RATING BASED LÉVY LIBOR MODEL

ERNST EBERLEIN AND ZORANA GRBAC

Abstract. In this paper we consider modeling of credit risk within the Libor market models. We extend the classical definition of the default-free forward Libor rate to defaultable bonds with credit ratings and develop the rating based Libor market model. As driving processes for the dynamics of the default-free and the pre-default term structure of Libor rates time-inhomogeneous Lévy processes are used. Credit migration is modeled by a conditional Markov chain, whose properties are preserved under different forward Libor measures. Conditions for absence of arbitrage in the model are derived and valuation formulae for some common credit derivatives in this setup are presented.

1. Introduction

Due to the recent credit crisis, interest rate markets have experienced some dramatic changes and a number of anomalies have appeared. In particular, the Libor rates that have always been assumed to be essentially default-free rates, in these days reflect also the credit risk of the interbanking sector (see recent papers by Mercurio (2009), Morini (2009), Henrard (2009), and many others).

However, in the present literature there exist many defaultable extensions of the Heath–Jarrow–Morton (HJM) framework for modeling of the term structure of instantaneous continuously compounded forward rates, whereas credit risk within the Libor market models seems to be far less studied. To mention just some of the papers proposing the defaultable HJM models, we begin with Bielecki and Rutkowski (2000, 2004), who introduced an extension of the Gaussian HJM model to defaultable bonds with credit migration. Eberlein and Özkan (2003) developed the defaultable HJM model with credit migration based on Lévy processes. More recently, Özkan and Schmidt (2005) and Jakubowski and Nieweglowski (2009a) consider infinite dimensional Lévy processes for credit risk modeling within the HJM framework. On the other side, the first extension of the log-normal Libor model to defaultable contracts is due to Lotz and Schlögl (2000), who use a deterministic hazard rate to model the time of default. Schönbucher (2000) extended the log-normal Libor model by adding defaultable forward Libor rates to the model for default-free Libor rates. Following his ideas, Eberlein, Kluge, and Schönbucher (2006) constructed the Lévy Libor model with default risk, driven by time-inhomogeneous Lévy processes. None of these models takes into account that in markets subject to credit risk, there usually exists a multitude of credit rating classes. A detailed account on different approaches to

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credit risk modeling can be found in Bielecki and Rutkowski (2002), Lando (2004), Duffie and Singleton (2003) and McNeil, Frey, and Embrechts (2005, Chapters 8 and 9).

In this paper we develop an arbitrage-free model for defaultable forward Libor rates related to defaultable bonds with credit ratings. As driving processes time-inhomogeneous Lévy processes are used. We call this model the *rating based Lévy Libor model*.

The modeling objects in any Libor market model are the discretely compounded forward Libor rates, whose dynamics are modeled under forward martingale measures with maturities corresponding to the tenor structure. To develop the rating based Libor model, we begin by constructing a family of default-free forward Libor rates and forward measures, according to the Lévy Libor model (Eberlein and Özkan (2005)). In addition, we model the pre-default term structure, i.e. we specify the dynamics of the forward Libor rates for every rating class. These rates are not modeled directly, instead the modeling objects are the *inter-rating spreads*, which are assumed to evolve as exponential semimartingales driven by time-inhomogeneous Lévy processes. By specifying the inter-rating spreads as positive processes, we ensure automatically that higher interest rates correspond to worse credit ratings, thus reflecting the increased investment risk.

Credit migration of a defaultable bond is modeled by a conditional Markov chain with a finite number of states representing different rating classes. This process is constructed in a canonical way by enlarging the reference probability space which carries the default-free information. Due to this canonical construction and the fact that any two forward measures are related via the Radon–Nikodym density process that is adapted to the reference filtration, we are able to show that the conditional Markov property is preserved under all forward measures. Moreover, we prove that the progressive enlargement of the (default-free) reference filtration with the natural filtration of the conditional Markov chain has the *immersion property* under all forward measures, i.e. local martingales with respect to the reference filtration remain local martingales with respect to the enlarged filtration.

The paper is structured as follows. In Section 2 we introduce the setting and the main ingredients for rating based Libor modeling, in particular we introduce the defaultable and the rating-dependent forward Libor rates and associated spreads. Section 3 presents a detailed construction of the pre-default term structure of the rating-dependent Libor rates under corresponding forward measures. The credit migration between rating classes is introduced in Section 4, using the classical conditional Markov chain approach, which is in this paper adapted to the modeling directly under forward measures. In Section 5 we put all these building blocks together and derive necessary and sufficient conditions for the absence of arbitrage in the model. Finally, Section 6 is devoted to the valuation of credit derivatives in the rating based Libor model. We derive expressions for the price of a defaultable bond and a credit default swap. Furthermore, we introduce the defaultable forward measures which are useful tools for valuation of interest rate derivatives such as forward rate agreements, swaps and caps/floors on the defaultable Libor rate. As an example we provide a formula for the defaultable Libor rate caps.

2. Definitions and notation

Let us consider a fixed time horizon T^* and a discrete tenor structure $0 = T_0 < \ldots < T_n = T^*$, where $\delta_k = T_{k+1} - T_k$, for $k = 0, \ldots, n-1$. Assume that default-free and defaultable zero coupon bonds with maturities $T_1, \ldots, T_n$ are traded in the market. We denote by $B(t, T_k)$ and $B_C(t, T_k)$ the time-t prices of a default-free and a defaultable zero coupon bond with maturity T_k, respectively. Note that $B(T_k, T_k) = 1$ and $B_C(T_k, T_k) \leq 1$, as the defaultable bond may default before maturity. Moreover, we assume that the defaultable bond is *rated*, i.e. at each time point it has a certain *credit rating* that reflects the credit quality of its issuer. The migration between various classes of a credit rating system will be described by a stochastic process C; the subscript C in the defaultable bond price emphasizes its dependence on the credit migration process.

The credit ratings are identified with elements of a finite set denoted by $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 stands for the best possible rating and K corresponds to the default event. The process C is assumed to be a continuous-time conditional Markov chain with state space $\mathcal{K}$. The default state K is an absorbing state for C and the default time τ is modeled as the first time when C reaches this state, i.e. $\tau = \inf\{t > 0 : C_t = K\}$. We assume $C_0 \neq K$ a.s.

A defaultable bond pays to its holder 1 unit of cash at maturity only if default does not occur before that date. In case of default, the holder of the bond receives a reduced payment called the recovery payment. There exist several different recovery schemes describing the amount and timing of the recovery payment (for a detailed overview see Bielecki and Rutkowski (2002, Sections 1.1.1. and 13.2.5) or McNeil, Frey, and Embrechts (2005, Section 9.4.1)). In this work we adopt the *fractional recovery of treasury value* scheme: in case of default prior to maturity, a fixed fraction of the face value of the bond is paid to the bond holder at the maturity date. This fraction depends on the rating class from which the bond has defaulted and is represented by a vector $q = (q_1, q_2, \ldots, q_{K-1})$ of recovery rates, where $q_i \in [0, 1]$ for every i. Therefore, the payoff of the defaultable bond at maturity is given by

$$\begin{aligned} B_C(T_k, T_k) &= \mathbf{1}_{\{\tau > T_k\}} + \mathbf{1}_{\{\tau \leq T_k\}} q_{C_{\tau-}} \\ &= \sum_{i=1}^{K-1} \mathbf{1}_{\{C_{T_k} = i\}} + \mathbf{1}_{\{C_{T_k} = K\}} q_{C_{\tau-}}, \end{aligned}$$

where $C_{\tau-}$ denotes the pre-default rating. The defaultable bond price process $(B_C(t, T_k))_{t \leq T_k}$ can be written as

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t = i\}} + q_{C_{\tau-}} B(t, T_k) \mathbf{1}_{\{C_t = K\}}, \tag{1}$$

where $B_i(t, T_k)$ denotes the pre-default price of the defaultable bond at time t given that the bond is in the rating class i during the time interval $[0, t]$, where $i \in \{1, \ldots, K-1\}$. We have $B_i(T_k, T_k) = 1$, for each i.

Our goal is to build up in this discrete tenor setting a model for the evolution of discretely compounded forward interest rates related to defaultable bonds. Let us first recall that the default-free *forward Libor rate* at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$ is defined as

$$L(t, T_k) := \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right).$$

In addition, we introduce the concept of a *discretely compounded forward interest rate* related to defaultable bond prices. The idea is to generalize the above definition by using the defaultable bond prices instead of the default-free bond prices. For a detailed discussion on this concept we refer to Bielecki and Rutkowski (2002, Section 14.1.4, page 431), where a defaultable forward rate agreement (FRA) which yields this rate is described. We call it the *defaultable forward Libor rate*. The default risk in this context means the risk of default of the underlying instrument. It does not mean the counterparty credit risk. The defaultable forward Libor rate is a rate that one can contract for at time $t \leq T_k$, on a defaultable forward investment of one unit of cash from T_k to T_{k+1}. The settlement scheme prescribes that default prior to the reset date T_k of the FRA results only in the reduction of the principal value and the contract then becomes default-free. The defaultable forward Libor rate is defined at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$ as

$$L_C(t, T_k) := \frac{1}{\delta_k}\left(\frac{B_C(t, T_k)}{B_C(t, T_{k+1})} - 1\right). \tag{2}$$

Note that it depends on the present state C_t of the migration process. Furthermore, making use of the bond price process $B_i(\cdot, T_k)$, we define the *forward Libor rate for rating class* i at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$ by

$$L_i(t, T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t, T_k)}{B_i(t, T_{k+1})} - 1\right), \tag{3}$$

for each $i = 1, \ldots, K-1$. The *discrete-tenor forward inter-rating spreads* between two rating classes are given by

$$H_i(t, T_k) := \frac{L_i(t, T_k) - L_{i-1}(t, T_k)}{1 + \delta_k L_{i-1}(t, T_k)}, \qquad i = 1, 2, \ldots, K-1, \tag{4}$$

where we set $L_0(\cdot, T_k) := L(\cdot, T_k)$.

Combining (3) and (4) we establish the following connection between the inter-rating spreads and the bond prices

$$H_i(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_i(t, T_k)}{B_{i-1}(t, T_k)}\frac{B_{i-1}(t, T_{k+1})}{B_i(t, T_{k+1})} - 1\right). \tag{5}$$

Remark 2.1. Observe that the quantities $H_i(t, T_k)$ represent the discrete-tenor analogs of the inter-rating spreads $g_i(t, T) - g_{i-1}(t, T)$ in the defaultable HJM framework, i.e. the differences between instantaneous continuously compounded forward rates for rating classes i and $i-1$ (see Bielecki and

Rutkowski 2002, page 406). In the HJM framework, the bond price $B_i(t, T_k)$ is given by the following formula

$$B_i(t, T_k) = \exp\left(-\int_t^{T_k} g_i(t, s)\mathrm{d}s \right),$$

where $g_i(t, s)$ is the instantaneous forward rate for the rating class $i = 1, \ldots, K-1$. Inserting this into (5) yields

$$\begin{aligned} H_i(t, T_k) &= \frac{1}{\delta_k}\left(\frac{\exp\left(\int_{T_k}^{T_{k+1}} g_i(t, s)\mathrm{d}s\right)}{\exp\left(\int_{T_k}^{T_{k+1}} g_{i-1}(t, s)\mathrm{d}s\right)} - 1 \right) \\ &= \frac{1}{\delta_k}\left(\exp\left(\int_{T_k}^{T_{k+1}} (g_i(t, s) - g_{i-1}(t, s))\mathrm{d}s \right) - 1 \right) \\ &\approx \frac{1}{\delta_k} \int_{T_k}^{T_{k+1}} (g_i(t, s) - g_{i-1}(t, s))\mathrm{d}s. \end{aligned}$$

Therefore, $H_i(t, T_k)$ can be thought of as the average inter-rating spread over the interval $[T_k, T_{k+1}]$, which explains why we refer to it as the *discrete-tenor inter-rating spread.*

3. Pre-default term structure of Libor rates

Our goal is to develop an arbitrage-free model for the evolution of defaultable forward Libor rates. In order to do so, we are going to specify the pre-default term structure of rating-dependent Libor rates $L_i(\cdot, T_k)$ for each credit rating i, where $i \in \{1, \ldots, K-1\}$. We require that $0 \leq L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k)$ to reflect the empirical fact that higher interest rates correspond to worse credit ratings, as a compensation for the increased investment risk. Making use of relation

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t, T_k)), \tag{6}$$

which follows from (4), we choose not to model the Libor rates $L_i(\cdot, T_k)$ directly. Instead, we model the forward inter-rating spreads $H_j(\cdot, T_k)$ as positive processes and therefore, by (6), ensure automatically the monotonicity of Libor rates with respect to credit ratings.

To model the default-free Libor rates $L(\cdot, T_k)$ we shall adopt the Lévy Libor model of Eberlein and Özkan (2005). Our construction is presented below in detail.

3.1. **The driving process.** Let $(\Omega, \mathcal{F} = \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis and let $(X_t)_{0 \leq t \leq T^*}$ be an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process, also known as PIIAC (process with independent increments and absolutely continuous characteristics). For a precise definition and main properties of these processes we refer the reader to Eberlein and Kluge (2006) and Eberlein, Jacod, and Raible (2005). We assume that the filtration $\mathbb{F}$ is the completed, natural filtration of X. The probability measure $\mathbb{P}_{T^*}$ plays the role of the forward measure associated with the terminal tenor date T^*. The triplet of local semimartingale characteristics of X is denoted by $(b_t, c_t, F_t^{T^*})_{0 \leq t \leq T^*}$ and we make the following

Assumption ($\mathbb{SUP}$). The triplets $(b_t, c_t, F_t^{T^*})$ satisfy

$$\sup_{0 \leq t \leq T^*} \left(|b_t| + \|c_t\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_t^{T^*}(\mathrm{d}x) \right) < \infty$$

and there exist constants $M, \varepsilon > 0$ such that

$$\sup_{0 \leq t \leq T^*} \left(\int_{|x|>1} \exp\langle u, x \rangle F_t^{T^*}(\mathrm{d}x) \right) < \infty$$

for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$.

The definition of the local characteristics of a semimartingale, as well as other results from general semimartingale theory that we use throughout the paper are taken from Jacod and Shiryaev (2003), whose notation we adopt. Other books such as Protter (2004) or Métivier (1982) can also be used as references for semimartingale theory.

Note that Assumption ($\mathbb{SUP}$) implies the existence of exponential moments of X (cf. Lemma 6 in Eberlein and Kluge (2006)). It also makes X a special semimartingale with the following canonical representation

$$X_t = \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where W^{T^*} denotes a standard Brownian motion with respect to $\mathbb{P}_{T^*}$, μ is the random measure of jumps of X and $\nu^{T^*}(\mathrm{d}s, \mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\mathrm{d}s$ is the compensator of μ. Note that we assumed that X is driftless, i.e. $b = 0$, as the drift term will be included separately in the model.

3.2. **The default-free Lévy Libor model.** Here we outline briefly the construction of the default-free Lévy Libor model. For details we refer to Eberlein and Özkan (2005). The model is driven by a time-inhomogeneous Lévy process and is built up using *backward induction* – a standard procedure for Libor market models; see the seminal papers by Miltersen, Sandmann, and Sondermann (1997), Brace, Gątarek, and Musiela (1997) and Musiela and Rutkowski (1997). The following assumptions are made:

($\mathbb{L}$.1) For every T_k there is a deterministic, Borel measurable function $\sigma(\cdot, T_k) : [0, T^*] \to \mathbb{R}_+^d$, which represents the volatility of the forward Libor rate $L(\cdot, T_k)$. We assume that

$$\sum_{k=1}^{n-1} \sigma^j(s, T_k) \leq M,$$

for all $s \in [0, T^*]$ and every coordinate $j \in \{1, \ldots, d\}$, where $M > 0$ is the constant from Assumption ($\mathbb{SUP}$). If $s > T_k$, then $\sigma(s, T_k) = 0$.

($\mathbb{L}$.2) The initial term structure $B(0,T_k)$ is strictly positive and strictly decreasing in k.

The backward induction is started by specifying the dynamics of the most distant Libor rate $L(\cdot,T_{n-1})$ under $\mathbb{P}_{T^*}$. In each step of the construction a new forward measure $\mathbb{P}_{T_{k+1}}$ is constructed and the next Libor rate is then specified under this measure as

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sigma(s,T_k)\mathrm{d}X_s^{T_{k+1}}\right) \tag{7}$$

with initial condition

$$L(0,T_k) = \frac{1}{\delta_k}\left(\frac{B(0,T_k)}{B(0,T_{k+1})} - 1\right).$$

The drift term $b^L(s,T_k)$ is chosen to make $L(\cdot,T_k)$ a $\mathbb{P}_{T_{k+1}}$- martingale, i.e.

$$\begin{aligned} b^L(s,T_k) &= -\frac{1}{2}\langle\sigma(s,T_k), c_s\sigma(s,T_k)\rangle \\ &\quad -\int_{\mathbb{R}^d}\left(e^{\langle\sigma(s,T_k),x\rangle} - 1 - \langle\sigma(s,T_k),x\rangle\right)F_s^{T_{k+1}}(\mathrm{d}x). \end{aligned} \tag{8}$$

The process $X^{T_{k+1}}$ is obtained from the driving process X in such a way that it is driftless under the forward measure $\mathbb{P}_{T_{k+1}}$ associated with the tenor date T_{k+1}. More precisely, the measure $\mathbb{P}_{T_{k+1}}$ is given by

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{B(0,T^*)}{B(0,T_{k+1})}\prod_{j=k+1}^{n-1}(1+\delta_j L(t,T_j)) = \frac{B(0,T^*)}{B(0,T_{k+1})}\frac{B(t,T_{k+1})}{B(t,T^*)}, \tag{9}$$

and $X^{T_{k+1}}$ is a special semimartingale with canonical decomposition

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^t\int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x), \tag{10}$$

where

$$W_t^{T_{k+1}} := W_t^{T^*} - \int_0^t \sqrt{c_s}\left(\sum_{j=k+1}^{n-1}\ell(s-,T_j)\sigma(s,T_j)\right)\mathrm{d}s \tag{11}$$

is a standard d-dimensional Brownian motion with respect to $\mathbb{P}_{T_{k+1}}$ and

$$\begin{aligned} \nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) &:= \prod_{j=k+1}^{n-1}\beta(s,x,T_j)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x) \\ &=: F_s^{T_{k+1}}(\mathrm{d}x)\mathrm{d}s \end{aligned} \tag{12}$$

is the $\mathbb{P}_{T_{k+1}}$-compensator of μ. Here we used for short

$$\beta(s,x,T_j) := 1 + \ell(s-,T_j)\left(e^{\langle\sigma(s,T_j),x\rangle} - 1\right) \tag{13}$$

with

$$\ell(s,T_j) := \frac{\delta_j L(s,T_j)}{1+\delta_j L(s,T_j)}. \tag{14}$$

This construction guarantees that the discounted processes $\frac{B(\cdot,T_j)}{B(\cdot,T_k)}$ are martingales with respect to the forward measure $\mathbb{P}_{T_k}$ for all j,k. The default-free Libor model is thus free of arbitrage and the time-t price π_t^Y of a contingent claim with payoff Y at maturity T_k is given by

$$\pi_t^Y = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[Y|\mathcal{F}_t].$$

3.3. **The pre-default term structure of rating-dependent Libor rates.** In this subsection we proceed by modeling the pre-default term structure of the rating-dependent Libor rates, or equivalently of the forward inter-rating spreads. We make the following additional assumptions:

($\mathbb{RL}$.1) For every rating class $i \in \{1,\dots,K-1\}$ and every maturity T_k there is a deterministic, Borel measurable function $\gamma_i(\cdot,T_k) : [0,T^*] \to \mathbb{R}_+^d$, which represents the volatility of the inter-rating spread $H_i(\cdot,T_k)$. We assume that $\gamma_i(s,T_k)=0$ for $s>T_k$ and that

$$\sum_{k=1}^{n-1}(\sigma^j(s,T_k)+\gamma_1^j(s,T_k)+\cdots+\gamma_{K-1}^j(s,T_k)) \le M,$$

for all $s\in[0,T^*]$ and every coordinate $j\in\{1,\dots,d\}$.

($\mathbb{RL}$.2) The initial term structure $L_i(0,T_k)$, $i=1,\dots,K-1$, of forward Libor rates satisfies

$$0 < L(0,T_k) \le L_1(0,T_k) \le \cdots \le L_{K-1}(0,T_k),$$

for all $k=0,1,\dots,n-1$, i.e.

$$0 < \frac{B(0,T_k)}{B(0,T_{k+1})} \le \frac{B_1(0,T_k)}{B_1(0,T_{k+1})} \le \cdots \le \frac{B_{K-1}(0,T_k)}{B_{K-1}(0,T_{k+1})}.$$

We postulate that the forward inter-rating spread $H_i(\cdot,T_k)$ for the rating class i, $i=1,\dots,K-1$, and the tenor date T_k, $k=1,\dots,n-1$, is an exponential semimartingale whose dynamics under the forward measure $\mathbb{P}_{T_{k+1}}$ is given by

$$H_i(t,T_k) = H_i(0,T_k)\exp\left(\int_0^t b^{H_i}(s,T_k)\mathrm{d}s + \int_0^t \gamma_i(s,T_k)\mathrm{d}X_s^{T_{k+1}}\right) \tag{15}$$

with initial condition

$$H_i(0,T_k) = \frac{1}{\delta_k}\left(\frac{B_i(0,T_k)B_{i-1}(0,T_{k+1})}{B_{i-1}(0,T_k)B_i(0,T_{k+1})} - 1\right).$$

The drift term $b^{H_i}(\cdot,T_k)$ will be specified in the forthcoming section. We assume that $b^{H_i}(s,T_k)=0$ for $s\ge T_k$, i.e. $H_i(t,T_k)=H_i(T_k,T_k)$ for $t\ge T_k$.

In the following theorem we deduce the dynamics of the rating-dependent forward Libor rates $L_i(\cdot,T_k)$ under the corresponding forward measures, which is implied by specification (15) .

Theorem 3.1. *Assume that* ($\mathbb{L}$.1), ($\mathbb{L}$.2), ($\mathbb{RL}$.1) *and* ($\mathbb{RL}$.2) *are in force. For each* $k=1,\dots,n-1$, *let* $L(\cdot,T_k)$ *and* $H_i(\cdot,T_k)$, $i\in\{1,\dots,K-1\}$, *be given by* (7) *and* (15), *respectively. Then:*

(a) *The rating-dependent forward Libor rates $L_i(\cdot, T_k)$ satisfy for every T_k and any $t \leq T_k$*

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k),$$

i.e. Libor rates are monotone with respect to credit ratings.

(b) *The dynamics of the Libor rate $L_i(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is given by*

$$L_i(t, T_k) = L_i(0, T_k) \exp\left(\int_0^t b^{L_i}(s, T_k) \mathrm{d}s + \int_0^t \sqrt{c_s} \sigma_i(s, T_k) \mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} S_i(s, x, T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \right), \tag{16}$$

where

$$\begin{aligned} \sigma_i(s, T_k) &:= \ell_i(s-, T_k)^{-1} \Big(\ell_{i-1}(s-, T_k) \sigma_{i-1}(s, T_k) + h_i(s-, T_k) \gamma_i(s, T_k) \Big) \\ &= \ell_i(s-, T_k)^{-1} \Big[\ell(s-, T_k) \sigma(s, T_k) + \sum_{j=1}^{i} h_j(s-, T_k) \gamma_j(s, T_k) \Big] \end{aligned} \tag{17}$$

represents the volatility of the Brownian part and

$$S_i(s, x, T_k) := \ln \Big(1 + \ell_i(s-, T_k)^{-1} (\beta_i(s, x, T_k) - 1) \Big)$$

controls the jump size. Here we have used

$$h_i(s, T_k) := \frac{\delta_k H_i(s, T_k)}{1 + \delta_k H_i(s, T_k)}, \tag{18}$$

$$\ell_i(s, T_k) := \frac{\delta_k L_i(s, T_k)}{1 + \delta_k L_i(s, T_k)}, \tag{19}$$

and

$$\begin{aligned} \beta_i(s, x, T_k) &:= \beta_{i-1}(s, x, T_k) \Big(1 + h_i(s-, T_k)(e^{\langle \gamma_i(s, T_k), x \rangle} - 1) \Big) \\ &= \Big(1 + \ell(s-, T_k)(e^{\langle \sigma(s, T_k), x \rangle} - 1) \Big) \\ &\quad \times \prod_{j=1}^{i} \Big(1 + h_j(s-, T_k)(e^{\langle \gamma_j(s, T_k), x \rangle} - 1) \Big). \end{aligned} \tag{20}$$

The drift term in (16) *is given by*

$$
\begin{aligned}
b^{L_i}(s,T_k) \;=\; & \ell_i^{-1}(s-,T_k)\sum_{j=1}^{i} h_j(s-,T_k) b^{H_j}(s,T_k) \\
& -\frac{1}{2}\ell_i^{-1}(s-,T_k)\ell(s-,T_k)\|\sqrt{c_s}\sigma(s,T_k)\|^2 \\
& +\frac{1}{2}\ell_i^{-1}(s-,T_k)\sum_{j=1}^{i}\Big((h_j(s-,T_k)-h_j(s-,T_k)^2)\|\sqrt{c_s}\gamma_j(s,T_k)\|^2\Big) \\
& +\frac{1}{2}(\ell_i(s-,T_k)-1)\|\sqrt{c_s}\sigma_j(s,T_k)\|^2 \\
& +\int_{\mathbb{R}^d}\Big(S_i(s,x,T_k)-\ell_i^{-1}(s-,T_k)\Big(\beta(s,x,T_k)-1 \\
& \qquad +\sum_{j=1}^{i} h_j(s-,T_k)\langle\gamma_j(s,T_k),x\rangle\Big)\Big)F_s^{T_{k+1}}(\mathrm{d}x). \qquad (21)
\end{aligned}
$$

Proof: The proof is deferred to the appendix. □

Remark 3.2. Let us compare the expressions for the dynamics of the rating-dependent Libor rate $L_i(\cdot,T_k)$ and the dynamics of the default-free Libor rate $L(\cdot,T_k)$ under $\mathbb{P}_{T_{k+1}}$. Note that (7) can be written as

$$
\begin{aligned}
L(t,T_k) \;=\; & L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s+\int_0^t\sqrt{c_s}\sigma(s,T_k)\mathrm{d}W_s^{T_{k+1}}\right. \qquad (22) \\
& \left.+\int_0^t\int_{\mathbb{R}^d} S(s,x,T_k)(\mu-\nu^{T_{k+1}})(\mathrm{d}t,\mathrm{d}x)\right),
\end{aligned}
$$

where

$$
S(s,x,T_k) := \ln\Big(1+\ell(s-,T_k)^{-1}(\beta(s,x,T_k)-1)\Big) = \langle\sigma(s,T_k),x\rangle,
$$

with $\beta(s,x,T_k)$ defined in (13) and $\ell(s,T_k)$ in (14). Therefore, we observe that the equation (16) describing the dynamics of the Libor rate for the rating i is of the same form as the default-free Libor rate dynamics (22), naturally with the appropriate specifications of $\sigma_i(\cdot,T_k)$ and $S_i(\cdot,\cdot,T_k)$.

In Figure 1 we represent graphically the connections between different rating-dependent Libor rates.

Having established the pre-default term structure, the next step is to study migration between different rating classes in order to obtain an arbitrage-free model for the evolution of defaultable Libor rates. The credit migration process is assumed to be a canonically constructed conditional Markov process C, which we study in the next section.

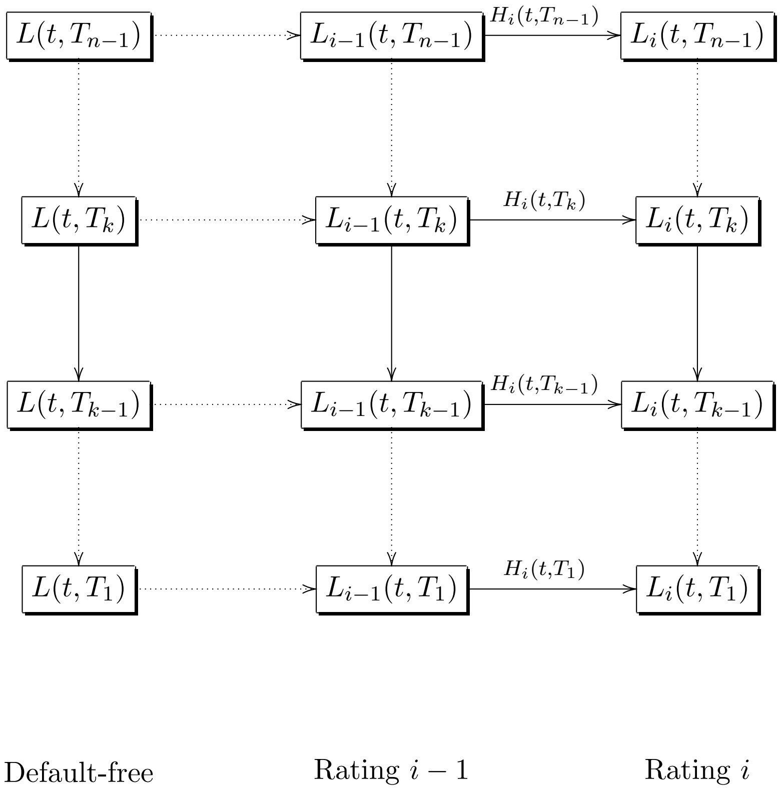

FIGURE 1. Connection between subsequent Libor rates

4. Credit migration under forward measures

4.1. **Conditional Markov chains and their main properties.** Let us describe the appropriate probabilistic setting required for a model that allows credit migration. As pointed out in the introduction, rating classes are typically identified with elements of a finite set, denoted by $\mathcal{K}$. We assume that $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 denotes the best possible rating and K corresponds to the default event. In credit risk theory credit migration is usually modeled by a conditional Markov chain C with continuous time parameter and the state space $\mathcal{K}$. We adopt the same idea here. Recall that in this setting the default state K is an absorbing state of C and the default time τ is modeled as the first hitting time of this state, i.e.

$$\tau = \inf \{t > 0 : C_t = K\}.$$

To construct such a process, we are going to use the canonical construction based on a given reference filtration $\mathbb{F}$ and a stochastic infinitesimal generator Λ. This construction can be found in Bielecki and Rutkowski (2002) or Eberlein and Özkan (2003). Our underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P}_{T^*})$ with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$, which is generated by the time-inhomogeneous Lévy process X driving the default-free and the predefault term structure of Libor rates. Furthermore, let $\Lambda = (\Lambda(t))_{0 \le t \le T^*}$ be a matrix-valued stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{23}$$

where $\lambda_{ij} : \Omega \times [0, T^*] \to \mathbb{R}_+$ are bounded, $\mathbb{F}$-progressively measurable stochastic processes. For every $i, j \in \mathcal{K}$, $i \neq j$, the processes λ_{ij} are non-negative and $\lambda_{ii}(t) = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_{ij}(t)$, for $t \in [0, T^*]$. The last row of Λ contains only zeros since the state K is an absorbing state of C.

Let $\mu = (\mu_1, \dots, \mu_K)$ be a probability distribution on $\mathcal{K}$, which is the initial distribution of the process C, i.e. the distribution of C_0. In credit risk applications μ is a one-point mass on the rating class observed at time $t = 0$.

The process C is constructed from the initial distribution and the $\mathbb{F}$-adapted infinitesimal generator Λ by enlarging the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P}_{T^*})$ to a probability space denoted in the sequel by $(\widetilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*})$. The new probability space is obtained as a product space of the underlying one with a probability space supporting the initial distribution μ of C and a probability space supporting a sequence of uniformly distributed random variables, which control, together with the entries of the infinitesimal generator Λ, the laws of jump times $(\tau_n)_{n \in \mathbb{N}}$ of C and jump heights. Note that by using a product space we obtain a certain independence which will be crucial for many properties of C.

We denote by $\mathbb{F}$ its trivial extension from the original probability space $(\Omega, \mathcal{F}, \mathbb{P}_{T^*})$ to $(\widetilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*})$. Moreover, all stochastic processes are extended to the new probability space by retaining their names and setting for example $X(\widetilde{\omega}) := X(\omega)$, and similarly for other processes.

Remark 4.1. We recall that this canonical construction is a generalization of the classical Cox construction, which is used in credit risk theory to model the default time with a given $\mathbb{F}$-intensity λ (see Jeanblanc and Rutkowski (2000) or Bielecki and Rutkowski (2002)). Indeed, when $K = 2$, the conditional Markov chain has only two states which have the interpretation of a non-default state 1 and the default state 2 and the above construction becomes the Cox construction of a default time with intensity $\lambda(t) = -\lambda_{11}(t)$.

The process C obtained by the canonical construction is an $\mathbb{F}$-*conditional Markov chain*, i.e. if we denote by $\mathbb{F}^C$ the natural filtration of the process C, the *conditional Markov property*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)] \tag{24}$$

is satisfied for every $0 \leq t \leq s \leq T^*$ and any function $h : \mathcal{K} \to \mathbb{R}$.

It is important to mention that the process C possesses also the following property:

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)], \tag{25}$$

for every $0 \leq t \leq s \leq u \leq T^*$ and any function $h : \mathcal{K} \to \mathbb{R}$.

We will refer to (25) too as the *conditional Markov property*, even though it is a stronger property which implies property (24). Note that in general, not all conditional Markov chains possess property (25), but the canonically constructed process C satisfies both (24) and (25) (compare Bielecki and Rutkowski (2002), formula (11.47) and comments thereafter).

For every fixed $t \geq 0$, the σ-algebra $\sigma(C_t)$ is finitely generated, as C_t takes its values in a finite set $\mathcal{K}$. This enables us to establish the following decomposition of conditional expectations (it is a counterpart in the conditional Markov setting of Corollary 5.1.1 in Bielecki and Rutkowski (2002) or Lemma 1 in Elliot, Jeanblanc, and Yor (2000)):

Lemma 4.2. *If Y is a $\mathcal{G}$-measurable random variable, then*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s \vee \sigma(C_t)] = \sum_{i=1}^{K} \mathbf{1}_{\{C_t=i\}} \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[Y\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s]}, \tag{26}$$

for every $0 \leq t \leq s$ and $i \in \mathcal{K}$.

Proof: The proof is straightforward combining the definition of conditional expectation and the aforementioned fact that $\sigma(C_t)$ is generated by atoms $\{C_t = j\}$, $j = 1, \dots, K$. Note that the right-hand side in (26) is well-defined since $\left\{\mathbb{E}_{Q_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s] = 0\right\} \subseteq \{C_t = i\}^c$ $\mathbb{Q}_{T^*}$-a.s. (cf. Last and Brandt (1995, Lemma A3.17)).

The conditional expectation on the left-hand side in (26) is equal to the right-hand side if and only if

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}Y] &= \mathbb{E}_{\mathbb{Q}_{T^*}}\left[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}\sum_{i=1}^{K}\mathbf{1}_{\{C_t=i\}}\frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}Y|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s]}\right]\\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}\left[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}\frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}Y|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}|\mathcal{F}_s]}\right],\end{aligned}$$

for every $F \in \mathcal{F}_s$ and every $\{C_t = j\}$, $j \in \mathcal{K}$, since the σ-algebra $\mathcal{F}_s \vee \sigma(C_t)$ is generated by finite intersections $F \cap \{C_t = j\}$. We have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{T^*}}\left[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}\frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}Y|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}|\mathcal{F}_s]}\right]\\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}\left[\mathbb{E}_{\mathbb{Q}_{T^*}}\left[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}\frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}Y|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}|\mathcal{F}_s]}\Big|\mathcal{F}_s\right]\right]\\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}\left[\frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}Y|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}|\mathcal{F}_s]}\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=j\}}|\mathcal{F}_s]\right]\\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}Y|\mathcal{F}_s]]\\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_F\mathbf{1}_{\{C_t=j\}}Y],\end{aligned}$$

which is what we had to show. □

In view of this result, the conditional Markov property takes the following form:

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \sum_{i=1}^{K} \mathbf{1}_{\{C_t=i\}} \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}, \tag{27}$$

for every $t \leq s$ and any function $h : \mathcal{K} \to \mathbb{R}$.

Let us now examine the most important properties of the process C. Due to the canonical construction, each random state C_s is actually influenced by information from the filtration $\mathbb{F}$ only up to time s. A precise formulation of this property is contained in the following proposition.

Proposition 4.3. *Let C be a conditional Markov chain obtained by the canonical construction. Then:*

(a) *For every $0 \le t \le s \le u \le T^*$ and $j \in \mathcal{K}$*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_s=j\}}|\mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_s=j\}}|\mathcal{F}_s \vee \mathcal{F}_t^C].$$

(b) *More generally, for any $0 \le t \le s_1 \le s_2 \le u \le T^*$ and $j_1, j_2 \in \mathcal{K}$*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_{s_1}=j_1\}}\mathbf{1}_{\{C_{s_2}=j_2\}}|\mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_{s_1}=j_1\}}\mathbf{1}_{\{C_{s_2}=j_2\}}|\mathcal{F}_{s_2} \vee \mathcal{F}_t^C].$$

Proof: The proof relies on the canonical construction of C and its jump times and the independence we mentioned earlier. We refer to Grbac (2010, Proposition 2.18) for details. □

The σ-algebra $\mathcal{F}_t^C$ can be omitted in the above results, as the following corollary shows.

Corollary 4.4. *Let C be a canonically constructed conditional Markov chain. Then:*

(a) *For every $0 \le s \le u \le T^*$ and $j \in \mathcal{K}$*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_s=j\}}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_s=j\}}|\mathcal{F}_s].$$

(b) *For any $0 \le s_1 \le s_2 \le u \le T^*$ and $j_1, j_2 \in \mathcal{K}$*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_{s_1}=j_1\}}\mathbf{1}_{\{C_{s_2}=j_2\}}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_{s_1}=j_1\}}\mathbf{1}_{\{C_{s_2}=j_2\}}|\mathcal{F}_{s_2}].$$

Proof: This follows by inserting $t = 0$ into the previous proposition and applying Lemma 4.2. The independence between the initial distribution μ of C and $\mathbb{F}$ implies

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_0=i\}}|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_0=i\}}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_0=i\}}] = \mu_i,$$

which yields both claims. □

In the remainder of the subsection we study the transition probabilities of a canonically constructed conditional Markov chain C. It turns out that the usual properties of transition probabilities of an ordinary Markov chain, such as the Chapman–Kolmogorov equation, will remain valid, but will be expressed in terms of $\mathbb{F}$-conditional expectations.

To fix the notation, let us denote

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s; C_t = i] := \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[Y\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_s]},$$

where Y is a $\mathcal{G}$-measurable random variable and $0 \le t \le s \le T^*$. Hence, by Lemma 4.2, $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s; C_t = i]$ is an $\mathcal{F}_s$-measurable random variable that agrees with $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s \vee \sigma(C_t)]$ on the set $\{C_t = i\}$. Bielecki and Rutkowski (2002) use a slightly different notation $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s \vee \{C_t = i\}]$ instead, but we prefer the above notation to make a clear distinction from conditioning with respect to the smallest σ-algebra generated by $\mathcal{F}_s$ and the set $\{C_t = i\}$.

For $Y = \mathbf{1}_{\{C_s=j\}}$, the expression

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_s=j\}}|\mathcal{F}_s; C_t = i] = \mathbb{Q}_{T^*}(C_s = j|\mathcal{F}_s; C_t = i)$$

denotes the *conditional probability with respect to* $\mathbb{F}$ of the process C being in state j at time s if it was in state i at time t.

Definition 4.5. The $\mathbb{F}$-*conditional transition probability matrix* of C is defined as

$$P(t,s) = [p_{ij}(t,s)]_{i,j=1,\dots,K}, \qquad 0 \le t \le s \le T^*,$$

where

$$p_{ij}(t,s) = \mathbb{Q}_{T^*}(C_s = j|\mathcal{F}_s; C_t = i) = \mathbb{Q}_{T^*}(C_s = j|\mathcal{F}_{T^*}; C_t = i). \tag{28}$$

Note that the second equality in (28) follows from Corollary 4.4.

Remark 4.6. The process C together with the family of stochastic matrices $P(t,s)_{0\le t\le s\le T^*}$ is an $\mathbb{F}$-*doubly stochastic Markov chain* in the sense of Definition 2.1 introduced in Jakubowski and Nieweglowski (2009b); see Remark 2.16 and Theorem 2.14 in their paper. $\mathbb{F}$-doubly stochastic Markov chains form a subclass of the class of $\mathbb{F}$-conditional Markov chains and are particularly suitable for applications in credit risk. Processes that are typically used for this purpose such as an ordinary Markov chain, a compound Poisson process, a Cox process and a canonically constructed conditional Markov chain belong to this class. The main properties of $\mathbb{F}$-doubly stochastic Markov chains are studied in Jakubowski and Nieweglowski (2009b).

We conclude this section by formulating the appropriate $\mathbb{F}$-conditional versions of the Chapman–Kolmogorov equation and the forward Kolmogorov equation for C.

Proposition 4.7. *Let C be a canonically constructed $\mathbb{F}$-conditional Markov chain and $(P(t,s))_{0\le t\le s\le T^*}$ its family of conditional transition probability matrices. Then:*

(a) $P(\cdot,\cdot)$ *satisfies the* $\mathbb{F}$-conditional Chapman–Kolmogorov equation

$$P(t,s) = P(t,u)P(u,s), \qquad t \le u \le s. \tag{29}$$

(b) $P(\cdot,\cdot)$ *satisfies the* $\mathbb{F}$-conditional forward Kolmogorov equation

$$\frac{\mathrm{d}P(t,s)}{\mathrm{d}s} = P(t,s)\Lambda(s), \quad P(t,t) = Id.$$

Proof: Part (a) follows from Theorem 2.7 and part (b) from Definition 2.8 in Jakubowski and Nieweglowski (2009b).

Clearly, both claims can also be established directly, without the notion of an $\mathbb{F}$-doubly stochastic Markov chain; see Grbac (2010, Proposition 2.24). □

4.2. **The immersion property in the conditional Markov chain setting.** Starting from the reference filtration $\mathbb{F}$ and constructing a conditional Markov chain C, we obtain an enlargement of $\mathbb{F}$ that we denote by $\mathbb{G}$, where $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T^*}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^C_t$ (and completed). A natural question, that we examine in this subsection, is if the immersion property (or a so-called $(\mathcal{H})$-hypothesis) is satisfied for this enlargement, i.e. if $\mathbb{F}$-(local) martingales remain $\mathbb{G}$-(local) martingales. In case when a conditional Markov chain is constructed canonically, the answer to this question is affirmative, as we show in the sequel.

Proposition 4.8. *Let C be a canonically constructed conditional Markov chain. Then for every $B \in \mathcal{F}_s^C$ and $0 \leq s \leq u \leq T^*$*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_s].$$

Proof: We must verify the claim for any $B \in \mathcal{F}_s^C = \sigma(\bigcup_{0 \leq t \leq s} \sigma(C_t))$. Since $\mathcal{F}_s^C$ is generated by finite intersections of sets of the form $\{C_t = j\}$, $j \in \mathcal{K}$, $t \in [0,s]$, it is enough to prove the claim for an arbitrary set of the form $\{C_{s_1} = i\} \cap \{C_{s_2} = j\}$, for some $s_1 \leq s_2 \leq s$ and $i, j \in \mathcal{K}$. But this is exactly Corollary 4.4 and now by standard arguments (monotone class theorem) the claim follows for any $B \in \mathcal{F}_s^C$. □

Theorem 4.9 (Hypotheses $(\mathcal{H}1)$, $(\mathcal{H}2)$ and $(\mathcal{H}3)$)**.** *Let C be a canonically constructed conditional Markov chain. Furthermore, let X be a bounded $\mathcal{F}_{T^*}$-measurable random variable and Y a bounded $\mathcal{F}_s^C$-measurable random variable, $s \in [0, T^*]$. Then the following three equivalent statements hold:*

($\mathcal{H}1$) $\mathbb{E}_{\mathbb{Q}_{T^*}}[XY|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\, \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s]$,
i.e. the σ-fields $\mathcal{F}_{T^*}$ and $\mathcal{F}_s^C$ are conditionally independent given $\mathcal{F}_s$.
($\mathcal{H}2$) $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s]$.
($\mathcal{H}3$) $\mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s \vee \mathcal{F}_s^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]$.

Proof: Making use of Proposition 4.8 we see that $(\mathcal{H}2)$ holds for $Y = \mathbf{1}_B$, where $B \in \mathcal{F}_s^C$. This implies that it is also true for any bounded $\mathcal{F}_s^C$-measurable Y since every $\mathcal{F}_s^C$-measurable random variable can be written as a limit of a sequence of elementary $\mathcal{F}_s^C$-measurable random variables. Applying the dominated convergence theorem for conditional expectation we establish $(\mathcal{H}2)$.

The equivalences between $(\mathcal{H}1)$, $(\mathcal{H}2)$ and $(\mathcal{H}3)$ follow from Theorem II.45 in Dellacherie and Meyer (1978). □

Consequently, we obtain the following result.

Theorem 4.10 ($(\mathcal{H})$- hypothesis)**.** *Let $(\widetilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*})$ be a probability space with a given filtration $\mathbb{F}$ and let C be a canonically constructed $\mathbb{F}$-conditional Markov chain. Then the immersion property holds for the filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^C$.*

Proof: Follows directly from the statements in Theorem 4.9 and the well-known fact that they are equivalent to the immersion property of the enlargement (see for example Brémaud and Yor (1978) or Jeanblanc and Rutkowski (2000)). □

Remark 4.11. Since $\sigma(C_s) \subseteq \mathcal{F}_s^C$, it immediately follows that all statements from Theorem 4.9 remain valid when we replace $\mathcal{F}_s^C$ with $\sigma(C_s)$.

The immersion property of the enlargement obviously implies that every $\mathbb{F}$-semimartingale X remains a $\mathbb{G}$-semimartingale. Moreover, if X is a time-inhomogeneous Lévy process with respect to $\mathbb{F}$, then it remains a time-inhomogeneous Lévy process with respect to $\mathbb{G}$.

Proposition 4.12. *Suppose that the assumptions of Theorem 4.10 are in force and let X be a time-inhomogeneous Lévy process with respect to the filtration $\mathbb{F}$ with the triplet of predictable characteristics (B, C, ν). Then X remains a time-inhomogeneous Lévy process with respect to $\mathbb{G}$ with the same predictable characteristics.*

Proof: Clearly, X is a $\mathbb{G}$-semimartingale such that $X_0 = 0$. According to Theorem II.2.42 and Corollary II.2.48 in Jacod and Shiryaev (2003), (B, C, ν) is its triplet of predictable characteristics if and only if the process

$$\frac{e^{\mathrm{i}\langle u, X\rangle}}{G(\mathrm{i}u)} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}),$$

with G defined in Theorem II.2.27 in Jacod and Shiryaev (2003). By assumption, (B, C, ν) is the triplet of predictable characteristics of X with respect to $\mathbb{F}$; hence, this process is in $\mathcal{M}_{\mathrm{loc}}(\mathbb{F})$. Due to the immersion property it is also in $\mathcal{M}_{\mathrm{loc}}(\mathbb{G})$, and therefore (B, C, ν) is the triplet of predictable characteristics with respect to $\mathbb{G}$.

Moreover, X has independent increments with respect to $\mathbb{G}$ if (B, C, ν) is deterministic – which is the case because its increments are independent with respect to $\mathbb{F}$ (cf. Jacod and Shiryaev (2003, Theorem II.4.15)). Finally, equation (4.16) from the same theorem shows that the characteristic function of X_t takes the form given in the definition of a time-inhomogeneous Lévy process in Eberlein and Kluge (2006, Section 2.1) and the proof is completed. □

We conclude this section by introducing a certain generalization of the $(\mathcal{H})$-hypothesis in the conditional Markov setting.

Definition 4.13. Let C be an $\mathbb{F}$-conditional Markov chain with the natural filtration $\mathbb{F}^C$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^C$. Furthermore, for a fixed $r \geq 0$, denote for every $s \geq r$

$$\tilde{\mathcal{F}}_s := \mathcal{F}_s \vee \mathcal{F}_r^C \qquad \text{and} \qquad \tilde{\mathbb{F}}^r := (\tilde{\mathcal{F}}_s)_{s \geq r}.$$

We say that the enlargement $\mathbb{G}$ of $\tilde{\mathbb{F}}^r$ satisfies the $(\mathcal{H}^r)$-*hypothesis* if

$(\mathcal{H}^r)$ Every $(\tilde{\mathcal{F}}_s)_{s \geq r}$-local martingale is a $(\mathcal{G}_s)_{s \geq r}$-local martingale.

Remark 4.14. Note that $\mathbb{G}$ is indeed an enlargement of $\tilde{\mathbb{F}}^r$ since for every $s \geq r$ we have

$$\tilde{\mathcal{F}}_s = \mathcal{F}_s \vee \mathcal{F}_r^C \subset \mathcal{F}_s \vee \mathcal{F}_s^C = \mathcal{G}_s.$$

Proposition 4.15. *Let C be a canonically constructed $\mathbb{F}$-conditional Markov chain. Then the $(\mathcal{H}^r)$-hypothesis holds between the filtrations $\tilde{\mathbb{F}}^r$ and $\mathbb{G}$.*

Moreover, hypothesis $(\mathcal{H}^r)$ implies hypothesis $(\mathcal{H})$ for this enlargement. If, in addition, we assume that the reference filtration $\mathbb{F}$ is a natural filtration of a time-inhomogeneous Lévy process, then the converse statement is also true, i.e. $(\mathcal{H})$ implies $(\mathcal{H}^r)$, for every $r \geq 0$.

Proof: Let us fix an arbitrary $r \geq 0$ and show that $(\mathcal{H}^r)$ holds for $\tilde{\mathbb{F}}^r$ and $\mathbb{G}$.

Note that for every $s \geq r$

$$\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{F}_s^C = \tilde{\mathcal{F}}_s \vee \mathcal{F}_s^C,$$

and therefore, $(\mathcal{H}^r)$ is equivalent to showing that

$$\mathbb{E}[Y|\tilde{\mathcal{F}}_{T^*}] = \mathbb{E}[Y|\tilde{\mathcal{F}}_s],$$

for every bounded $\mathcal{F}_s^C$-measurable random variable. By definition of $\tilde{\mathbb{F}}^r$, this is actually

$$\mathbb{E}[Y|\mathcal{F}_{T^*} \vee \mathcal{F}_r^C] = \mathbb{E}[Y|\mathcal{F}_s \vee \mathcal{F}_r^C].$$

Now we are done since this follows from Proposition 4.3 exactly in the same way as $(\mathcal{H}2)$ follows from Corollary 4.4.

Moreover, the implication $(\mathcal{H}^r) \Rightarrow (\mathcal{H})$ is obvious, since it can be easily proved that every $\mathbb{F}$-local martingale is an $\mathbb{F} \vee \mathcal{F}_r^C$-local martingale, which is then, by hypothesis $(\mathcal{H}^r)$, a $\mathbb{G}$-local martingale. We simply have to note that, due to the canonical construction of C, we have

$$\mathcal{F}_r^C \subset \mathcal{F}_r \vee \mathcal{E}_r, \tag{30}$$

where $\mathcal{E}_r$ is a σ-algebra independent from $\mathbb{F}$. This means that $\mathcal{F}_s \vee \mathcal{F}_r^C = \mathcal{F}_s \vee \mathcal{E}_r$, for $s \geq r$. Thus, the enlargement of $\mathbb{F}$ to $\tilde{\mathbb{F}}^r$, given as $\mathbb{F} \vee \mathcal{F}_r^C = \mathbb{F} \vee \mathcal{E}_r$, clearly possesses the immersion property (this is simply the initial enlargement of $\mathbb{F}$ with an independent σ-algebra $\mathcal{E}_r$).

Conversely, let us assume that the filtration $\mathbb{F}$ is the natural filtration of a time-inhomogeneous Lévy process X and let us prove that $(\mathcal{H})$ implies $(\mathcal{H}^r)$, for every $r \geq 0$.

The proof relies on the representation theorem for local martingales and the fact that X remains a time-inhomogeneous Lévy process with the same characteristics also with respect to the enlarged filtration $\mathbb{G}$.

Let us fix some $r \geq 0$ and establish $(\mathcal{H}^r)$. As we pointed out above, the filtration $\tilde{\mathbb{F}}^r$ may be thought of as the initial enlargement of $\mathbb{F}$ with an independent σ-algebra $\mathcal{E}_r$. This allows us to make use of the representation theorem for local martingales in filtrations generated by processes with independent increments (Jacod and Shiryaev 2003, Theorem III.4.34). More precisely, if M^r is a local martingale with respect to $\tilde{\mathbb{F}}^r$, it can be written as

$$M^r = M_r^r + H^r \cdot X^c + W^r * (\mu^X - \nu^X),$$

for some processes $H^r \in L^2_{\text{loc}}(X^c)$ and $W^r \in G_{\text{loc}}(\mu)$, where X^c is the continuous martingale part of X and μ^X the random measure of jumps of X with compensator ν^X (for the definitions of $L^2_{\text{loc}}(X^c)$ and $G_{\text{loc}}(\mu)$ see pages 48 and 72 of Jacod and Shiryaev (2003)). Hence, thanks to the Proposition 4.12, which ensures that both the continuous martingale part and the purely discontinuous martingale part of X remain the same with respect to $\mathbb{G}$, M^r is a $\mathbb{G}$-local martingale and $(\mathcal{H}^r)$ is verified. □

4.3. **Conditional Markov property under forward Libor measures.** This subsection is devoted to the study of the conditional Markov property under different forward Libor measures. To develop a rating based Lévy Libor model with a migration process which is a canonically constructed conditional Markov process, we have to know how changes of the forward measure affect this process and the immersion property of the associated enlargement.

Recall that the conditional Markov chain C is constructed starting with $(\Omega, \mathcal{F}, \mathbb{P}_{T^*})$ and using the canonical construction. The enlarged probability space on which C is defined is denoted by $(\widetilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*})$. Moreover, we denote by $\mathbb{Q}_{T_k}$ the forward measure defined on $(\widetilde{\Omega}, \mathcal{G}_{T_k})$ which is obtained from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_k}$ is constructed from $\mathbb{P}_{T^*}$. The Radon–Nikodym derivative of $\mathbb{Q}_{T_k}$ with respect to $\mathbb{Q}_{T^*}$ is therefore

$$\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}} =: \psi^k, \tag{31}$$

where ψ^k is a positive, $\mathcal{F}_{T_k}$-measurable random variable with expectation 1 (more precisely it is given in equation (9)).

By construction, C satisfies the $\mathbb{F}$-conditional Markov property under the terminal forward measure $\mathbb{Q}_{T^*}$. In the theorem below we show that the conditional Markov property of C is preserved under all forward measures $\mathbb{Q}_{T_k}$, $k = 1, \dots, n-1$.

Theorem 4.16. *Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$ and let $\mathbb{Q}_{T_k}$, $k = 1, \dots, n-1$, be the forward measures given by* (31)*. Then C is a conditional Markov chain with respect to every $\mathbb{Q}_{T_k}$, $k = 1, \dots, n-1$, i.e.*

$$\mathbb{E}_{\mathbb{Q}_{T_k}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T_k}}[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)], \tag{32}$$

for all $0 \le t \le s \le u \le T_k$ and any function $h : \mathcal{K} \to \mathbb{R}$.

Furthermore, the matrices of conditional transition probabilities under $\mathbb{Q}_{T^*}$ and $\mathbb{Q}_{T_k}$ are the same, i.e.

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s), \tag{33}$$

for all $i, j \in \mathcal{K}$ and $0 \le t \le s \le T_k$, where $p_{ij}^{\mathbb{Q}_{T_k}}(t,s)$ and $p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$ are defined by (28)*.*

Proof: Let us fix a $k \in \{1, 2, \dots, n-1\}$ and establish the conditional Markov property (32). By assumption C is a conditional Markov chain under $\mathbb{Q}_{T^*}$.

Therefore, we obtain the following sequence of equalities:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}_{T_k}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C] &= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u \vee \mathcal{F}_t^C]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k h(C_s)|\mathcal{F}_u \vee \mathcal{F}_s^C]|\mathcal{F}_u \vee \mathcal{F}_t^C]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u \vee \mathcal{F}_s^C]|\mathcal{F}_u \vee \mathcal{F}_t^C]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]|\mathcal{F}_u \vee \mathcal{F}_t^C]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi^k|\mathcal{F}_u]} \\
&= \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C] \\
&= \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)] \\
&= \ldots \text{same reasoning backwards} \ldots \\
&= \mathbb{E}_{\mathbb{Q}_{T_k}}[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)],
\end{aligned}
$$

where we have applied the abstract Bayes' rule for the first equality and the second one follows from $(\mathcal{H}3)$ in Theorem 4.9 (plus the dominated convergence theorem). The third equality is obvious since $h(C_s)$ is $\mathcal{F}_s^C$-measurable, the fourth one is again a consequence of $(\mathcal{H}3)$, and finally, the fifth equality is the conditional Markov property (25). For the remaining equalities we use Remark 4.11 and the same reasoning backwards. Thus, we have shown (32).

It remains to prove the second claim in the proposition. From the above calculation, it is obvious that

$$\mathbb{Q}_{T_k}(C_s = j|\mathcal{F}_s \vee \sigma(C_t)) = \mathbb{Q}_{T^*}(C_s = j|\mathcal{F}_s \vee \sigma(C_t))$$

and then in particular also

$$\mathbb{Q}_{T_k}(C_s = j|\mathcal{F}_s; C_t = i) = \mathbb{Q}_{T^*}(C_s = j|\mathcal{F}_s; C_t = i),$$

on the set $\{C_t = i\}$, for all $i, j \in \mathcal{K}$ and $0 \leq t \leq s \leq T_k$. These are by definition the transition probabilities under the measures $\mathbb{Q}_{T_k}$ and $\mathbb{Q}_{T^*}$ and hence, (33) is proved. □

Generally speaking, the immersion property of some filtration enlargement is not always preserved under an equivalent change of probability measure. This usually depends on the component of the Radon–Nikodym density corresponding to the filtration $\mathbb{F}^C$, which has to satisfy some conditions. However, in the case of the forward Libor measures, this component is trivial since the Radon–Nikodym densities are adapted to $\mathbb{F}$. Thus, it turns out that the immersion property of the enlargement is indeed preserved under all forward measures.

Theorem 4.17. *Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$ and let $\mathbb{Q}_{T_k}$, $k = 1, \ldots, n-1$, be the forward measures given by* (31)*. Then the immersion property holds under all $\mathbb{Q}_{T_k}$, i.e. every $(\mathbb{F}, \mathbb{Q}_{T_k})$-local martingale is a $(\mathbb{G}, \mathbb{Q}_{T_k})$-local martingale.*

Proof: The proof relies on the following result which can be found for example in Coculescu and Nikeghbali (2007):

> Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures and assume that the $(\mathcal{H})$-hypothesis holds under $\mathbb{P}$. Denote
> $$\psi := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}, \quad \psi_t := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad \psi_t^{\mathbb{G}} := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{G}_t}.$$
> Then hypothesis $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if for every $X \in \mathcal{F}_{T^*}$, $X \geq 0$
> $$\frac{\mathbb{E}_{\mathbb{P}}[X\psi|\mathcal{G}_t]}{\psi_t^{\mathbb{G}}} = \frac{\mathbb{E}_{\mathbb{P}}[X\psi|\mathcal{F}_t]}{\psi_t}.$$

In our case, $\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}}$ is $\mathcal{F}_{T_k}$-measurable, which implies $\psi_t = \psi_t^{\mathbb{G}}$ and hence, the condition is trivially fulfilled since

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[X\psi|\mathcal{G}_t] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X\psi|\mathcal{F}_t],$$

by $(\mathcal{H}3)$. Therefore, we conclude that the immersion property is satisfied under all $\mathbb{Q}_{T_k}$. □

5. Absence of arbitrage in the rating based Lévy Libor model

From arbitrage pricing theory we know that in order to have an arbitrage-free model, the forward prices of defaultable bonds $\frac{B_C(\cdot,T_k)}{B(\cdot,T^*)}$, where the default-free bond with maturity T^* is used as a numeraire, must be local martingales with respect to the forward measure $\mathbb{Q}_{T^*}$. When the default-free bonds with other maturities are used as numeraires, the forward defaultable bond price processes have to be local martingales with respect to the corresponding forward measures as well. It can be shown that it is enough to require that $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$, for every $k = 1, \ldots, n-1$. To see this, let us fix some $k, l \in \{1, \ldots, n\}$ and assume that $l \geq k$ (the other case is treated similarly). We have

$$\frac{B_C(t,T_k)}{B(t,T_l)} = \frac{B_C(t,T_k)}{B(t,T_k)} \frac{B(t,T_k)}{B(t,T_l)},$$

where $\left(\frac{B(t,T_k)}{B(t,T_l)}\right)_{0 \leq t \leq T_k}$ is the density process of the change of measure from $\mathbb{Q}_{T_k}$ to $\mathbb{Q}_{T_l}$ up to a norming constant; compare equation (9). Hence, $\frac{B_C(\cdot,T_k)}{B(\cdot,T_l)}$ is a $\mathbb{Q}_{T_l}$-local martingale if and only if $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a $\mathbb{Q}_{T_k}$-local martingale (cf. Proposition III.3.8(a) in Jacod and Shiryaev (2003)).

So far we have not specified directly the bond prices in the model, but as we have already specified the inter-rating spreads $H_j(\cdot, T_k)$, $j = 1, \ldots, K-1$, any bond price specification we make must be consistent with relation (5) connecting the bond prices and the inter-rating spreads. Let us explore the consequences of this for the bond prices. For a fixed $t \in [0, T_k]$, we obtain recursively from (5)

$$\frac{B_j(t,T_k)}{B_{j-1}(t,T_k)} = \left(\prod_{l=l_0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)}\right) \frac{B_j(t,T_{l_0})}{B_{j-1}(t,T_{l_0})},$$

where T_{l_0} is a tenor date such that $t \in (T_{l_0 - 1}, T_{l_0}]$. Consequently, for each rating i it follows

$$\frac{B_i(t,T_k)}{B(t,T_k)} = \frac{B_1(t,T_k)}{B(t,T_k)} \prod_{j=2}^{i} \frac{B_j(t,T_k)}{B_{j-1}(t,T_k)} = \left(\prod_{j=1}^{i} \prod_{l=l_0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)} \right) \frac{B_i(t,T_{l_0})}{B(t,T_{l_0})}.$$

Since every bond price specification must be consistent with the above relation, we postulate

Assumption ($\mathbb{B}$). For every $i \in \{1, 2, \ldots, K-1\}$

$$\frac{B_i(t,T_k)}{B(t,T_k)} = \left(\prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)} \right) e^{\int_0^t \lambda_i(s)\mathrm{d}s}, \tag{34}$$

for some integrable, $\mathbb{F}$-adapted stochastic process $\lambda_i(\cdot)$ that satisfies

$$e^{\int_0^{T_k} \lambda_i(s)\mathrm{d}s} = \prod_{j=1}^{i} \prod_{l=0}^{k-1} (1+\delta_l H_j(T_l,T_l)). \tag{35}$$

It is easily checked that the above specification is indeed consistent and moreover, $\frac{B_i(T_k,T_k)}{B(T_k,T_k)} = 1$ due to (35). Recall that by assumption $H_j(t,T_l) = H_j(T_l,T_l)$, for $t \geq T_l$.

Under Assumption ($\mathbb{B}$), the forward bond price process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is given by

$$\begin{aligned}
\frac{B_C(t,T_k)}{B(t,T_k)} &= \sum_{i=1}^{K-1} \left(\prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)} \right) e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau_-}} \mathbf{1}_{\{C_t=K\}} \\
&= \sum_{i=1}^{K-1} \mathbb{H}(t,T_k,i) e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau_-}} \mathbf{1}_{\{C_t=K\}},
\end{aligned} \tag{36}$$

where

$$\mathbb{H}(t,T_k,i) := \prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)}.$$

In the sequel we are going to provide necessary and sufficient conditions for the local martingality of the forward defaultable bond price process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$. Let us begin by stating the main result.

Theorem 5.1. *Let T_k be a tenor date. Assume that the processes $H_j(\cdot,T_k)$, $j = 1, \ldots, K-1$, are given by* (15) *and that Assumption* ($\mathbb{B}$) *holds. Then the process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ defined in* (36) *is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and the filtration $\mathbb{G}$ if and only if the following condition is satisfied:*

for almost all $t \leq T_k$ on the set $\{C_t \neq K\}$

$$b^{\mathbb{H}}(t,T_k,C_t)+\lambda_{C_t}(t) = \left(1-q_{C_t}\frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-,T_k,C_t)}\right)\lambda_{C_tK}(t) \qquad (37)$$
$$+\sum_{j=1,j\neq C_t}^{K-1}\left(1-\frac{\mathbb{H}(t-,T_k,j)e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-,T_k,C_t)e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right)\lambda_{C_tj}(t).$$

Before proving this theorem, we need some auxiliary results. In the following lemma we deduce the dynamics of the process $\mathbb{H}(\cdot,T_k,i)$ for each i under the measure $\mathbb{Q}_{T_k}$.

Lemma 5.2. *Let T_k be a tenor date and assume that $H_j(\cdot,T_k)$ are given by (15). The process $\mathbb{H}(\cdot,T_k,i)$ defined in (36) has the following dynamics under $\mathbb{Q}_{T_k}$*

$$\mathbb{H}(t,T_k,i) = \mathbb{H}(0,T_k,i)$$
$$\times\mathcal{E}_t\Bigg(\int_0^{\cdot} b^{\mathbb{H}}(s,T_k,i)\mathrm{d}s - \int_0^{\cdot}\sum_{j=1}^{i}\sum_{l=1}^{k-1} h_j(s-,T_l)\sqrt{c_s}\gamma_j(s,T_l)\mathrm{d}W_s^{T_k} \qquad (38)$$
$$+\int_0^{\cdot}\int_{\mathbb{R}^d}\left(\prod_{j=1}^{i}\prod_{l=1}^{k-1}\Big(1+h_j(s-,T_l)(e^{\langle\gamma_j(s,T_l),x\rangle}-1)\Big)^{-1}-1\right)(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg),$$

where $h_j(s,T_l)$ is defined in (18) and

$$b^{\mathbb{H}}(s,T_k,i) := -\sum_{j=1}^{i}\sum_{l=1}^{k-1} h_j(s-,T_l)b^{H_j}(s,T_l)$$
$$+\sum_{j=1}^{i}\sum_{l=1}^{k-1} h_j(s-,T_l)\Big\langle\gamma_j(s,T_l),\sum_{m=l+1}^{k-1}\ell(s-,T_m)c_s\sigma(s,T_m)\Big\rangle$$
$$-\sum_{j=1}^{i}\sum_{l=1}^{k-1}\frac{1}{2}(h_j(s-,T_l)-h_j(s-,T_l)^2)\|\sqrt{c_s}\gamma_j(s,T_l)\|^2$$
$$+\frac{1}{2}\Big\|\sum_{j=1}^{i}\sum_{l=1}^{k-1} h_j(s-,T_l)\sqrt{c_s}\gamma_j(s,T_l)\Big\|^2 \qquad (39)$$
$$+\int_{\mathbb{R}^d}\Bigg[\prod_{j=1}^{i}\prod_{l=1}^{k-1}\Big(1+h_j(s-,T_l)(e^{\langle\gamma_j(s,T_l),x\rangle}-1)\Big)^{-1}-1$$
$$+\sum_{j=1}^{i}\sum_{l=1}^{k-1} h_j(s-,T_l)\langle\gamma_j(s,T_l),x\rangle$$
$$\times\left(\prod_{m=l+1}^{k-1}\Big(1+\ell(s-,T_m)(e^{\langle\sigma(s,T_m),x\rangle}-1)\Big)\right)\Bigg]F_s^{T_k}(\mathrm{d}x).$$

Proof: The proof is deferred to the appendix. □

Furthermore, we make the following observation: the processes $\mathbb{H}(\cdot, T_k, i)$ and C *do not jump simultaneously*, i.e. $\Delta\mathbb{H}(t, T_k, i)\Delta C_t = 0$ $\mathbb{Q}_{T_k}$-a.s. for $t \in [0, T^*]$. This property is a consequence of the canonical construction of C. A similar result concerning a canonically constructed default time is stated in Jakubowski and Nieweglowski (2009a, Proposition 2). The proposition below is a slight generalization since we deal with a series of jump times and, in addition, require the property to hold under all forward measures $\mathbb{Q}_{T_k}$.

Proposition 5.3. *Let $(Y_t)_{0\leq t\leq T^*}$ be an $\mathbb{F}$-adapted semimartingale and (τ_n) a sequence of random times representing the jump times of a conditional Markov chain constructed by the canonical construction. Then*

$$\mathbb{Q}_{T_k}(\Delta Y_{\tau_n} \neq 0) = 0, \qquad n \in \mathbb{N},$$

for every forward measure $\mathbb{Q}_{T_k}$ $(1 \leq k \leq n)$.

Proof: See Appendix B. □

Using these results, we are now able to prove the main theorem.

Proof of Theorem 5.1: Recall from Theorem 4.16 that C is a conditional Markov chain under every forward measure $\mathbb{Q}_{T_k}$, $k = 1, \ldots, n$.

According to Bielecki and Rutkowski (2002, Proposition 11.3.1), for each i the process

$$M_t^i = \mathbf{1}_{\{C_t=i\}} - \int_0^t \lambda_{C_s i}(s)\mathrm{d}s \tag{40}$$

is a $\mathbb{Q}_{T_k}$-martingale.

Moreover, we will make use of an auxiliary process H^{ij}, for $i, j \in \mathcal{K}, i \neq j$, defined in Bielecki and Rutkowski (2002, page 333):

$$H_t^{ij} = \sum_{0<u\leq t} \mathbf{1}_{\{C_{u-}=i\}}\mathbf{1}_{\{C_u=j\}}.$$

This process counts the number of jumps of C from the state i to the state j in the time interval $[0, t]$ and it is known that

$$M_t^{ij} = H_t^{ij} - \int_0^t \lambda_{ij}(u)\mathbf{1}_{\{C_u=i\}}\mathrm{d}u \tag{41}$$

is a $\mathbb{Q}_{T_k}$-martingale; see Bielecki and Rutkowski (2002, page 407).

In particular, the process H^{iK} is useful for us since it takes the following values: it equals 1 if and only if C jumped from i to the default state K (remember that this state is absorbing) during the time interval $[0, t]$ and otherwise it equals zero. Therefore, we can use it to rewrite the defaultable bond price process in the following way:

$$B_C(t, T_k) = \sum_{i=1}^{K-1} \left(B_i(t, T_k)\mathbf{1}_{\{C_t=i\}} + q_i B(t, T_k)H_t^{iK}\right).$$

The forward defaultable bond price process is then given by

$$\begin{aligned}\frac{B_C(t,T_k)}{B(t,T_k)} &= \sum_{i=1}^{K-1}\left(\frac{B_i(t,T_k)}{B(t,T_k)}\mathbf{1}_{\{C_t=i\}} + q_i H_t^{iK}\right)\\ &= \sum_{i=1}^{K-1}\left(\mathbb{H}(t,T_k,i)e^{\int_0^t \lambda_i(s)\mathrm{d}s}\mathbf{1}_{\{C_t=i\}} + q_i H_t^{iK}\right).\end{aligned}$$

Let us calculate its dynamics under $\mathbb{Q}_{T_k}$ and extract the drift part.

For each $i = 1, \dots, K-1$, making use of (41) we have

$$q_i H_t^{iK} = q_i M_t^{iK} + \int_0^t q_i \lambda_{iK}(u)\mathbf{1}_{\{C_u=i\}}\mathrm{d}u. \tag{42}$$

Next, we apply the integration by parts formula which yields

$$\begin{aligned}\mathbb{H}(t,T_k,i)e^{\int_0^t \lambda_i(s)\mathrm{d}s}\mathbf{1}_{\{C_t=i\}} &= \int_0^t \mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\mathrm{d}\mathbf{1}_{\{C_u=i\}} \qquad (43)\\ &\quad + \int_0^t \mathbf{1}_{\{C_{u-}=i\}}\mathrm{d}\left(\mathbb{H}(u,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\right),\end{aligned}$$

since the quadratic covariation process $\left[\mathbb{H}(\cdot,T_k,i)e^{\int_0^\cdot \lambda_i(s)\mathrm{d}s}, \mathbf{1}_{\{C_\cdot=i\}}\right]$ vanishes. To see this, we remark that for any two semimartingales X, Y the quadratic covariation is given by $[X,Y]_t = \langle X^c, Y^c\rangle_t + \sum_{s\leq t}\Delta X_s \Delta Y_s$. In our case the continuous martingale part of $\mathbf{1}_{\{C_\cdot=i\}}$ is zero since this process has finite variation, and hence the covariation of the two continuous martingale parts equals zero. As far as the jumps are concerned, this sum is also zero, since $\mathbb{H}(\cdot,T_k,i)$, which is $\mathbb{F}$-adapted, and $\mathbf{1}_{\{C_\cdot=i\}}$ have no common jumps by virtue of Proposition 5.3 (obviously the indicator process jumps only when a jump of C happens and then the claim follows directly from the proposition).

Using (40) it follows

$$\mathrm{d}\mathbf{1}_{\{C_u=i\}} = \mathrm{d}M_u^i + \lambda_{C_u i}(u)\mathrm{d}u. \tag{44}$$

Furthermore, since $e^{\int_0^\cdot \lambda_i(s)\mathrm{d}s}$ is a continuous process with finite variation, we obtain

$$\begin{aligned}\mathrm{d}\left(\mathbb{H}(u,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\right) &= \mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\lambda_i(u)\mathrm{d}u\\ &\quad + e^{\int_0^u \lambda_i(s)\mathrm{d}s}\mathrm{d}\mathbb{H}(u,T_k,i), \qquad (45)\end{aligned}$$

where

$$\mathrm{d}\mathbb{H}(u,T_k,i) = \mathbb{H}(u-,T_k,i)\left(b^{\mathbb{H}}(u,T_k,i)\mathrm{d}u + \mathrm{d}M_u^{\mathbb{H}}\right), \tag{46}$$

by Lemma 5.2 (we denote by $M^{\mathbb{H}}$ the local martingale part in (38)). Note that $M^{\mathbb{H}}$ is $\mathbb{F}$-adapted and a local martingale with respect to $\mathbb{F}$, but due to the immersion property, it remains a local martingale with respect to the enlarged filtration $\mathbb{G}$ as well (see Theorem 4.17).

Consequently, combining (42), (43), (44), (45) and (46) we obtain

$$\begin{aligned}\frac{B_C(t,T_k)}{B(t,T_k)} &= \sum_{i=1}^{K-1}\int_0^t \mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\mathrm{d}M_u^i \\ &+ \sum_{i=1}^{K-1}\int_0^t \mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\lambda_{C_u i}(u)\mathrm{d}u \\ &+ \sum_{i=1}^{K-1}\int_0^t \mathbf{1}_{\{C_u=i\}}\mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\lambda_i(u)\mathrm{d}u \\ &+ \sum_{i=1}^{K-1}\int_0^t \mathbf{1}_{\{C_u=i\}}e^{\int_0^u \lambda_i(s)\mathrm{d}s}\mathbb{H}(u-,T_k,i)b^{\mathbb{H}}(u,T_k,i)\mathrm{d}u \\ &+ \sum_{i=1}^{K-1}\int_0^t \mathbf{1}_{\{C_{u-}=i\}}e^{\int_0^u \lambda_i(s)\mathrm{d}s}\mathbb{H}(u-,T_k,i)\mathrm{d}M_u^{\mathbb{H}} \\ &+ \sum_{i=1}^{K-1} q_i M_t^{iK} + \sum_{i=1}^{K-1}\int_0^t q_i\lambda_{iK}(u)\mathbf{1}_{\{C_u=i\}}\mathrm{d}u.\end{aligned}$$

The drift part, denoted by $D(t,T_k)$, is therefore given by

$$\begin{aligned}D(t,T_k) &= \int_0^t \sum_{i=1}^{K-1}\mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\lambda_{C_u i}(u)\mathrm{d}u \\ &\quad+ \int_0^t \sum_{i=1}^{K-1}\mathbf{1}_{\{C_u=i\}}\mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\bigg(\lambda_i(u)+b^{\mathbb{H}}(u,T_k,i) \\ &\qquad\qquad + q_i\lambda_{iK}(u)\frac{1}{\mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}}\bigg)\mathrm{d}u \\ &= \int_0^t \sum_{i=1}^{K-1}\mathbb{H}(u-,T_k,i)e^{\int_0^u \lambda_i(s)\mathrm{d}s}\lambda_{C_u i}(u)\mathrm{d}u \\ &\quad+ \int_0^t \mathbf{1}_{\{C_u\neq K\}}\mathbb{H}(u-,T_k,C_u)e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}\bigg(\lambda_{C_u}(u)+b^{\mathbb{H}}(u,T_k,C_u) \\ &\qquad\qquad + q_{C_u}\lambda_{C_u K}(u)\frac{1}{\mathbb{H}(u-,T_k,C_u)e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}}\bigg)\mathrm{d}u,\end{aligned}$$

since $\mathbb{H}(u, T_k, j) > 0$, for every $j \in \mathcal{K} \setminus \{K\}$. We have

$$\sum_{i=1}^{K-1} \frac{\mathbb{H}(u-, T_k, i) e^{\int_0^u \lambda_i(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} \lambda_{C_u i}(u)$$
$$= \sum_{i=1, i\neq C_u}^{K-1} \frac{\mathbb{H}(u-, T_k, i) e^{\int_0^u \lambda_i(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} \lambda_{C_u i}(u) + \lambda_{C_u C_u}(u)$$
$$= \sum_{i=1, i\neq C_u}^{K-1} \left(\frac{\mathbb{H}(u-, T_k, i) e^{\int_0^u \lambda_i(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} - 1 \right) \lambda_{C_u i}(u) - \lambda_{C_u K}(u),$$

where we have used the property $\lambda_{jj}(u) = -\sum_{i=1, i\neq j}^{K} \lambda_{ji}(u)$ that holds for every row of the intensity matrix Λ. Hence, we obtain

$$\begin{aligned} D(t, T_k) = \int_0^t & \mathbf{1}_{\{C_u \neq K\}} \mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s} \\ & \times \Bigg(\lambda_{C_u}(u) + b^{\mathbb{H}}(u, T_k, C_u) \\ & + q_{C_u} \lambda_{C_u K}(u) \frac{1}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} \\ & + \sum_{i=1, i\neq C_u}^{K-1} \left(\frac{\mathbb{H}(u-, T_k, i) e^{\int_0^u \lambda_i(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} - 1 \right) \lambda_{C_u i}(u) \\ & - \lambda_{C_u K}(u) \Bigg) \mathrm{d}u. \end{aligned}$$

Now the proof is finished since condition (37) implies that the drift term $D(\cdot, T_k)$ vanishes and thus, $\frac{B_C(\cdot, T_k)}{B(\cdot, T_k)}$ is a $\mathbb{Q}_{T_k}$-local martingale. Conversely, if $\frac{B_C(\cdot, T_k)}{B(\cdot, T_k)}$ is a $\mathbb{Q}_{T_k}$-local martingale, then its drift term $D(\cdot, T_k)$ (i.e. the predictable process with finite variation in its semimartingale decomposition) vanishes, which implies that on the set $\{C_u \neq K\}$

$$\begin{aligned} b^{\mathbb{H}}(u, T_k, C_u) + \lambda_{C_u}(u) = & \left(1 - q_{C_u} \frac{e^{-\int_0^u \lambda_{C_u}(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u)} \right) \lambda_{C_u K}(u) \\ & + \sum_{j=1, j\neq C_u}^{K-1} \left(1 - \frac{\mathbb{H}(u-, T_k, j) e^{\int_0^u \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(u-, T_k, C_u) e^{\int_0^u \lambda_{C_u}(s)\mathrm{d}s}} \right) \lambda_{C_u j}(u), \end{aligned}$$

for almost all $u \in [0, T_k]$, which is exactly condition (37). □

Remark 5.4. Condition (37) which we have just established can be compared with the HJM drift condition given in Jakubowski and Nieweglowski (2009a, Theorem 5) and with the consistency conditions in the defaultable HJM models by Bielecki and Rutkowski (2000) and Eberlein and Özkan (2003). The latter two papers treat these conditions as conditions on the intensity matrix Λ, and thus in general obtain different migration processes for different maturities (see the discussion in Bielecki and Rutkowski (2002, page 407)). In our Libor modeling framework we want the migration process to be the same for different maturities in order to exploit formula (2) for the defaultable Libor rates. Therefore, our condition is treated as a drift condition on $b^{H_j}(\cdot, T_l)$, for $j = 1, \dots, i$, $l = 1, \dots, k-1$ (recall that $b^{\mathbb{H}}(\cdot, T_k, i)$ is given by (39)). The drift terms $b^{H_j}(\cdot, T_l)$ are then obtained as solutions of the given SDEs and Assumption ($\mathbb{SUP}$) is needed for the existence of the solution (see Eberlein, Kluge, and Schönbucher (2006) for a detailed discussion of the case with no credit migration).

6. Valuation of credit derivatives

The purpose of this section is to illustrate the valuation of credit derivatives in the model. By the term credit derivative we mean a derivative linked to some credit risk sensitive underlying asset. Thus the only credit risk involved in such a contract is the risk related to the *underlying asset*, while the *counterparty risk* (i.e. the risk that one of the parties in the contract might default) is considered negligible. Very often this underlying asset is a defaultable corporate bond. Therefore, we begin by calculating the arbitrage-free price $B_C(t, T_k)$, $t \leq T_k$, of a defaultable bond with fractional recovery q in the rating based Lévy Libor model.

Proposition 6.1. *The price at time $t \leq T_k$ of a defaultable bond with maturity T_k and fractional recovery q is given by*

$$B_C(t,T_k)\mathbf{1}_{\{C_t\neq K\}} = B(t,T_k)\sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\Bigg[\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(t,T_k)|\mathcal{F}_t] + \sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{\tau-}=j\}}q_j|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\Bigg], \tag{47}$$

or equivalently, by

$$B_C(t,T_k)\mathbf{1}_{\{C_t\neq K\}} = B(t,T_k)\sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}} \times\Bigg[1+\sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i,C_{T_k}=K\}}\mathbf{1}_{\{C_{\tau-}=j\}}(q_j-1)|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\Bigg]. \tag{48}$$

Proof: The promised payoff of such a bond at maturity time T_k equals

$$B_C(T_k,T_k) = \mathbf{1}_{\{C_{T_k}\neq K\}} + \mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}.$$

Using the risk-neutral valuation formula, its time-t value is given as the conditional expectation with respect to the forward measure $\mathbb{Q}_{T_k}$

$$\begin{aligned} B_C(t,T_k) &= B(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}} + \mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{G}_t] \\ &= B(t,T_k)\left(\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{G}_t] + \mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{G}_t]\right). \end{aligned}$$

Therefore,

$$B_C(t,T_k)\mathbf{1}_{\{C_t\neq K\}} = B(t,T_k)\Big(\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{G}_t] \qquad (49)$$

$$+\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t\neq K\}}\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{G}_t]\Big),$$

since $\mathbf{1}_{\{C_t\neq K\}}\mathbf{1}_{\{C_{T_k}\neq K\}} = \mathbf{1}_{\{C_{T_k}\neq K\}}$ for $t \leq T_k$, because K is an absorbing state.

Using the conditional Markov property of C (equation (24)), for the first summand in (49) we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{G}_t] &= \mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{F}_t\vee\sigma(C_t)]\\ &= \sum_{i=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}}\\ &= \sum_{i=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-\mathbf{1}_{\{C_{T_k}=K\}})\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}},\end{aligned}$$

where we have applied Lemma 4.2 to obtain the second equality and the third one is obvious. On the set $\{C_t = i\}$ we have

$$\begin{aligned}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}=K\}}\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]} &= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}=K\}}\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_{T_k}]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_{T_k}]}\Big|\mathcal{F}_t\right]\\ &= \mathbb{E}_{\mathbb{Q}_{T_k}}[p_{iK}(t,T_k)|\mathcal{F}_t], \qquad (50)\end{aligned}$$

where the first equality follows from Proposition 4.8 and the chain rule for conditional expectations and the second one is simply the definition of conditional transition probabilities (Definition 4.5). Therefore, we obtain

$$\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{G}_t] = \sum_{i=1}^{K-1}\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(t,T_k)|\mathcal{F}_t]\mathbf{1}_{\{C_t=i\}}. \qquad (51)$$

The second summand in (49) is given by

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t\neq K\}}\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{G}_t]\\ &\quad= \mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t\neq K\}}\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{F}_t\vee\sigma(C_t)]\\ &\quad= \sum_{i=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}} \qquad (52)\\ &\quad= \sum_{i=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}q_{C_{\tau-}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}}\\ &\quad= \sum_{i=1}^{K-1}\sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{\tau-}=j\}}q_j|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}},\end{aligned}$$

where the second equality follows again by Proposition 4.8 and the subsequent equalities are obvious when we note that

$$\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{T_k}=K\}} = \mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}.$$

The first equality in (52) still has to be justified. Recall that by the conditional Markov property

$$\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_A|\mathcal{G}_t] = \mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_A|\mathcal{F}_t \vee \sigma(C_t)], \tag{53}$$

for every $A \in \sigma(C_u)$ with $u \in [t, T_k]$. We show that (53) holds for every $A \in \mathcal{F}^C_{t,T_k}$, where $\mathcal{F}^C_{t,T_k} := \bigvee_{t \le u \le T_k} \sigma(C_u)$. The proof follows by a monotone class argument. More precisely, property (53) can be proved for the generator of $\mathcal{F}^C_{t,T_k}$, which consists of the finite intersections of the sets from $\sigma(C_u), u \in [t, T_k]$. In addition, the family $\mathcal{A}$ of the sets A satisfying (53) is a monotone class.

Property (53) can be extended to bounded $\mathcal{F}^C_{t,T_k}$-measurable random variables X:

$$\mathbb{E}_{\mathbb{Q}_{T_k}}[X|\mathcal{G}_t] = \mathbb{E}_{\mathbb{Q}_{T_k}}[X|\mathcal{F}_t \vee \sigma(C_t)]. \tag{54}$$

Now we just have to note that the random variable $\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{T_k}=K\}}q_{C_{\tau-}}$ is $\mathcal{F}^C_{t,T_k}$-measurable, then use (54) and we have established (52).

Finally, combining (51) and (52) we get (47). The second equality (48) follows by inserting (50) into (47). □

Let us consider now the valuation of a *credit default swap*. It is a financial contract offering protection against default of an underlying asset. The protection buyer A periodically pays a fixed amount S, called the *credit swap premium* (or the *credit swap rate*), to the protection seller B until default of the underlying asset or the maturity date of the contract, whichever comes first. The protection seller agrees in turn to make a payment that covers the loss of A if default occurs. The underlying asset is issued by some third party C and both counterparties A and B are assumed to be risk-free.

Here we consider a defaultable bond with maturity date T_m (where T_m is one of the tenor dates) and fractional recovery of treasury value q as the underlying asset and assume the following fee payment scheme: the protection buyer pays a fixed amount S periodically at tenor dates $T_1, \dots, T_{m-1}$ until default of the underlying bond. The protection seller is obliged to make a payment that covers the loss if default happens, i.e. the amount

$$1 - q_{C_{\tau-}}$$

has to be paid at T_{k+1} if default occurs in $(T_k, T_{k+1}]$.

Proposition 6.2. *The swap rate S for the credit default swap described above is given by*

$$S = \frac{\sum_{k=2}^{m} B(0,T_k) \sum_{j=1}^{K-1} \mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k, C_{\tau-}=j\}}]}{\sum_{k=1}^{m-1} B(0,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[1 - p_{iK}(0,T_k)]}, \tag{55}$$

where i denotes the rating class observed at time 0.

Proof: The value of the premium leg at any time point $t \le T_1$ is given by

$$\begin{aligned}&\sum_{k=1}^{m-1} SB(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}}|\mathcal{G}_t]\\&\quad=\sum_{k=1}^{m-1} SB(t,T_k)\sum_{l=1}^{K-1}\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{lK}(t,T_k)|\mathcal{F}_t]\mathbf{1}_{\{C_t=l\}},\end{aligned}$$

where we have used formula (51).

The value of the default leg at $t \le T_1$ is given by

$$\sum_{k=2}^{m} B(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_{C_{\tau-}})\mathbf{1}_{\{T_{k-1}<\tau\le T_k\}}|\mathcal{G}_t],$$

and similarly to the proof of Proposition 6.1, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_{C_{\tau-}})\mathbf{1}_{\{T_{k-1}<\tau\le T_k\}}|\mathcal{G}_t]\\&\quad=\sum_{l=1}^{K-1}\mathbf{1}_{\{C_t=l\}}\sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k,C_{\tau-}=j\}}\mathbf{1}_{\{C_t=l\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=l\}}|\mathcal{F}_t]}.\end{aligned}$$

The swap rate S is by definition the rate that makes the value of the credit default swap at time 0 equal to zero. The value of the premium leg at time 0 is given by

$$S\sum_{k=1}^{m-1}\sum_{l=1}^{K-1} B(0,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{lK}(0,T_k)]\mathbf{1}_{\{C_0=l\}},$$

and the value of the default leg at time 0 equals

$$\sum_{k=2}^{m} B(0,T_k)\sum_{l=1}^{K-1}\mathbf{1}_{\{C_0=l\}}\sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k,C_{\tau-}=j\}}\mathbf{1}_{\{C_0=l\}}]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_0=l\}}]}.$$

By assumption the observed rating class at time 0 is i. Hence, $\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_0=i\}}] = 1$ and we obtain the swap rate S by solving

$$\begin{aligned}&S\sum_{k=1}^{m-1} B(0,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(0,T_k)]\\&\quad=\sum_{k=2}^{m} B(0,T_k)\sum_{j=1}^{K-1}\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k,C_{\tau-}=j\}}],\end{aligned}$$

which produces (55). □

The rating based Lévy Libor model is particularly useful for pricing of interest rate derivatives such as *forward rate agreements* (FRAs), *interest rate swaps* and *caps* and *floors*, where the underlying interest rate is the defaultable forward Libor rate.

We begin by introducing a new set of probability measures which represent the defaultable counterparts of forward measures and are convenient tools for pricing of these derivatives. Such a concept was first introduced by Schönbucher (2000) and further explored for credit derivative valuation in Eberlein, Kluge, and Schönbucher (2006). The novelty in our definition below is credit migration and non-zero recovery of the defaultable bond. Due to the non-zero recovery assumption our defaultable measures are *not* survival measures, as in the aforementioned papers.

Definition 6.3. The *defaultable forward measure* $\mathbb{Q}_{C,T_k}$ for the settlement date T_k is defined on $(\Omega, \mathcal{G}_{T_k})$ by

$$\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{T_k}} := \frac{B(0,T_k)}{B_C(0,T_k)} B_C(T_k,T_k).$$

This corresponds to the choice of $B_C(\cdot,T_k)$ as a numeraire (remember that $B_C(\cdot,T_k)$ is a strictly positive process as long as the recovery rate q is not zero, and thus a valid choice for a numeraire).

Restricted to the σ-field $\mathcal{G}_t$ the defaultable forward measure becomes

$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} = \frac{B(0,T_k)}{B_C(0,T_k)} \frac{B_C(t,T_k)}{B(t,T_k)}, \tag{56}$$

since $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a $\mathbb{Q}_{T_k}$-martingale.

It can be easily proved that the defaultable forward Libor rate $L_C(\cdot,T_k)$ is a martingale under the corresponding defaultable forward measure $\mathbb{Q}_{C,T_{k+1}}$, just as the default-free forward Libor rates are martingales under their own forward measures (cf. Proposition 3.19 in Grbac (2010)).

Therefore, we are able to calculate the prices of the aforementioned defaultable forward Libor rate derivatives as conditional expectations with respect to the defaultable forward measures. More precisely, we have the following result providing a valuation formula for a defaultable contingent claim with a promised $\mathcal{G}_{T_k}$-measurable payoff Y at maturity T_k and fractional recovery of treasury value q upon default. It generalizes Proposition 15.2.3 in Bielecki and Rutkowski (2002) and Proposition 7 in Eberlein, Kluge, and Schönbucher (2006), where models without credit migration were considered.

Proposition 6.4. *Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$. The time-t value of such a claim is given by*

$$\pi_t(Y) = B_C(t,T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

Proof: The payoff at maturity T_k of the given defaultable claim equals

$$Y\mathbf{1}_{\{C_{T_k}\neq K\}} + q_{C_{\tau-}}Y\mathbf{1}_{\{C_{T_k}=K\}} = YB_C(T_k,T_k),$$

and, using the risk-neutral valuation formula under the forward measure $\mathbb{Q}_{T_k}$, its time-t value is given by

$$\pi_t(Y) = B(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[YB_C(T_k,T_k)|\mathcal{G}_t].$$

Recalling Definition 6.3 of the defaultable forward measure $\mathbb{Q}_{C,T_k}$ and applying the abstract Bays' rule we obtain

$$\begin{aligned}\pi_t(Y) &= B(t,T_k)\frac{\mathbb{E}_{\mathbb{Q}_{C,T_k}}[YB_C(T_k,T_k)\frac{B(T_k,T_k)}{B_C(T_k,T_k)}|\mathcal{G}_t]}{\mathbb{E}_{\mathbb{Q}_{C,T_k}}[\frac{B(T_k,T_k)}{B_C(T_k,T_k)}|\mathcal{G}_t]} \\ &= B(t,T_k)\frac{B_C(t,T_k)}{B(t,T_k)}\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t] \\ &= B_C(t,T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].\end{aligned}$$

□

Example 6.5 (**Defaultable Libor rate caps and floors**). Recall that an *interest rate cap* (resp. *floor*) is a financial contract in which the buyer receives payments at the end of each period in which the interest rate exceeds (resp. falls below) a mutually agreed level called the *strike*. The payment that the seller has to make covers exactly the difference between the strike and the interest rate at the end of each period (settlement in arrears). Every cap (resp. floor) is a series of *caplets* (resp. *floorlets*), each of which is a call (resp. put) option on the subsequent forward rate.

Suppose that we want to price a cap on the defaultable forward Libor rate. The payoff of a caplet with strike K and maturity T_k is given by $B_C(T_{k+1},T_{k+1})(L_C(T_k,T_k)-K)^+$ (this corresponds to the settlement scheme which assumes the reduction of the principal value of the contract in case of default that we adopted at the beginning of the paper; see comments before equation (2) and Bielecki and Rutkowski (2002, Section 14.1.4, page 431)). Then the time-t price of the caplet is given by

$$C_t(T_k,K) = \delta_k B_C(t,T_{k+1})\mathbb{E}_{\mathbb{Q}_{C,T_{k+1}}}[(L_C(T_k,T_k)-K)^+|\mathcal{G}_t]$$

and the price of the defaultable forward Libor rate cap at time $t \le T_1$ is given as a sum

$$\mathbb{C}_t(K) = \sum_{k=1}^{n}\delta_{k-1}B_C(t,T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[(L_C(T_{k-1},T_{k-1})-K)^+|\mathcal{G}_t].$$

Appendix A

Assume that a complete stochastic basis $(\widetilde{\Omega},\mathcal{G},(\mathcal{G}_t)_{0\le t\le T^*},\mathbb{Q}_{T^*})$ is given and all processes we consider are defined on this basis.

Lemma A.1. *Let U be a real-valued special semimartingale such that $U_0 = 0$ with canonical representation given by*

$$U = U^c + x * (\mu^U - \nu^U) + A,$$

where U^c is the continuous martingale part, μ^U is the random measure of jumps of U with compensator ν^U and A is the predictable, finite-variation process.

Put $V_t := V_0 \exp U_t$ and let $\delta > 0$ be a real number. Define two semimartingales Z^1 and Z^2 by setting

(A1) $Z^1_t := 1 + \delta V_t = 1 + \delta V_0 \exp U_t,\ t \ge 0$
(A2) $Z^2_t := \frac{1}{\delta}(V_t - 1) = \frac{1}{\delta}(V_0 \exp U_t - 1),\ t \ge 0.$

In (A2) we impose an additional assumption that $V, V_- > 1$ to ensure that Z^2 is positive. Then

$$\begin{aligned} Z_t^i \;=\; & Z_0^i \exp\Big((v_-^i \cdot U^c)_t + \ln\big(1 + v_-^i (e^x - 1)\big) * (\mu^U - \nu^U)_t + (v_-^i \cdot A)_t \\ & + (\tfrac{1}{2}(v_-^i - (v_-^i)^2) \cdot \langle U^c, U^c \rangle)_t + \big[\ln\big(1 + v_-^i (e^x - 1)\big) - v_-^i x\big] * \nu_t^U \Big), \end{aligned}$$

where $i = 1, 2$ and

$$v_t^1 := \frac{\delta V_t}{1 + \delta V_t}, \qquad v_t^2 := \frac{V_t}{V_t - 1}, \qquad t \geq 0.$$

Proof: The statement for the semimartingale Z^1 can be found in Eberlein, Grbac, and Schmidt (2010, Lemma A.1) and the result for Z^2 follows similarly. A detailed proof is given in Grbac (2010, Lemma 1.10). □

Lemma A.2. *Consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P})$ and denote by W a standard d-dimensional Brownian motion and by μ the random measure of jumps of some semimartingale with compensator $\nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\mathrm{d}s$. Fix an $m \in \mathbb{N}$ and let V^k, $k = 1, \ldots, m$, be given by*

$$V_t^k \;=\; V_0^k \exp\left(\int_0^t b^k(s)\mathrm{d}s + \int_0^t \sigma^k(s)\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} S^k(s,x)(\mu - \nu)(\mathrm{d}s, \mathrm{d}x) \right),$$

for some $\sigma^k \in L(W)$ and $S^k \in G_{\mathrm{loc}}(\mu)$. Further let $\delta_k > 0$ be real numbers, for $k = 1, \ldots, m$. Then

$$\begin{aligned} \prod_{k=1}^m \frac{1}{1 + \delta_k V_t^k} \;=\; & \left(\prod_{k=1}^m \frac{1}{1 + \delta_k V_0^k} \right) \exp\Bigg(- \int_0^t \sum_{k=1}^m a^k(s)\mathrm{d}s - \int_0^t \sum_{k=1}^m v_{s-}^k \sigma^k(s)\mathrm{d}W_s \\ & - \int_0^t \int_{\mathbb{R}^d} \ln \prod_{k=1}^m \Big(1 + v_{s-}^k (e^{S^k(s,x)} - 1)\Big) (\mu - \nu)(\mathrm{d}s, \mathrm{d}x) \Bigg), \end{aligned}$$

where

$$v_s^k := \frac{\delta_k V_s^k}{1 + \delta_k V_s^k}$$

and

$$\begin{aligned} a^k(s) \;:=\; & v_{s-}^k b^k(s) + \tfrac{1}{2}(v_{s-}^k - (v_{s-}^k)^2)\|\sigma^k(s)\|^2 \\ & + \int_{\mathbb{R}^d} \Big(\ln\Big(1 + v_{s-}^k (e^{S^k(s,x)} - 1)\Big) - v_{s-}^k S^k(s,x) \Big) F_s(\mathrm{d}x). \end{aligned}$$

Proof: One has to apply Lemma A.1 to each process $1 + \delta_k V^k$ and then to calculate the product of exponentials. □

Proof of Theorem 3.1: Part (a) is a direct consequence of relation (6) and specification (15) for $H_j(\cdot, T_k)$, which ensures that $H_j(\cdot, T_k) \geq 0$.

Let us now prove part (b) and calculate the dynamics of $L_i(\cdot, T_k)$. We make use of the representation (6) and rely on Lemma A.1 and the connection between the ordinary and the stochastic exponential of a semimartingale given in Kallsen and Shiryaev (2002, Lemma 2.6).

Firstly, for the Libor rate $L(\cdot, T_k)$ given in (7), we apply Lemma A.1 to $Z_t^1 = 1 + \delta_k L(t, T_k)$ and the special semimartingale

$$U_t = \int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma(s,T_k)\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} \langle \sigma(s,T_k), x\rangle (\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x)$$

to obtain

$$\begin{aligned} 1 + \delta_k L(t,T_k) = {} & (1+\delta_k L(0,T_k)) \\ & \times \exp\Bigg(\int_0^t a^L(s,T_k)\mathrm{d}s + \int_0^t \ell(s-,T_k)\sqrt{c_s}\sigma(s,T_k)\mathrm{d}W_s^{T_{k+1}} \\ & + \int_0^t \int_{\mathbb{R}^d} \ln\Big(1 + \ell(s-,T_k)(e^{\langle \sigma(s,T_k),x\rangle} - 1)\Big)(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg), \end{aligned} \tag{57}$$

where the drift term is

$$\begin{aligned} a^L(s,T_k) := {} & \ell(s-,T_k) b^L(s,T_k) \\ & + \frac{1}{2}(\ell(s-,T_k) - \ell(s-,T_k)^2)\|\sqrt{c_s}\sigma(s,T_k)\|^2 \\ & + \int_{\mathbb{R}^d} \Big(\ln\Big(1 + \ell(s-,T_k)(e^{\langle\sigma(s,T_k),x\rangle} - 1)\Big) \\ & \qquad - \ell(s-,T_k)\langle \sigma(s,T_k), x\rangle \Big) F_s^{T_{k+1}}(\mathrm{d}x) \\ = {} & -\frac{1}{2}\ell(s-,T_k)^2 \|\sqrt{c_s}\sigma(s,T_k)\|^2 \\ & + \int_{\mathbb{R}^d} \Big(\ln\Big(1 + \ell(s-,T_k)(e^{\langle\sigma(s,T_k),x\rangle} - 1)\Big) \\ & \qquad - \ell(s-,T_k)(e^{\langle \sigma(s,T_k), x\rangle} - 1) \Big) F_s^{T_{k+1}}(\mathrm{d}x), \end{aligned} \tag{58}$$

with $\ell(s, T_k)$ defined in (14). Inserting (8) for $b^L(s, T_k)$ yields the second equality.

Similarly, for $H_j(\cdot, T_k)$, $j = 1, \ldots, i$, we have

$$\begin{aligned} 1 + \delta_k H_j(t,T_k) = {} & (1+\delta_k H_j(0,T_k)) \\ & \times \exp\Bigg(\int_0^t a^{H_j}(s,T_k)\mathrm{d}s + \int_0^t h_j(s-,T_k)\sqrt{c_s}\gamma_j(s,T_k)\mathrm{d}W_s^{T_{k+1}} \\ & + \int_0^t \int_{\mathbb{R}^d} \ln\Big(1 + h_j(s-,T_k)(e^{\langle \gamma_j(s,T_k),x\rangle} - 1)\Big)(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg), \end{aligned}$$

with the drift term

$$\begin{aligned} a^{H_j}(s,T_k) := \ & h_j(s-,T_k) b^{H_j}(s,T_k) \\ & + \frac{1}{2}(h_j(s-,T_k) - h_j(s-,T_k)^2) \| \sqrt{c_s} \gamma_j(s,T_k) \|^2 \\ & + \int_{\mathbb{R}^d} \Big(\ln \Big(1 + h_j(s-,T_k)(e^{\langle \gamma_j(s,T_k),x \rangle} - 1) \Big) \\ & \qquad - h_j(s-,T_k) \langle \gamma_j(s,T_k), x \rangle \Big) F_s^{T_{k+1}}(\mathrm{d}x). \end{aligned} \tag{59}$$

Multiplying these two expressions we get

$$\begin{aligned} 1 + \delta_k L_i(t,T_k) = \ & (1 + \delta_k L(t,T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t,T_k)) \\ = \ & (1 + \delta_k L_i(0,T_k)) \\ & \times \exp \Bigg(\int_0^t \Big(a^L(s,T_k) + \sum_{j=1}^{i} a^{H_j}(s,T_k) \Big) \mathrm{d}s \\ & \quad + \int_0^t \sqrt{c_s} \Big(\ell(s-,T_k)\sigma(s,T_k) + \sum_{j=1}^{i} h_j(s-,T_k)\gamma_j(s,T_k) \Big) \mathrm{d}W_s^{T_{k+1}} \\ & \quad + \int_0^t \int_{\mathbb{R}^d} \ln \Bigg(\Big(1 + \ell(s-,T_k)(e^{\langle \sigma(s,T_k),x \rangle} - 1) \Big) \\ & \qquad \times \prod_{j=1}^{i} \Big(1 + h_j(s-,T_k)(e^{\langle \gamma_1(s,T_k),x \rangle} - 1) \Big) \Bigg) (\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x) \Bigg) \\ = \ & (1 + \delta_k L_i(0,T_k)) \\ & \times \exp \Bigg(\int_0^t \Big(a^L(s,T_k) + \sum_{j=1}^{i} a^{H_j}(s,T_k) \Big) \mathrm{d}s \\ & \quad + \int_0^t \ell_i(s-,T_k) \sqrt{c_s} \sigma_i(s,T_k) \mathrm{d}W_s^{T_{k+1}} \\ & \quad + \int_0^t \int_{\mathbb{R}^d} \ln \beta_i(s,x,T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x) \Bigg), \end{aligned}$$

with $\sigma_i(s,T_k)$ given in (17) and $\beta_i(s,x,T_k)$ in (20).

To establish (16), another application of Lemma A.1 is needed, this time to the semimartingale $Z_t^2 = L_i(t,T_k)$ with $V_t = 1 + \delta_k L_i(t,T_k)$ and

$$\begin{aligned} U_t = \ & \int_0^t \Big(a^L(s,T_k) + \sum_{j=1}^{i} a^{H_j}(s,T_k) \Big) \mathrm{d}s + \int_0^t \ell_i(s-,T_k) \sqrt{c_s} \sigma_i(s,T_k) \mathrm{d}W_s^{T_{k+1}} \\ & + \int_0^t \int_{\mathbb{R}^d} \ln \beta_i(s,x,T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

The process v^2 from Lemma A.1 equals

$$v_s^2 = \frac{1+\delta_k L_i(s,T_k)}{\delta_k L_i(s,T_k)} = \ell_i^{-1}(s,T_k),$$

where $\ell_i(s,T_k)$ is defined in (19). Thus, it follows

$$\begin{aligned} L_i(t,T_k) \; = \; & L_i(0,T_k)\exp\left(\int_0^t b^{L_i}(s,T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma_i(s,T_k)\mathrm{d}W_s^{T_{k+1}}\right. \\ & \left. + \int_0^t\int_{\mathbb{R}^d} \ln\Big(1+\ell_i(s-,T_k)^{-1}(\beta_i(s,x,T_k)-1)\Big)(\mu-\nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\right), \end{aligned}$$

which is exactly (16). A tedious calculation, carried out by making use of Lemma A.1 and inserting (58) for $a^L(s,T_k)$ and (59) for $a^{H_j}(s,T_k)$, yields the drift term $b^{L_i}(s,T_k)$ given in (21). □

Proof of Lemma 5.2: Let us fix a rating $j \in \{1,\dots,i\}$ and for every $l = 1,2,\dots,k-1$ express the dynamics of $H_j(\cdot,T_l)$ under the measure $\mathbb{Q}_{T_k}$. Recall that

$$W_s^{T_{l+1}} = W_s^{T_k} - \int_0^s \sqrt{c_u}\left(\sum_{m=l+1}^{k-1} \ell(u-,T_m)\sigma(u,T_m)\right)\mathrm{d}u$$

and

$$\nu^{T_{l+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{m=l+1}^{k-1}\beta(s,x,T_m)\nu^{T_k}(\mathrm{d}s,\mathrm{d}x) = \prod_{m=l+1}^{k-1}\beta(s,x,T_m)F_s^{T_k}(\mathrm{d}x)\mathrm{d}s,$$

with $\beta(s,x,T_m)$ defined in (13). Therefore, equation (15) becomes

$$\begin{aligned} H_j(t,T_l) \; = \; & H_j(0,T_l)\exp\left(\int_0^t b^{H_j}(s,T_l,T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma_j(s,T_l)\mathrm{d}W_s^{T_k}\right. \\ & \left. + \int_0^t\int_{\mathbb{R}^d}\langle\gamma_j(s,T_l),x\rangle(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\right), \end{aligned} \tag{60}$$

where

$$\begin{aligned} b^{H_j}(s,T_l,T_k) \; := \; & b^{H_j}(s,T_l) - \left\langle\gamma_j(s,T_l), \sum_{m=l+1}^{k-1}\ell(s-,T_m)c_s\sigma(s,T_m)\right\rangle \\ & - \int_{\mathbb{R}^d}\langle\gamma_j(s,T_l),x\rangle\left(\prod_{m=l+1}^{k-1}\beta(s,x,T_m)-1\right)F_s^{T_k}(\mathrm{d}x). \end{aligned}$$

An application of Lemma A.2 to the processes $H_j(\cdot, T_l)$ given by (60), for $j = 1, \dots, i$ and $l = 1, \dots, k-1$, yields

$$\begin{aligned}
\mathbb{H}(t, T_k, i) &= \prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{1 + \delta_l H_j(t, T_l)} \\
&= \mathbb{H}(0, T_k, i) \exp\Bigg(- \int_0^t \sum_{j=1}^{i} \sum_{l=1}^{k-1} a^{H_j}(s, T_l, T_k) \mathrm{d}s \\
&\quad - \int_0^t \sum_{j=1}^{i} \sum_{l=1}^{k-1} h_j(s-, T_l) \sqrt{c_s} \gamma_j(s, T_l) \mathrm{d}W_s^{T_k} \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \ln \prod_{j=1}^{i} \prod_{l=1}^{k-1} \Big(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s, T_l), x\rangle} - 1)\Big) (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x)\Bigg)
\end{aligned}$$

with $h_j(s, T_l)$ defined in (18) and

$$\begin{aligned}
a^{H_j}(s, T_l, T_k) &:= h_j(s-, T_l) b^{H_j}(s, T_l, T_k) \\
&\quad + \frac{1}{2}(h_j(s-, T_l) - h_j(s-, T_l)^2) \|\sqrt{c_s}\gamma_j(s, T_l)\|^2 \\
&\quad + \int_{\mathbb{R}^d} \Bigg(\ln\Big(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s, T_l), x\rangle} - 1)\Big) \\
&\qquad - h_j(s-, T_l) \langle \gamma_j(s, T_l), x\rangle \Bigg) F_s^{T_k}(\mathrm{d}x).
\end{aligned}$$

Finally, we express the ordinary exponential as the stochastic exponential, which yields

$$\begin{aligned}
&\mathbb{H}(t, T_k, i) = \mathbb{H}(0, T_k, i) \\
&\times \mathcal{E}_t\Bigg(\int_0^{\cdot} b^{\mathbb{H}}(s, T_k, i) \mathrm{d}s - \int_0^{\cdot} \sum_{j=1}^{i} \sum_{l=1}^{k-1} h_j(s-, T_l) \sqrt{c_s} \gamma_j(s, T_l) \mathrm{d}W_s^{T_k} \\
&+ \int_0^{\cdot} \int_{\mathbb{R}^d} \Bigg(\prod_{j=1}^{i} \prod_{l=1}^{k-1} \Big(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s, T_l), x\rangle} - 1)\Big)^{-1} - 1 \Bigg) (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x)\Bigg),
\end{aligned}$$

where $b^{\mathbb{H}}(s, T_k, i)$ is given in (39) and obtained by plugging in the expressions for $a^{H_j}(s, T_l, T_k)$ and $b^{H_j}(s, T_l, T_k)$. □

Appendix B

Proof of Proposition 5.3: Let us fix an arbitrary jump time of Y and denote it by ξ. Recall that for the jump times of a canonically constructed conditional Markov chain we have

$$\tau_n = \sum_{i=1}^{n} \eta_i,$$

where

$$\eta_i = \inf\left\{ t \geq 0 : e^{\int_{\tau_{i-1}}^{\tau_{i-1}+t} \lambda_{\overline{C}_{i-1}, \overline{C}_{i-1}}(u) \mathrm{d}u} \leq U_{1,i} \right\},$$

for an $\mathbb{F}$-adapted matrix of stochastic intensities Λ and a sequence of random variables $(U_{1,i})$ that are uniformly distributed on $[0,1]$ and mutually independent. Moreover, they are independent from $\mathbb{F}$ by construction.

We claim that

$$\mathbb{Q}_{T^*}(\tau_n = \xi) = 0, \qquad n \in \mathbb{N}. \tag{61}$$

For $n = 1$ we have

$$\begin{aligned}\mathbb{Q}_{T^*}(\tau_1 = \xi) &= \mathbb{Q}_{T^*}(\eta_1 = \xi) = \mathbb{Q}_{T^*}\left(e^{\int_0^{\eta_1} \lambda_{\overline{C}_0,\overline{C}_0}(u)\mathrm{d}u} = e^{\int_0^{\xi} \lambda_{\overline{C}_0,\overline{C}_0}(u)\mathrm{d}u}\right) \\ &= \mathbb{Q}_{T^*}\left(U_{1,1} = e^{\int_0^{\xi} \lambda_{\overline{C}_0,\overline{C}_0}(u)\mathrm{d}u}\right) = 0,\end{aligned}$$

which follows by a simple calculation since $U_{1,1}$ is an absolutely continuous random variable independent from the $\mathcal{F}_\xi$-measurable random variable $e^{\int_0^{\xi} \lambda_{\overline{C}_0,\overline{C}_0}(u)\mathrm{d}u}$. This result also follows directly from Proposition 2 in Jakubowski and Nieweglowski (2009a).

For $n \geq 2$ we show the claim by writing

$$\mathbb{Q}_{T^*}(\tau_n = \xi) = \mathbb{Q}_{T^*}\left(\eta_n = \xi - \sum_{i=1}^{n-1} \eta_i\right) = \mathbb{Q}_{T^*}\left(U_{1,n} = e^{\int_{\tau_{n-1}}^{\xi} \lambda_{\overline{C}_{n-1},\overline{C}_{n-1}}(u)}\mathrm{d}u\right)$$

and noting that $U_{1,n}$ is independent from $e^{\int_{\tau_{n-1}}^{\xi} \lambda_{\overline{C}_{n-1},\overline{C}_{n-1}}(u)\mathrm{d}u}$ due to the canonical construction of C.

Since Y is a semimartingale, its trajectories are càdlàg functions on $[0,T^*]$ and therefore the set of all jump times of Y is at most countable. Denote these jump times by $\xi_m, m \in \mathbb{N}$. Then it easily follows that for every $n \in \mathbb{N}$

$$\mathbb{Q}_{T^*}(\Delta Y_{\tau_n} \neq 0) = \mathbb{Q}_{T^*}\left(\bigcup_{m \geq 1} \{\tau_n = \xi_m\}\right) \leq \sum_{m=1}^{\infty} \mathbb{Q}_{T^*}(\tau_n = \xi_m) = 0,$$

by the claim which we proved above.

Finally, since every forward measure $\mathbb{Q}_{T_k}$ is equivalent to the terminal forward measure $\mathbb{Q}_{T^*}$, we get

$$\mathbb{Q}_{T_k}(\Delta Y_{\tau_n} \neq 0) = 0, \qquad n \in \mathbb{N}.$$

□

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Ernst Eberlein, University of Freiburg, Department of Mathematical Stochastics, Eckerstr. 1, 79104 Freiburg, Germany
E-mail address: eberlein@stochastik.uni-freiburg.de

Zorana Grbac, University of Freiburg, Department of Mathematical Stochastics, Eckerstr. 1, 79104 Freiburg, Germany
E-mail address: zorana@stochastik.uni-freiburg.de