k$ and \mathcal{F}_t (independent increments of the process L) and the \mathcal{F}_t -measurability of $\frac{B_t^k(T_j^k)}{B_t^k(T_{j-1}^k)}$, we obtain from [Kallenberg \(2002, Theorem 6.4\)](#) that

$$\begin{aligned} \text{DD}_t^k(T_{j-1}^k, T_j^k, T, B, w) &= B_t^d(T) \cdot g_w^k \left(\frac{B_t^k(T_j^k)}{B_t^k(T_{j-1}^k)} \right) \\ &= B_t^d(T) \cdot g_w^k \left(F^k(t, T_{j-1}^k, T_j^k)^{-1} \right), \end{aligned}$$

where $F^k(t, T_{j-1}^k, T_j^k) = \frac{B_t^k(T_{j-1}^k)}{B_t^k(T_j^k)}$ and the function $g_w^k : \mathbb{R} \rightarrow [0, 1]$ is defined by

$$g_w^k(y) := \mathbb{E}_{P_T^d} \left[\mathbb{1}_{\{w(1+\delta^k B)^{-1} > wyH^k(t, T_{j-1}^k)\}} \right].$$

For every $y > 0$, we have

$$\begin{aligned}
 g_w^k(y) &= P_T^d \left(w \exp \left(X_{t,j}^k \right) < w \frac{\exp \left(\int_t^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1 + \delta^k B)y} \right) \\
 &= \begin{cases} P_T^{d, X_{t,j}^k} \left(x < \log \left(\frac{\exp \left(\int_t^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1 + \delta^k B)y} \right) \right) & \text{for } w = 1 \\ 1 - P_T^{d, X_{t,j}^k} \left(x \leq \log \left(\frac{\exp \left(\int_t^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1 + \delta^k B)y} \right) \right) & \text{for } w = -1, \end{cases}
 \end{aligned} \tag{5.15}$$

where we denote by $P_T^{d, X_{t,j}^k}$ the distribution of $X_{t,j}^k$. The extended characteristic function of $X_{t,j}^k$ under P_T^d can be determined by

$$\varphi_{X_{t,j}^k}^T(z) = \exp \left(\int_t^{T_{j-1}^k} \left[\theta_s(\Sigma^d(s, T) + iz \Sigma^k(s, T_{j-1}^k, T_j^k)) - \theta_s(\Sigma^d(s, T)) \right] ds \right)$$

for every $z \in \mathbb{C}$ where this function is defined.

Now, we calculate the present value ($t = 0$) of a delayed digital option by applying the Fourier-based valuation method. Let us write

$$\begin{aligned}
 &DD_0^k(T_{j-1}^k, T_j^k, T, B, w) \\
 &= B_0^d(T) \mathbb{E}_{P_T^d} \left[\mathbb{1}_{\{w \exp(X_j^k - \xi_j^k) < w(1 + \delta^k B)^{-1}\}} \right] \\
 &= \begin{cases} B_0^d(T) P_T^d \left(X_j^k < \log((1 + \delta^k B)^{-1}) + \xi_j^k \right) & \text{for } w = 1 \\ B_0^d(T) P_T^d \left(X_j^k > \log((1 + \delta^k B)^{-1}) + \xi_j^k \right) & \text{for } w = -1 \end{cases} \\
 &=: V_w^{j,k}(\xi_j^k),
 \end{aligned}$$

where we used $\xi_j^k = -\log(D_j^k) \in \mathbb{R}$.

Let us consider the map $\xi_j^k \mapsto V_w^{j,k}(\xi_j^k) \in \mathbb{R}_+$. Clearly, $V_w^{j,k}$ has locally bounded variation. We assume that the distribution of X_j^k under P_T^d is atomless. Then, by using the symbol $F_{X_j^k}^T$ for the cumulative distribution function of a random variable X under P_T^d , we get

$$V_w^{j,k}(\xi_j^k) = \begin{cases} B_0^d(T) F_{X_j^k}^T(\log((1 + \delta^k B)^{-1}) + \xi_j^k) & \text{for } w = 1 \\ B_0^d(T) (1 - F_{X_j^k}^T(\log((1 + \delta^k B)^{-1}) + \xi_j^k)) & \text{for } w = -1. \end{cases}$$

It follows that $V_w^{j,k}$ is a continuous function. The payoff function of a digital call option with barrier $B \in \mathbb{R}_+$ is given by

$$f_w(x) = \mathbb{1}_{\{w e^x < w \tilde{B}_k^{-1}\}},$$

where we set $\tilde{B}_k = 1 + \delta^k B$. Easy calculations lead to the following form of the Fourier transform of f_w for every $z \in \mathbb{C}$ where it is defined by

$$\hat{f}_w(z) = \begin{cases} \frac{\tilde{B}_k^{-iz}}{iz}, & \text{for } w = 1 \text{ and } \text{Im}(z) \in (-\infty, 0) \\ -\frac{\tilde{B}_k^{-iz}}{iz}, & \text{for } w = -1 \text{ and } \text{Im}(z) \in (0, \infty). \end{cases}$$

Let us consider the dampened payoff function of the digital option given by $d_w(x) = e^{-Rx} f_w(x)$. We easily verify that $d_w \in L^1(\mathbb{R})$ for every

$$\begin{cases} R \in (-\infty, 0), & \text{for } w = 1, \\ R \in (0, \infty), & \text{for } w = -1. \end{cases}$$

Observe that we can find an R that satisfies the prerequisites of [Eberlein, Glau, and Papapantoleon \(2010, Theorem 2.7.\)](#). Then, we conclude from this Theorem that the value of a delayed digital option at point ξ_j^k can be expressed as

$$\begin{aligned} & \text{DD}_0^k(T_{j-1}^k, T_j^k, T, B, w) \\ &= \begin{cases} B_0^d(T) \cdot \lim_{A \rightarrow \infty} \frac{e^{-R\xi_j^k}}{\pi} \int_0^A \text{Re} \left(\frac{e^{-iu\xi_j^k} \varphi_{X_j^k}^T(u-iR) \tilde{B}_k^{R+iu}}{-R-iu} \right) du, & \text{for } w = 1 \text{ and } R < 0 \\ B_0^d(T) \cdot \lim_{A \rightarrow \infty} \frac{e^{-R\xi_j^k}}{\pi} \int_0^A \text{Re} \left(\frac{e^{-iu\xi_j^k} \varphi_{X_j^k}^T(u-iR) \tilde{B}_k^{R+iu}}{R+iu} \right) du, & \text{for } w = -1 \text{ and } R > 0. \end{cases} \end{aligned}$$

Note that the price of a delayed range digital option with barriers \underline{B}, \bar{B} satisfying $0 < \underline{B} < \bar{B}$ can be determined by the formula

$$\text{DRD}_t^i(T, T_1, T_2, \underline{B}, \bar{B}) := \text{DD}_t^i(T_1, T_2, T, \underline{B}, 1) + \text{DD}_t^i(T_1, T_2, T, \bar{B}, -1) - B_t^d(T).$$

5.5. Two-Price Theory. In this section we extend the multiple-curve model according to the two-price approach. We provide bid and ask valuation formulas for some interest rate derivatives. We consider again m risky curves and the tenor structure $\mathcal{T}^k = \{T_0^k, \dots, T_{n_k}^k\}$ with tenor δ^k and $k \in \{0, 1, \dots, m\}$.

5.5.1. The Two-Price Theory based on concave distortions. Let us briefly recall the basic concept as developed in [Cherny and Madan \(2010\)](#). In order to get bid and ask prices, instead of a single pricing measure a whole set \mathcal{M} of pricing measures is considered. We assume that \mathcal{M} contains at least one risk-neutral probability measure. The bid price of a discounted claim X is defined as

$$b(X) = \inf_{Q \in \mathcal{M}} \mathbb{E}_Q[X]$$

whereas the ask price is given by

$$a(X) = \sup_{Q \in \mathcal{M}} \mathbb{E}_Q[X].$$

Under slight additional assumptions, namely co-monotonicity and law-invariance, these two values can be obtained by using an increasing concave function $\Psi : [0, 1] \rightarrow [0, 1]$ (concave distortion) in the form

$$b(X) = - \int_{-\infty}^0 \Psi(F_X(y)) dy + \int_0^{\infty} 1 - \Psi(F_X(y)) dy$$

and

$$a(X) = \int_0^{\infty} \Psi(1 - F_X(y)) dy - \int_{-\infty}^0 1 - \Psi(1 - F_X(y)) dy.$$

Observe that these bid and ask prices depend on an underlying probability measure. In our situation, this will be a basic pricing measure which is distorted by Ψ (it is assumed to be contained in the set \mathcal{M}).

Hereafter, we consider a parameterised family $(\Psi_\gamma)_{\gamma \geq 0}$ of distortion functions. For technical reasons we assume that the map $\gamma \mapsto \Psi_\gamma(y)$ is increasing on $[0, \infty)$ and continuous on $(0, \infty)$ for every $y \in (0, 1)$. Further, we only consider values for $\gamma \geq 0$ that satisfy

$$\int_{-\infty}^0 \Psi_\gamma(F_X(y)) dy < \infty \quad \text{and} \quad \int_0^{\infty} \Psi_\gamma(1 - F_X(y)) dy < \infty. \quad (5.16)$$

The parameter γ represents the current level of (il)liquidity in the market. Note that Ψ_γ distorts the cumulative distribution function of a maturity-dependent payoff. Consequently γ becomes a maturity-dependent parameter in the interest rate models where we will consider cash flows along a sequence of maturities.

The family of distortion functions which we will use (see [Cherny and Madan \(2009, 2010\)](#)) is called MINMAXVAR and given by

$$\Psi_\gamma^{\text{mmv}} : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto 1 - (1 - y^{\frac{1}{1+\gamma}})^{1+\gamma}. \end{cases}$$

Other standard families are the MINVAR family defined by

$$\Psi_\gamma^{\text{mv}} : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto 1 - (1 - y)^{1+\gamma}, \end{cases}$$

the MAXVAR family defined by

$$\Psi_\gamma^{\text{mav}} : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto y^{\frac{1}{1+\gamma}}, \end{cases}$$

and the MAXMINVAR family given by

$$\Psi_\gamma^{\text{mamv}} : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto (1 - (1 - y)^{1+\gamma})^{\frac{1}{1+\gamma}}. \end{cases}$$

5.5.2. Bid and Ask Price of Caplets and Floorlets. Our first aim is to derive explicit valuation formulas of bid and ask prices of caplets and floorlets. In order to determine these formulas we have to distort the cumulative distribution function of the discounted payoff

$$\beta_{T_j^k}^d (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ \quad (5.17)$$

with respect to measure P^d (cf. relation (5.7)). Although this can be done analytically, this calculation is, in general, a challenging task. In most cases, it has to be determined numerically and its numerical evaluation is extremely time-consuming. The main reason for this lies in the joint appearance of the

random variables $\beta_{T_j^k}^d$ and X_j^k in (5.17). To handle this issue, we switch to the forward martingale measure and consider the more tractable discounted payoff

$$\text{Cpl}_{j,K}^k := B_0^d(T_j^k)(\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+$$

(cf. pricing formula (5.8)). Analogously, we deal with the corresponding floorlets quantity

$$\text{Flt}_{j,K}^k := B_0^d(T_j^k)(\tilde{K}^k - \eta_j^k \exp(-X_j^k))^+.$$

Recall that F_Y^T denotes the cumulative distribution function of a random variable Y under P_T^d with $T \in [0, T^*]$.

Lemma 5.2. (1) *Let us assume that X_j^k has exponential moments of order $M_j^k > 1$ under $P_{T_j^k}^d$. Then, we get*

(i) *for any $\gamma \geq 0$*

$$a_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = \int_{-\infty}^0 \Psi_\gamma^{\text{mv}}(F_{-\text{Cpl}_{j,K}^k}^{T_j^k}(x)) dx$$

and

$$b_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = \int_0^\infty \left(1 - \Psi_\gamma^{\text{mv}}(F_{\text{Cpl}_{j,K}^k}^{T_j^k}(x))\right) dx.$$

(ii) *for the family $\Psi = (\Psi_\gamma)_{\gamma \geq 0} \in \{\Psi^{\text{mav}}, \Psi^{\text{mmv}}, \Psi^{\text{mamv}}\}$ of distortion functions and every $\gamma \in [0, u_1 - 1)$, where u_1 has to satisfy $1 < u_1 \leq M_j^k$, we obtain*

$$a_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = \int_{-\infty}^0 \Psi_\gamma(F_{-\text{Cpl}_{j,K}^k}^{T_j^k}(x)) dx$$

and

$$b_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = \int_0^\infty \left(1 - \Psi_\gamma(F_{\text{Cpl}_{j,K}^k}^{T_j^k}(x))\right) dx.$$

(2) *For every $\Psi = (\Psi_\gamma)_{\gamma \geq 0} \in \{\Psi^{\text{mv}}, \Psi^{\text{mav}}, \Psi^{\text{mmv}}, \Psi^{\text{mamv}}\}$ and $\gamma \geq 0$, we obtain*

$$a_\gamma^{T_j^k}(\text{Flt}_{j,K}^k) = \int_{-\infty}^0 \Psi_\gamma(F_{-\text{Flt}_{j,K}^k}^{T_j^k}(x)) dx$$

and

$$b_\gamma^{T_j^k}(\text{Flt}_{j,K}^k) = \int_0^\infty \left(1 - \Psi_\gamma(F_{\text{Flt}_{j,K}^k}^{T_j^k}(x))\right) dx.$$

Proof. We need to verify for which $\gamma \geq 0$ condition (5.16) is satisfied. We frequently use the change-of-variable formula here. Since $\text{Cpl}_{j,K}^k \geq 0$, we only have to consider the second integral in (5.16).

- (1) An application of Bernoulli's inequality leads to $\Psi_\gamma^{\text{mv}}(y) \leq (1 + \gamma)y$. For any $\gamma \geq 0$, we obtain

$$\begin{aligned} & \int_0^\infty \Psi_\gamma^{\text{mv}}(1 - F_{\text{Cpl}_{j,K}^{T_j^k}}^{T_j^k}(y))dy \leq (1 + \gamma) \int_0^\infty 1 - F_{\text{Cpl}_{j,K}^{T_j^k}}^{T_j^k}(y)dy \\ & = (1 + \gamma)B_0^{\text{d}}(T_j^k) \int_0^\infty 1 - F_{(\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+}^{T_j^k}(y)dy \\ & = (1 + \gamma)B_0^{\text{d}}(T_j^k) \int_{\tilde{K}^k}^\infty F_{X_j^k}^{T_j^k}(-\log(yD_j^k))dy \\ & \leq (1 + \gamma)B_0^{\text{d}}(T_j^k)C_1 \int_{\tilde{K}^k}^\infty e^{-u_1 \log(yD_j^k)}dy < \infty, \end{aligned}$$

where $C_1 > 0$ and $1 < u_1 \leq M_j^k$.

- (2) Since

$$\begin{aligned} \Psi_\gamma^{\text{mamv}}(y) &= \Psi_\gamma^{\text{mv}}(\Psi_\gamma^{\text{mv}}(y)) \leq \Psi_\gamma^{\text{mv}}((1 + \gamma)y) = (1 + \gamma)^{\frac{1}{1+\gamma}} \Psi_\gamma^{\text{mv}}(y) \\ \Psi_\gamma^{\text{mmv}}(y) &= \Psi_\gamma^{\text{mv}}(\Psi_\gamma^{\text{mv}}(y)) \leq (1 + \gamma)\Psi_\gamma^{\text{mv}}(y), \end{aligned}$$

we only need to check the condition for the distortion function Ψ^{mv} . In a similar way as above, we get

$$\int_0^\infty \Psi_\gamma^{\text{mv}}(1 - F_{\text{Cpl}_{j,K}^{T_j^k}}^{T_j^k}(y))dy \leq B_0^{\text{d}}(T_j^k)C_1^{\frac{1}{1+\gamma}}(\eta_j^k)^{\frac{u_1}{1+\gamma}} \int_{\tilde{K}^k}^\infty y^{-\frac{u_1}{1+\gamma}}dy < \infty$$

for every γ satisfying $0 \leq \gamma < u_1 - 1$.

- (3) Clearly, we have

$$M_{\text{Flt}_{j,K}^{T_j^k}}^{T_j^k}(u) = \mathbb{E}_{P_{T_j^k}^{\text{d}}} \left[\exp \left(u \text{Flt}_{j,K}^k \right) \right] < \exp \left(u B_0^{\text{d}}(T_j^k) \tilde{K}^k \right) < \infty$$

for every $u \in \mathbb{R}$.

□

The following Proposition states integral representations for bid and ask prices of caplets and floorlets with reset date T_{j-1}^k , settlement date $T_j^k = T_{j-1}^k + \delta^k$ and strike rate K at a permitted level γ . Here we consider no longer the distribution functions of payoffs of caplet and floorlets, but the distribution function of the random term X_j^k in the exponent.

Proposition 5.3. *Let $(\Psi_\gamma)_{\gamma \geq 0}$ be a family of distortion functions and $\gamma \geq 0$ be chosen such that condition (5.16) is satisfied for the caplet and floorlet. Then, the ask price of the caplet is given by*

$$a_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = B_0^{\text{d}}(T_j^k) \int_{\tilde{K}^k}^\infty \Psi_\gamma(F_{X_j^k}^{T_j^k}(-\log(xD_j^k)))dx \quad (5.18)$$

and the bid price of the caplet has the form

$$b_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) = B_0^d(T_j^k) \int_{\tilde{K}^k}^{\infty} \left[1 - \Psi_\gamma(1 - F_{X_j^k}^{T_j^k}(-\log(xD_j^k))) \right] dx. \quad (5.19)$$

The ask price of the floorlet is determined by

$$a_\gamma^{T_j^k}(\text{Flt}_{j,K}^k) = B_0^d(T_j^k) \int_0^{\tilde{K}^k} \Psi_\gamma(1 - F_{X_j^k}^{T_j^k}(-\log(xD_j^k))) dx$$

and the bid price of the floorlet is represented by

$$b_\gamma^{T_j^k}(\text{Flt}_{j,K}^k) = B_0^d(T_j^k) \int_0^{\tilde{K}^k} \left[1 - \Psi_\gamma(F_{X_j^k}^{T_j^k}(-\log(xD_j^k))) \right] dx.$$

Proof. By applying the change-of-variable formula, we get the ask price as

$$\begin{aligned} a_\gamma^{T_j^k}(\text{Cpl}_{j,K}^k) &= \int_{-\infty}^0 \Psi_\gamma(F_{-\text{Cpl}_{j,K}^k}^{T_j^k}(x)) dx \\ &= \int_{-\infty}^0 \Psi_\gamma(1 - F_{(\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+}^{T_j^k}(-xB_0^d(T_j^k)^{-1})) dx \\ &= B_0^d(T_j^k) \int_0^{\infty} \Psi_\gamma(1 - F_{(\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+}^{T_j^k}(x)) dx \\ &= B_0^d(T_j^k) \int_{\tilde{K}^k}^{\infty} \Psi_\gamma(F_{X_j^k}^{T_j^k}(-\log(xD_j^k))) dx. \end{aligned}$$

The other formulas follow by the same arguments. \square

On the basis of the analysis we made above, in order to determine the bid and ask prices, we have to identify the cumulative distribution function $F_{X_j^k}^T$.

We proceed as follows

- (1) We determine the characteristic function of X_j^k under P_T^d .
- (2) As an approximation of the cumulative distribution function, we consider

$$F_{X_j^k}^T(y) \approx \lim_{M \rightarrow \infty} \frac{1}{\pi} \cdot \int_0^M \text{Re} \left(\frac{e^{-iux} - e^{-iuy}}{iu} \varphi_{X_j^k}^T(u) \right) du \quad (5.20)$$

for a suitable $x \in \mathbb{R}$ satisfying $x < y$ and $P_T^{\text{d}, X_j^k}(\{x\}) = P_T^{\text{d}, X_j^k}(\{y\}) = 0$, where P_T^{d, X_j^k} denotes the distribution of X_j^k with respect to P_T^d .

5.5.3. *Bid and Ask Price of Digital Options.* The discounted payoff of a delayed digital option is

$$\text{DD}_T^k(w) := B_0^d(T) \cdot \mathbb{1}_{\{w \exp(X_j^k) < w \tilde{B}_k^{-1} \exp(\xi_j^k)\}}.$$

We state its bid and ask price.

Proposition 5.4. *The ask price of the delayed digital option at level $\gamma \geq 0$ is given by*

$$a_\gamma^T(\text{DD}_T^k(w)) = B_0^d(T) \Psi_\gamma \left(P_T^d(w e^{X_j^k} < w \tilde{B}_k^{-1} \exp(\xi_j^k)) \right)$$

and the bid price at level $\gamma \geq 0$ can be expressed as

$$b_\gamma^T(\text{DD}_T^k(w)) = B_0^d(T) \left[1 - \Psi_\gamma \left(P_T^d(w e^{X_j^k} \geq w \tilde{B}_k^{-1} \exp(\xi_j^k)) \right) \right].$$

Proof. One verifies that the cumulative distribution functions of the random variables $-\text{DD}_T^k(w)$ and $\text{DD}_T^k(w)$ under P_T^d are given by

$$F_{-\text{DD}_T^k(w)}^T(y) = \begin{cases} 1, & y \geq 0 \\ P_T^d(w \exp(X_j^k) < w \tilde{B}_k^{-1} \exp(\xi_j^k)), & y \in [-B_0^d(T), 0) \\ 0, & y < -B_0^d(T) \end{cases}$$

and

$$F_{\text{DD}_T^k(w)}^T(y) = \begin{cases} 1, & y \geq B_0^d(T) \\ P_T^d(w \exp(X_j^k) \geq w \tilde{B}_k^{-1} \exp(\xi_j^k)), & y \in [0, B_0^d(T)) \\ 0, & y < 0. \end{cases}$$

Then, by considering the bid and ask price formulas above, we immediately get the statement. \square

The discounted payoff of a delayed range digital option with barriers \underline{B} and \bar{B} is given by

$$\text{DRD}_T^k := B_0^d(T) \cdot \mathbb{1}_{\left\{ \frac{\exp(\xi_j^k)}{\bar{B}_k} < \exp(X_j^k) < \frac{\exp(\xi_j^k)}{\underline{B}_k} \right\}}.$$

Proposition 5.5. *The ask price of the delayed range digital option at level $\gamma \geq 0$ is given by*

$$a_\gamma^T(\text{DRD}_T^k) = B_0^d(T) \Psi_\gamma \left(P_T^d \left(\frac{\exp(\xi_j^k)}{\bar{B}_k} < \exp(X_j^k) \right) + P_T^d \left(\exp(X_j^k) < \frac{\exp(\xi_j^k)}{\underline{B}_k} \right) \right)$$

and the bid price at level $\gamma \geq 0$ can be expressed as

$$b_\gamma^T(\text{DRD}_T^k) = B_0^d(T) \times \left[1 - \Psi_\gamma \left(P_T^d \left(\exp(X_j^k) \leq \frac{\exp(\xi_j^k)}{\bar{B}_k} \right) + P_T^d \left(\frac{\exp(\xi_j^k)}{\underline{B}_k} \leq \exp(X_j^k) \right) \right) \right].$$

Proof. Here, the cumulative distribution functions result in

$$F_{-\text{DRD}_T^k}^T(y) = \begin{cases} 1, & y \geq 0 \\ P_T^d \left(\frac{\exp(\xi_j^k)}{\bar{B}_k} < \exp(X_j^k) \right) + P_T^d \left(\exp(X_j^k) < \frac{\exp(\xi_j^k)}{\underline{B}_k} \right), & y \in [-B_0^d(T), 0) \\ 0, & y < -B_0^d(T) \end{cases}$$

and

$$F_{\text{DRD}_T^k}^T(y) = \begin{cases} 1, & y \geq B_0^d(T) \\ P_T^d\left(\exp(X_j^k) \leq \frac{\exp(\xi_j^k)}{\bar{B}_k}\right) + P_T^d\left(\frac{\exp(\xi_j^k)}{\bar{B}_k} \leq \exp(X_j^k)\right), & y \in [0, B_0^d(T)) \\ 0, & y < 0. \end{cases}$$

Then, the claim follows by the pricing formulas. \square

6. MODEL CALIBRATION

6.1. Data and Approach. We calibrate the model to European market data observed in the post crisis period which is provided by Bloomberg. Market rates of deposits, forward rate agreements and swaps based on different tenors as well as cap quotes indexed on Euribor for a number of maturities and strikes will be used. The cap quotes are given in form of the corresponding implied volatilities. More specifically we will consider cap quotes indexed on three-month and six-month Euribor on September 16, 2013 and September 15, 2016. Consequently, three term structures will be taken into account, namely the basic, the three-month and the six-month curve. For the data set from September 16, 2013 we also consider bid and ask quotes to calibrate the model in a two-price economy.

Based on bootstrapping (see [Ametrano and Bianchetti \(2013\)](#)) we will construct tenor-dependent bond price curves. The resulting basic, three-month and six-month curves for the two dates are shown on the left side of figure 4. The basic curve is constructed by taking the quoted OIS rates. For each of the tenors we use the corresponding market quotes of deposit rates for the short-term maturities, rates from forward rate agreements for the mid-term part and swap rates for mid- and long-term maturities. Exact cubic spline interpolations are used during the bootstrap procedure. This approach guarantees also enough smoothness of the curves. Smoothness of the curves is necessary in order to derive the instantaneous forward rates from the bond prices (see equation (3.11)). The resulting instantaneous forward rates are shown on the right side of figure 4.

To derive the bid, mid and ask market prices of caps from the quoted volatilities on September 16, 2013, the multiple-curve Gaussian market model is used (cf. [Mercurio \(2009\)](#)). For the volatility data from September 15, 2016, the task is more delicate since in 2016 various interest rates went deep into negative territory up to some maturity (cf. figure 4). The standard log-normal market model can no longer be used in this case. Following market practice we use a multiple-curve form of the Bachelier model in this situation to derive mid market prices of caps. Finally the market prices of the caplets are derived from the cap prices.

We highlight that the bootstrapped bond prices are of fundamental importance since they are used in three ways. First of all they are used in the process of converting the quoted volatilities into market prices. Secondly they are model inputs according to assumptions (D.1) and (MC.1) and thus go directly into the computation of model prices (see (5.10) together with (5.1)). Finally the derived tenor-dependent instantaneous forward rates enter implicitly in form of initial spreads in the assumptions that guarantee the monotonicity of the curves (see conditions (4.12) and (4.13)).

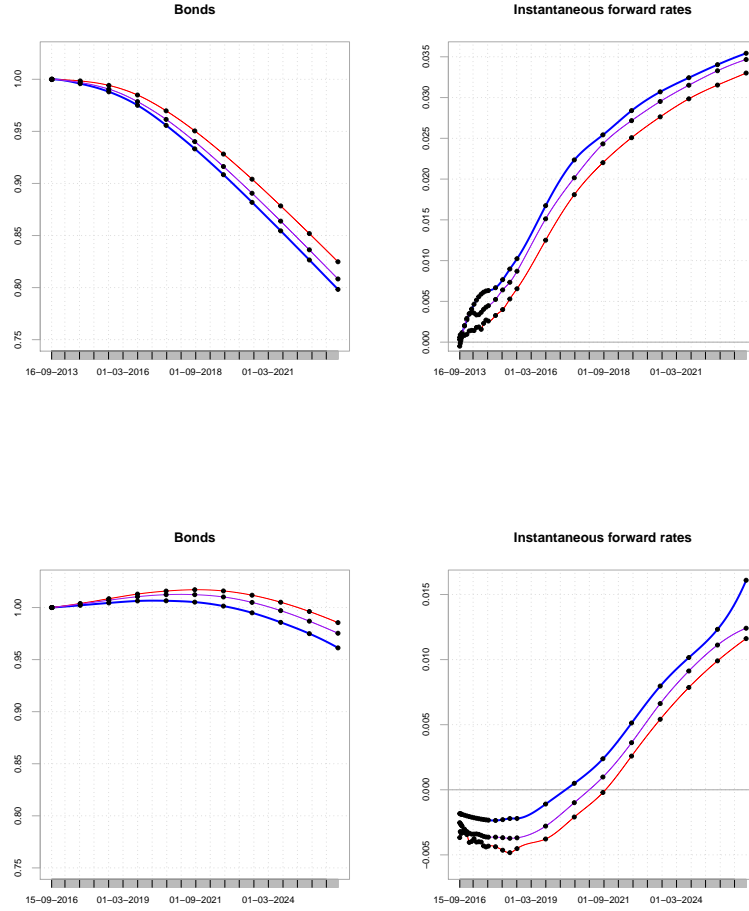


FIGURE 4. Bootstrapped term structure curves up to ten year maturities where Bloomberg data of deposits, forward rate agreements and interest rate swaps for the considered dates was used. Red represents the basic curve, purple the three months curve and blue the six months curve.

Hereafter we describe the calibration procedure. Let Θ be the set of admissible model parameters, \mathcal{T} the maturities and \mathcal{K} the strike rates of the considered caps. We minimise the sum of the squared relative errors between model and mid market cap prices

$$\sum_{T \in \mathcal{T}, K \in \mathcal{K}} \left(\frac{\text{cap model price}(\vartheta, T, K) - \text{cap market price}(T, K)}{\text{cap market price}(T, K)} \right)^2$$

with respect to $\vartheta \in \Theta$. This optimisation is done by using a randomised Powell algorithm (see Powell (1978)). Then we use the calibrated model parameters $\hat{\vartheta}$ to determine the model implied volatilities. The differences between the implied volatilities of the model prices and the implied volatilities of the mid market

prices specify the accuracy of the calibration. For the data set from September 2013 we also investigate market liquidity by taking into account bid and ask prices to calibrate the level of (il)liquidity γ (see Figure 5 for the differences between bid and ask market quotes of cap volatilities in September 2013).

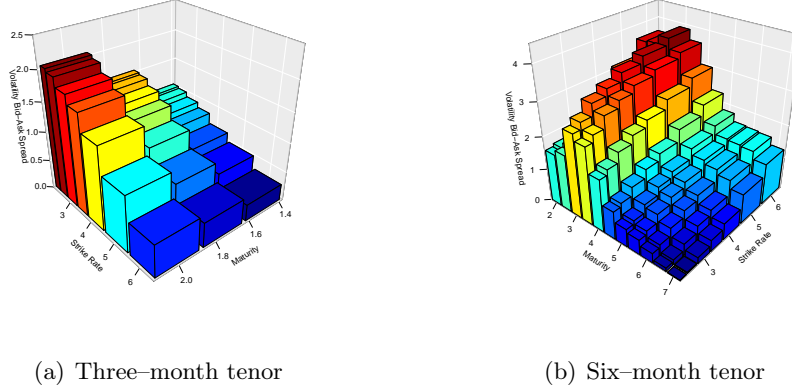


FIGURE 5. Bid-ask spreads of cap volatilities from September 16, 2013.

In order to do that we minimise the squared relative errors between bid and ask model and market caplet prices

$$\sum_{K \in \mathcal{K}} \left[\left(\frac{\text{bid caplet model price}(\gamma_T, \hat{\vartheta}, T, K) - \text{bid caplet price}(T, K)}{\text{bid caplet price}(T, K)} \right)^2 + \left(\frac{\text{ask caplet model price}(\gamma_T, \hat{\vartheta}, T, K) - \text{ask caplet price}(T, K)}{\text{ask caplet price}(T, K)} \right)^2 \right]$$

for a fixed maturity $T \in \mathcal{T}$ with respect to $\gamma_T \geq 0$. We emphasize that we used the calibrated parameter $\hat{\vartheta}$ of the first step of the calibration procedure here. This approach is justified by the following relationship

$$\begin{aligned} \text{caplet model price}(\hat{\vartheta}, T, K) &= \text{bid caplet model price}(0, \hat{\vartheta}, T, K) \\ &= \text{ask caplet model price}(0, \hat{\vartheta}, T, K). \end{aligned}$$

This procedure reduces the complexity of the calibration to bid and ask prices considerably.

6.2. Model Specification. We follow the design as exposed in section 4.2 and consider the case where $d = 3$, $l = 1$ and $m = 2$. We use a Vasicek type volatility structure of the form

$$\Sigma_1^d(t, T) = \text{sign}(a_d)(1 - e^{-a_d(T-t)})$$

and

$$\Sigma_{k+1}^{k, k-1}(t, T) = \text{sign}(a_k)(1 - e^{-a_k(T-t)})$$

with $a_d, a_k \neq 0$ for $k \in \{1, 2\}$. The other components of the volatility structure are set equal to zero. Note that to keep the number of parameters as small

as possible we consider a one parameter Vasicek structure only. As liquidity function we choose

$$l^k(t, T) = \sigma^k(T - t)e^{-b^k(T-t)}$$

where $\sigma^k, b^k > 0$ for $k \in \{1, 2\}$. In accordance with assumption (MC.DET), we require the parameters a_d, a_1 and a_2 to be restricted to values such that the volatility function Σ^2 (see definition (4.7)) is bounded in each component by a constant \hat{M} satisfying $0 < \hat{M} < M$. This guarantees the existence of the cumulant function θ at $\Sigma^2(t, T)$ for all $t, T \in [0, T^*]$ with $t \leq T$. Then, by definition, this function exists also at $\Sigma^d(t, T)$ and $\Sigma^1(t, T)$. Clearly, the parameters of the liquidity function can be restricted to a set such that this function is bounded. Observe that the considered functions satisfy assumption (VL). By (3.19) the drift terms are

$$A^{1,d}(t, T) = \theta(\Sigma^1(t, T)) - \theta(\Sigma^d(t, T)) + l^1(t, T)$$

and

$$A^{2,1}(t, T) = \theta(\Sigma^2(t, T)) - \theta(\Sigma^1(t, T)) + l^{2,1}(t, T).$$

Let N be a normal inverse Gaussian Lévy motion with parameters $\alpha, \beta, \delta, \mu$ satisfying $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$ (see e.g. Eberlein (2009)) and Z^j be a Gamma process with parameters $\alpha_j, \beta_j > 0$ for every $j \in \{1, 2, 3\}$ (see e.g. Sato (1999)). These processes are assumed to be stochastically independent. The driving process $L = (L^1, L^2, L^3)$ is then specified by $L^1 = N + Z^3$, $L^2 = -(Z^1 + Z^3)$ and $L^3 = -(Z^2 + Z^3)$. Clearly, the components of L are stochastically dependent. Furthermore the processes N, Z^1, Z^2, Z^3 do not possess a continuous martingale part, i.e. they are purely discontinuous. By using the stochastic independence of the processes N, Z^1, Z^2, Z^3 we can express the cumulant function θ of L under the measure P^d as

$$\begin{aligned} \theta(z) &= \theta^N(z_1) + \theta^{Z^1}(-z_2) + \theta^{Z^2}(-z_3) + \theta^{Z^3}(z_1 - z_2 - z_3) \\ &= \mu z_1 + \delta \sqrt{\alpha^2 - \beta^2} - \delta \sqrt{\alpha^2 - (\beta + z_1)^2} - \beta_1 \log \left(1 + \frac{z_2}{\alpha_1} \right) \\ &\quad - \beta_2 \log \left(1 + \frac{z_3}{\alpha_2} \right) - \beta_3 \log \left(1 - \frac{z_1 - z_2 - z_3}{\alpha_3} \right) \end{aligned}$$

for every $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ such that all the terms are well-defined. The existence of the cumulant process is guaranteed for any $z = (z_1, z_2, z_3) \in \mathbb{R}^3$ satisfying

- (1) $|z_1| < \min\{|\alpha - \beta|, \alpha - \beta\}$.
- (2) $z_k \in (-\alpha_k, \infty)$ for each $k \in \{2, 3\}$.
- (3) $z_1 - z_2 - z_3 < \alpha_3$.

The restrictions on the model parameters that ensure the positivity of the forward spreads can be derived from Proposition 4.3 (cf. conditions (4.12) and (4.13)). As mentioned above for the calibration to the bid and ask prices we use the family $(\Psi_\gamma^{\text{mmv}})_{\gamma \geq 0}$ of MINMAXVAR distortion functions.

6.3. Calibration Results. In the Figures 6 and 7 we show the calibration results for the data from September 16, 2013 and September 15, 2016, respectively. The surface given by the black grid represents in each case the market

volatilities for the different strikes and maturities. The model volatilities obtained from the calibration procedure are plotted as red points. The vertical lines show the distances between the two volatilities and thus the accuracy of the calibration. Note that in Figure 7 we have to consider negative strike values in addition to the strike value zero.

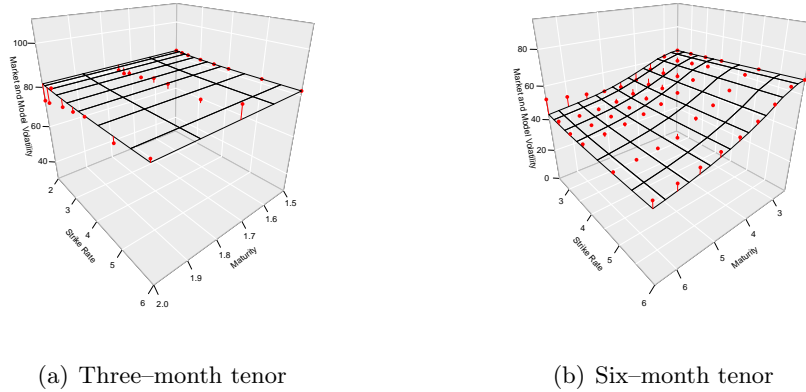


FIGURE 6. September 16, 2013.

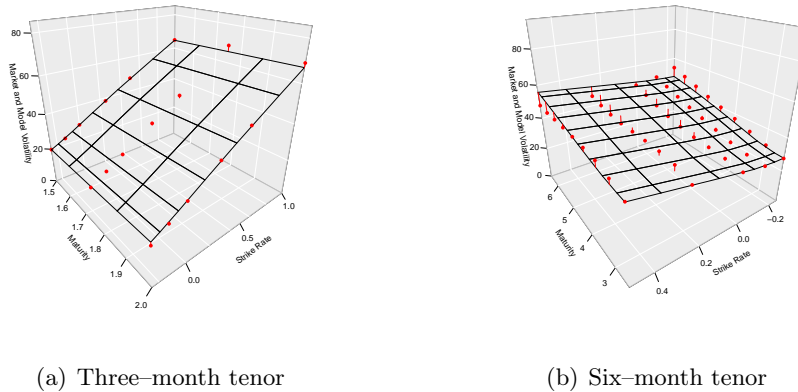


FIGURE 7. September 15, 2016.

The following Figure 8 shows the accuracy of the calibration to bid and ask caplet prices for the maturity of three years (six-month tenor). The purple line represents the mid prices, the red line the ask and the dark yellow line the bid prices. The blue resp. green line shows the calibrated model bid and ask prices.

We list the calibrated parameter values in Table 1 and 2. As far as the market liquidity parameter γ (distortion parameter) is concerned we chose the maturity of 3 years. Thus γ_3 is listed. The corresponding parameters for other maturities could be derived in the same way. As one sees from the figures the model developed here is able to reproduce market valuations with satisfactory

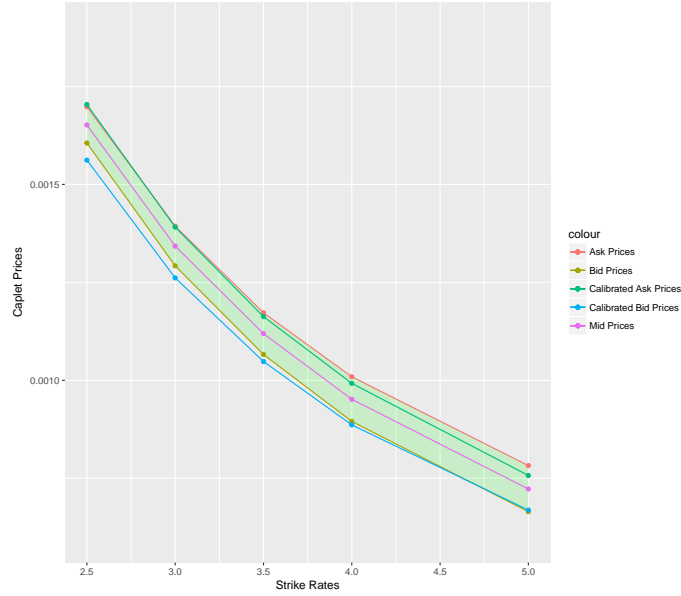


FIGURE 8. Caplet prices with maturity of 3 years on September 16, 2013.

accuracy. We highlight again that this approach handles at the same time the recent development to multiple curves, negative interest rates as well as bid and ask market prices. The deviations in terms of volatilities which can still be observed deep in or deep out of the money are explained by the numerically extremely small prices (order of magnitude 10^{-6} to 10^{-19}) on one side and the lack of liquidity in these ranges on the other side.

Calibrated Parameter			
NIG		Volatility Structure	
α	7616.2484	a_d	$-1.276678 \cdot 10^{-6}$
β	-3356.2188	a_{1d}	$-1.533437 \cdot 10^{-5}$
δ	802.4759	a_{21}	$-1.138113 \cdot 10^{-5}$
Gamma		β_2	0.0167380990
α_1	0.0083944687	α_3	0.0003146037
β_1	0.0012470450	β_3	0.0175950274
α_2	0.0010895854		
Liquidity Function		b^2	0.0362895556
b^1	0.0284998383	σ^2	0.0022651920
σ^1	0.0007718387		
Market Liquidity			
γ_3	0.011382		

TABLE 1. September 16, 2013.

Calibrated Parameter			
NIG		Volatility Structure	
α	7616.2159	a_d	$1.887946 \cdot 10^{-6}$
β	-3356.2107	a_{1d}	$-3.888884 \cdot 10^{-5}$
δ	802.4697	a_{21}	$-2.383391 \cdot 10^{-5}$
Gamma			
α_1	0.056857135	β_2	0.018390765
β_1	0.037967711	α_3	0.001729938
α_2	0.052680919	β_3	0.021147026
Liquidity Function			
b^1	0.0297731719	b^2	0.0269828887
σ^1	0.0005305050	σ^2	0.0007685246

TABLE 2. September 15, 2016.

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