Chapter 1
Fourier based valuation methods in mathematical finance

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1.1 Introduction

A fundamental problem of mathematical finance is the explicit computation of expectations which arise as prices of derivatives. What leads to simple formulas in the classical setting when the underlying random quantity is modeled by a geometric Brownian motion, turns out to be rather nontrivial in more sophisticated modeling approaches. There is overwhelming statistical evidence that Brownian motion as the driver of models in equity, fixed income, credit and foreign exchange markets produces distributions which are far from reality and can be considered as first approximations at best. Lévy processes are a much more flexible class of drivers. They can be parametrized with a low-dimensional set and at the same time generate distributions which are more realistic from a statistical point of view. However in Lévy models simple closed-form valuation formulas are typically not available even in the case of plain vanilla European options. The situation is worse for more complicated exotic options.

Efficient methods to compute prices of derivatives are crucial in particular for calibration purposes. During a calibration procedure in each iteration step typically a large number of model prices has to be computed and compared to market prices. Models which cannot be calibrated within reasonable time limits are useless for most applications. A method which almost always works to get expectations is Monte Carlo simulation. Its disadvantage is that it is computer intensive and therefore too slow for many purposes. Another classical approach is to represent prices as solutions of partial differential equations (PDEs) which in the case of Lévy processes with jumps become partial integro-differential equations (PIDEs). This approach applies to a wide range of valuation problems, in particular it allows to compute prices of American options as well. Nevertheless the numerical solution of PIDEs rests on sophisticated discretization methods and corresponding programs. It is the purpose of this article to discuss the state of the art of a third approach which is based on Fourier methods and which is relatively simple.

The initial references for Fourier based methods to compute option prices are Carr and Madan (1999) and Raible (2000). Whereas the first mentioned authors consider Fourier transforms of appropriately modified call prices and then invert these, the second author starts with representing the option price as a convolution of the modified payoff and the log return density, then derives the bilateral Laplace transform and finally inverts the resulting product. In both cases the result is an integral which can be evaluated numerically fast. Let us mention that the approach is closely related to Parseval's formula in harmonic analysis although this classical formula does not apply directly in this context. From a number of subsequent papers we mention just Borovkov and Novikov (2002) who consider pricing formulas for certain exotic options and Hubalek, Kallsen, and Krawczyk (2006) where hedging formulas were derived. Another remarkable reference if Hurd and Zhou (2009). These authors develop an algorithm to price spread options which is a notoriously difficult task.

The following presentation of Fourier based methods for pricing equity derivatives is based on Eberlein, Glau, and Papapantoleon (2010) and Eberlein, Glau, and Papapantoleon (2011a). In these two closely connected papers we asked the question: What are the precise mathematical assumptions such that the Fourier approach works? It turned out that convolutions are not an essential ingredient. Instead it is just sufficient integrability of an appropriately dampened payoff function as well as of the relevant distribution and then Fubini's theorem is applied. The key point which makes the method computationally efficient is the separation of the payoff function and the distribution of the underlying process. These two ingredients enter into the integral representation formula as Fourier transform and as characteristic function or equivalently as moment generating function. The Fourier transform of the payoff is a trivial object. The characteristic function is also easily available if the option depends only on the distribution of the driving Lévy process at a fixed time point. For options which depend on the running supremum or the running infimum we show in Section 1.5 that there exist reasonable representations for the characteristic functions of the corresponding distributions. The computational effort is much higher for those cases.

The development of a Lévy interest rate theory started with Eberlein and Raible (1999) and was pushed further in a number of subsequent papers. The rates in those interest rate models are typically driven by stochastic integrals with respect to Lévy processes and not just by the processes themselves. Consequently the need for efficient numerical procedures to compute prices of interest rate derivatives such as caps, floors and swaptions is evidently higher and the power of the Fourier based method becomes even more visible. Since the underlying distribution enters only in the form of its moment generating or characteristic function, the result in Theorem 4 is crucial which shows that these quantities are easily available for the type of stochastic integrals which is used here.

In Section 1.6 we introduce first the Lévy forward rate model. It is the proper generalization of the Heath–Jarrow–Morton (HJM) framework. It is natural to use in interest rate theory immediatley a more general class of driving processes, namely time-inhomogeneous Lévy processes also called processes with independent increments and absolutely continuous characteristics (PIIAC) in Jacod and Shiryaev (1987). Fourier based integral representations for prices of options on zero coupon bonds (see (1.95)) have the same form as the formulas for equity options. We show here that the results can be obtained under the same integrability assumptions which were introduced in Eberlein et al. (2010) for equity models. Initially we had derived these formulas (see Eberlein and Kluge (2006a,b)) by using convolution representations in the spirit of Raible (2000). The payoffs of caps and floors, the basic interest rate derivatives, can be interpreted as payoffs of put and call options on zero coupon bonds. Therefore for model calibration the formulas for the latter options are the right tool. The more flexible class of time-inhomogeneous Lévy processes is important for the accurate calibration of interest models across different strikes and maturities. The shape of the volatility surface produced by cap and floor prices is too sophisticated along the maturity axis to be matched by a model which is driven by a (time-homogeneous) Lévy process.

The second important interest rate modeling approach is the Libor or market model where the forward Libor rates are taken as basic quantities. The Lévy Libor model was introduced in Eberlein and Özkan (2005). We sketch the backward construction of the Libor rates in Section 1.7 and derive the integral formulas for the standard derivatives. In some sense it is more natural to take instead of the numerically small Libor rates the closely connected forward processes as basic quantity to be modeled. The Libor rates vary in the range of 0.01 to 0.1, whereas the forward processes have values close to 1. Since the random quantity is always described via $\exp(x)$ and the variation of the exponential function is much higher near the origin than in the range where the argument is between 0.01 and 0.1, one can expect better results by modeling the forward processes. As an alternative approach this has also be done in Eberlein and Özkan (2005) (see also Eberlein and Kluge (2007)). Another advantage of the forward process approach is that there is no approximation necessary since up to a constant the forward process is itself the density process which is used for the measure change. The substantially simplified expressions which one gets from the backward induction in this case, speed up the numerical procedures and avoid any approximation error. Nevertheless it should be mentioned that the forward process model – similar to the HJM approach – produces negative rates as well. They occur with such a small probability that practitioners do not care about it. For the sake of brevity we do not reproduce here the Lévy forward process model. Note that it can be embedded in the forward rate model (see Eberlein and Kluge (2007)). Fourier based pricing formulas for derivatives can again be derived without any consideration of convolutions.

For completeness we mention some further results. Pricing formulas for digital as well as fixed and floating strike range options have been developed in Eberlein and Kluge (2006b). An extension to a credit risk model and pricing of credit derivatives such as credit default swaptions is the topic of Eberlein, Kluge, and Schönbucher (2006). A model extension to price cross-currency derivatives such as foreign forward caps and floors, cross-currency swaps and quanto caplets, was achieved in Eberlein and Koval (2006).

Comparing Fourier based methods to the use of PIDEs for option pricing, one should be aware that although the two approaches come from totally different mathematical fields they nevertheless have a lot in common as well. This becomes clear if one looks for an explicit solution of the PIDE as given in (Eberlein and Glau, 2011, Theorem 6.1). The PIDE for a European option can be interpreted as a pseudo differential equation. Its Fourier transform is an ordinary differential equation. The explicit solution of this equation is an integral which coincides with the integral representations which are discussed in this paper.

1.2 The driving process

Although Fourier based valuation methods can be applied in the general framework of models which are driven by semimartingales (see Eberlein et al. (2010)) we will present the approach in the following within a more restrictive setting. The main reason not to consider semimartingales in full generality in this context is that they cannot be parametrized in a low dimensional space and therefore the implementation and calibration of a general semimartingale model is not really practicable in finance. Nevertheless our treatment of stochastic processes is in the spirit of semimartingale theory with the only difference that some semimartingale components simplify considerably. Lévy processes and the larger class of time-inhomogeneous Lévy processes constitute suitable subclasses which offer on one side enough distributional flexibility and on the other side they are tractable from an analytic and from a statistical point of view.

We denote by $(\Omega, \mathscr{F}, \mathbb{F}, P)$ a complete stochastic basis, where the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T^*]}$ is assumed to satisfy the usual conditions. The later means that $\mathscr{F}$ is complete with respect to P and every $\mathscr{F}_t$ contains all P-null sets of $\mathscr{F}$. $T^* > 0$ is a finite time horizon and we assume that $\mathscr{F} = \mathscr{F}_{T^*}$. A *Lévy process* $L = (L_t)_{t\geq 0}$ is a process with stationary and independent increments defined on $(\Omega, \mathscr{F}, \mathbb{F}, P)$. Implicitly this means that L is adapted to $(\mathscr{F}_t)_{t\geq 0}$. We will assume that the process has càdlàg paths, i.e. the paths of L are right-continuous functions with left limits. It can be shown that there exists always a version with càdlàg paths. A Lévy process can be decomposed in the following way

$$L_t = bt + \sqrt{c}W_t + Z_t + \sum_{s\leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s|>1\}}. \tag{1.1}$$

Here b and $c \geq 0$ are real numbers, $(W_t)_{t\geq 0}$ in a standard Brownian motion, $(Z_t)_{t\geq 0}$ is a purely discontinuous martingale which is independent of $(W_t)_{t\geq 0}$ and $\Delta L_s = L_s - L_{s-}$ denotes the jump of L at time $s > 0$ if there is a jump at this time point. (1.1) is called the *canonical representation of the Lévy process* and it is also known as the *Lévy–Itô decomposition*. The last term in (1.1) represents the sum of the jumps of the process up to time t with absolute jump size bigger than 1. As a consequence of the assumption about càdlàg paths one gets that for any $\varepsilon > 0$ along any finite time interval there can be only a finite number of jumps which are bigger than ε in absolute value. Thus the last term in (1.1) is a finite sum.

In order to explain $(Z_t)_{t\geq 0}$ let us introduce some semimartingale notation. A semimartingale is a process $X = (X_t)_{t\geq 0}$ which admits a decomposition $X = X_0 + M + V$ where M is a local martingale starting at 0 and V is an adapted process of finite variation. For simplicity we shall assume $X_0 = 0$. If one takes the big jumps of X away the remaining process

$$X_t - \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}} \tag{1.2}$$

has bounded jumps and therefore is a *special semimartingale* (see Jacod and Shiryaev (1987, I.4.24)). A special semimartingale by definition admits a unique decomposition into a local martingale M and a *predictable* process with finite variation V. For Lévy processes the finite variation component V turns out to be the (deterministic) linear function bt. Any local martingale M with $M_0 = 0$ can be decomposed in a unique way $M = M^c + M^d$ when M^c is a local martingale with continuous paths and M^d is a *purely discontinuous local martingale* which is the process denoted by $Z = (Z_t)_{t\geq 0}$ in (1.1). For Lévy processes the continuous process M^c is nothing but a standard Brownian motion $W = (W_t)_{t\geq 0}$ which is scaled with a constant $\sqrt{c}$. There are many Lévy processes which are important for applications in finance where $c = 0$. These are purely discontinuous processes. Examples are hyperbolic (Eberlein and Keller (1995)), normal inverse Gaussian (Barndorff-Nielsen (1998)), variance gamma (Madan and Seneta (1990)), CGMY (Carr, Geman, Madan, and Yor (2002)), and generalized hyperbolic Lévy motions (Eberlein and Prause (2002)).

As we mentioned above the sum of the big jumps converges since there are only finitely many such jumps in any finite time interval. This is not true for the sum of the small jumps

$$\sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\leq 1\}}. \tag{1.3}$$

Nevertheless by compensating one can force this sum to converge too. Compensating means to subtract the average increase by the small jumps along the time interval $[0,t]$. This average is given by the *intensity measure* $F(dx)$ with which the jumps arrive. In order to introduce the intensity measure let us first introduce the *random measure of jumps* of X which is denoted by μ^X. If a path of the process given by ω has a jump of size $\Delta X_s(\omega) = x$ at time point s, the random measure $\mu^X(\omega;\cdot,\cdot)$ places a unit mass $\varepsilon_{(s,x)}$ at the point (s,x) in $\mathbb{R}_+ \times \mathbb{R}$. In other words

$$\mu^X(\omega;dt,dx) = \sum_{s>0} \mathbb{1}_{\{\Delta X_s \neq 0\}} \varepsilon_{(s,\Delta X_s(\omega))}(dt,dx). \tag{1.4}$$

For a time span $[0,t]$ and a Borel set $A \subset \mathbb{R}$

$$\mu^X(\omega;[0,t]\times A) = |\{(s,x) \in [0,t]\times A \mid \Delta X_s(\omega) = x\}| \tag{1.5}$$

counts the number of jumps with jump size within A which occur for the path given by ω from time 0 to t. Because of the stationarity and the independence of the increments of a Lévy process L the expectation of this random quantity is linear in t

$$E[\mu^L(\cdot;[0,t]\times A] = tF(A). \tag{1.6}$$

$F(A)$ is the intensity measure F applied to A. One can show that the following limit exists in the sense of convergence in probability

$$\lim_{\varepsilon\to 0}\left(\sum_{s\le t}\Delta L_s \mathbb{1}_{\{\varepsilon\le|\Delta L_s|\le 1\}} - t\int x\mathbb{1}_{\{\varepsilon\le|x|\le 1\}}F(dx)\right). \tag{1.7}$$

The sum represents the increase by jumps of absolute jump size between ε and 1 within time 0 and t. The integral is the average increase by jumps of size within the same range which happen along an interval of length 1. In general non of the two expressions has a finite limit as $\varepsilon \to 0$. Consequently the difference cannot be separated. Making use of the random measure of jumps μ^L we can write this limit in the form

$$\int_0^t\int_{\mathbb{R}} x\mathbb{1}_{\{|x|\le 1\}}(\mu^L(ds,dx) - dsF(dx)). \tag{1.8}$$

This is the more explicit form of the purely discontinuous martingale $Z = (Z_t)_{t\ge 0}$ in the canonical representation of a Lévy process given by (1.1). Z describes the compensated jumps of absolute size less than 1. Of course one could use any other threshold than 1 to separate the jumps according to their size. Note that the sum of the big jumps in (1.1) can now equivalently be expressed in the form

$$\sum_{s\le t}\Delta L_s \mathbb{1}_{\{|\Delta L_s|>1\}} = \int_0^t\int_{\mathbb{R}} x\mathbb{1}_{\{|x|>1\}}\mu^L(ds,dx). \tag{1.9}$$

To summarize this brief introduction of the components of a Lévy process we note that from the distributional point of view a Lévy process is characterized by the three quantities (b,c,F), the so-called *triplet of local characteristics*, which appear in the representation

$$\begin{aligned} L_t = bt + \sqrt{c}W_t &+ \int_0^t\int_{\mathbb{R}} x\mathbb{1}_{\{|x|\le 1\}}\big(\mu^L(ds,dx) - dsF(dx)\big) \\ &+ \int_0^t\int_{\mathbb{R}} x\mathbb{1}_{\{|x|>1\}}\mu^L(ds,dx). \end{aligned} \tag{1.10}$$

It is the same triplet which determines the Fourier transform of the distribution of L_1 in its Lévy–Khintchine form

$$\begin{aligned} E[\exp(iuL_1)] &= \exp\left[iub - \frac{1}{2}u^2c + \int_{\mathbb{R}}(e^{iux} - 1 - iux\mathbb{1}_{\{|x|\le 1\}})F(dx)\right] \\ &= \exp(\psi(u)). \end{aligned} \tag{1.11}$$

The intensity measure F is also called the *Lévy measure* and satisfies

$$\int_{\mathbb{R}} \min(1,x^2)F(dx) < \infty. \tag{1.12}$$

ψ as defined in (1.11) is called the *characteristic exponent*. A property which follows again from the independence and the stationarity of the increments of the process is that the distribution of L_1 (see (1.11)) determines the distribution of L_T for any $T > 0$ via

$$E[\exp(iuL_T)] = \exp(T\psi(u)). \tag{1.13}$$

This is an important fact which will be used later when we have to compute option prices which are given as expectations $E[f(L_T)]$ for some function f which is derived from the payoff. The parameters of the Lévy process which is used to drive the model are typically the parameters of the distribution of L_1. For L_T to possess an easy connection with L_1 as in (1.13) is crucial for the computation.

The Lévy measure F contains information on the finiteness of the moments of the process as well as on certain path properties. Finiteness of moments can be seen from the tails of F. Sato (1999, Theorem 25.3) shows that for a Lévy process L, L_t has finite absolute p-th moment for $p \in \mathbb{R}_+$ if and only if

$$\int_{\{|x|>1\}} |x|^p F(dx) < \infty \tag{1.14}$$

and L_t has finite exponential moment of order p for $p \in \mathbb{R}$ if and only if

$$\int_{\{|x|>1\}} \exp(px)F(dx) < \infty. \tag{1.15}$$

The equivalence expressed in (1.14) has an immediate consequence. If the expectation of L_1, is finite, then $\int_{\{|x|>1\}} xF(dx)$ is finite as well. Therefore we can add $\int iux\mathbb{1}_{\{|x|>1\}}F(dx)$ to the integral in (1.11) and end up with the simpler representation

$$E[\exp(iuL_1)] = \exp\left[iub - \frac{1}{2}u^2c + \int_{\mathbb{R}} (e^{iux} - 1 - iux)F(dx)\right] \tag{1.16}$$

where of course the parameter b is now different from (1.11). The same argument allows to simplify (1.10). If L_1 has finite expectation then $\int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|>1\}} ds F(dx)$ is finite. We add this to the second integral in (1.10), merge the two resulting integrals and get the following simpler representation where again the b differs from the one in (1.10)

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L(ds,dx) - dsF(dx)). \tag{1.17}$$

From this representation one also sees that L is a *martingale* iff $b = E[L_1] = 0$. Since the expectation of the generating variable L_1 of all Lévy processes which we actually use in finance is finite, we will work with the more convenient forms (1.16) resp. (1.17) instead of (1.11) resp. (1.10).

Whereas the information on the existence of the moments $E[|L_t|^p]$ of the process sits in the tails of F, the path properties depend on the distribution of the mass of F around the origin. Sato (1999, Theorem 21.9) shows that almost all paths of L have *finite variation* if $c = 0$ and

$$\int_{\{|x|\leq 1\}} |x| F(dx) < \infty. \tag{1.18}$$

Almost all paths have infinite variation if $c \neq 0$ or if the integral in (1.18) is not finite.

If the integral in (1.18) is finite, this has consequences for the representation given in (1.17). In this case the sum of the small jumps converges and can be given in the form

$$\sum_{s\leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s|\leq 1\}} = \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x\leq 1\}} \mu^L(ds, dx) \tag{1.19}$$

and one can separate the integral in (1.17)

$$\int_0^t \int_{\mathbb{R}} x(\mu^L(ds, dx) - ds F(dx)) = \int_0^t \int_{\mathbb{R}} x \mu^L(ds, dx) - t \int_{\mathbb{R}} x F(dx). \tag{1.20}$$

Let us illustrate the decomposition of a Lévy process into drift, Gaussian component and compensated jumps as given by (1.17) by looking at the simplest process with jumps, the standard Poisson process. The jumps of size 1 occur with a rate λ per unit time. The Lévy measure is $F = \lambda \varepsilon_1$, the point mass in 1 scaled by the intensity parameter λ. There is no Gaussian part, therefore $c = 0$. The canonical representation is

$$L_t = \lambda t + (L_t - \lambda t) = \lambda t + \left(\sum_{n\geq 1} \mathbb{1}_{\{T_n\leq t\}} - \lambda t \right) \tag{1.21}$$

where $(T_n)_{n\geq 1}$ denotes the successive random times where the jumps occur.

The Poisson process is the simplest example of a process L with *finite activity*. Finite activity means that almost all paths of L have only a finite number of jumps along every compact interval. This is the case if $F(\mathbb{R}) < \infty$. If $F(\mathbb{R}) = \infty$ then almost all paths of L have an infinite number of jumps along every compact interval. In this case the process has *infinite activity*. The infinite mass of F sits around the origin. In both tails F has finite mass (see (1.12)). Most of the non-Gaussian Lévy processes which are used in modelling in finance are purely discontinuous, infinite activity processes. Prominent examples are hyperbolic, normal inverse Gaussian, variance gamma, and CGMY (for $Y > 0$) Lévy motions.

1.3 Exponential Lévy models

In order to price derivatives depending on a financial asset such as a stock, an index or an FX rate we model the underlying price process by

$$S_t = S_0 \exp(L_t) \tag{1.22}$$

where $L = (L_t)_{t\geq 0}$ is a Lévy process which is generated by the distribution $\mathscr{L}(L_1) = \nu$. Equation (1.22) is called an *exponential Lévy model*. The main reason to start with an ordinary exponential instead of a stochastic exponential or equivalently a stochastic differential equation is the statistical aspect. Taking log returns, $\log S_{t+1} - \log S_t$, along a time grid with span 1, from the price process in (1.22) one gets the generating distribution ν of the Lévy process. Therefore plugging in the Lévy process generated by an (infinitely divisible) distribution which one got out of analysing a time series of price date, one can be sure that the model has the right distribution at least for that time horizon. Equation (1.22) can be described alternatively by the following stochastic differential equation

$$dS_t = S_{t-}\left(dL_t + \frac{c}{2}dt + \int_{\mathbb{R}} (e^x - 1 - x)\mu^L(dt,dx)\right) \tag{1.23}$$

where S_{t-} denotes the left limit at time point t. If one writes (1.23) for short in the form

$$dS_t = S_{t-} d\widetilde{L}_t \tag{1.24}$$

then $(\widetilde{L}_t)_{t\geq 0}$ is a Lévy process with jumps bigger than -1, i.e. not a general Lévy process from any of the classes which we want to consider.

For pricing derivatives which depend on the underlying price process given by (1.22), we want $(S_t)_{t\geq 0}$ to be a martingale. For simplicity we assume here that the interest rate r is 0. If one wants to make the discount factor $\exp(-rt)$ explicit, one can just use the drift parameter $b + r$ instead of b in (1.17).

A necessary assumption for a martingale is that each variable has a finite expectation $E[S_t] = S_0 E[\exp(L_t)] < \infty$. Due to the equivalence (1.15) the finiteness of exponential moments of order 1 of the Lévy process can be achieved by

Assumption $(\mathbb{EM})$**:** There exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} \exp(ux) F(dx) < \infty \qquad \text{for all } u \in [-M, M]. \tag{1.25}$$

In the following we will always assume that the driving Lévy process satisfies Assumption $(\mathbb{EM})$. Note that this excludes a priori the class of stable Lévy processes in general. The lack of martingality of exponential Lévy models driven by stable processes seems to be the main reason why stable distributions did not become more popular in pricing models. On the contrary all the processes mentioned above like hyperbolic, normal inverse Gaussian, generalized hyperbolic, variance gamma, and CGMY Lévy processes satisfy ($\mathbb{EM}$). Since $E[\exp(L_t)] < \infty$ implies in particular that $E[L_t] < \infty$, we can and will use the simpler decomposition (1.17) for L. From the stochastic differential equation (1.23) one can derive that $(S_t)_{\geq 0}$ is a martingale if the drift parameter b coincides with the exponential compensator of the Gaussian and the pure jump part of L, i.e.

$$b = -\frac{c}{2} - \int_{\mathbb{R}} (e^x - 1 - x) F(dx). \tag{1.26}$$

We note here that if we would start with a historical measure P, because of the rich structure of Lévy processes, the set of equivalent martingale measures would in general be very large. It is shown in Eberlein and Jacod (1997) that under slight regularity assumptions for purely discontinuous exponential Lévy models the prices of call options under all equivalent martingale measures (EMMs) span the whole no-arbitrage interval. In this survey we do not enter in a discussion on the choice of EMMs but consider a priori a martingale model which is determined by (1.26).

A number of payoff functions of options do not only depend on the value of the underlying at maturity T but on the whole price path from 0 to T. Typical examples are lookback or barrier options. In this case it is the running supremum $\overline{S}_t = \sup_{0 \leq u \leq t} S_u$ or the running infimum $\underline{S}_t = \inf_{0 \leq u \leq t} S_u$ which is compared to a strike price K or a barrier B. Since the exponential function is monotone and increasing we get

$$\overline{S}_T = \sup_{0 \leq t \leq T} (S_0 e^{L_t}) = S_0 e^{\overline{L}_T} \tag{1.27}$$

and similarly $\underline{S}_T = S_0 e^{\underline{L}_T}$. Therefore it is the distribution of the running supremum and the running infimum of the driving process L which enters into the valuation formulas. There are also other functionals of the whole price path which have to be considered. For example in the case of Asian options a discrete or continuous average value is compared to the strike.

1.4 The Fourier approach to derivative pricing

The computational efficiency of the Fourier or Laplace based approach to valuation formulas in exponential Lévy models is essentially due to the separation of the payoff function and the underlying process. Let us illustrate the first step of this separation by looking at a fixed strike lookback option with maturity T. The payoff in this case is $(\overline{S}_T - K)^+$ where $(S_t)_{t \geq 0}$ is assumed to be an exponential Lévy process. We write this as

$$(\overline{S}_T - K)^+ = (S_0 e^{\overline{L}_T} - K)^+ = \left(e^{\overline{L}_T + \log S_0} - K\right)^+. \tag{1.28}$$

Now we can identify the function $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = (e^x - K)^+$ into which the supremum of the log-asset price process plus a constant is inserted.

In general we have to consider a functional ϕ of the whole price path which we write in the form

$$\phi(S_0 e^{L_t}, 0 \leq t \leq T) = f(X_T - s) \tag{1.29}$$

where $s = -\log S_0$ and the driving process X can be L, $\overline{L}$, $\underline{L}$ or other functions of the path of the underlying Lévy process. Assuming the interest rate r to be 0 we get the time-0-price of this option as a function of the process X and the value s in the form

$$\mathbb{V}_f(X;s) = E[\phi(S_t, 0 \leq t \leq T)] = E[f(X_T - s)]. \tag{1.30}$$

Expectation is taken with respect to the martingale measure which was discussed in the previous section.

The functions f as in the example above are typically not bounded. To enforce some degree of integrability or boundedness one has to *dampen* f. Define

$$g(x) = e^{-Rx} f(x) \tag{1.31}$$

for some suitably chosen real value R. We denote by M_{X_T} the moment generating function and by φ_{X_T} the characteristic function of the random variable X_T. Thus

$$M_{X_T}(u) = E[e^{uX_T}] = \varphi_{X_T}(-iu) \tag{1.32}$$

for $u \in \mathbb{C}$. Note that both, M_{X_T} and φ_{X_T}, are extended to the complex plane where this is possible. Furthermore we denote by $L^1_{bc}(\mathbb{R})$ the space of bounded, continuous function in $L^1(\mathbb{R})$ and by $\widehat{g}$ the Fourier transform of a function g.

The following Fourier based valuation formula can be derived under two alternative sets of assumptions.

Assumptions $(\mathbb{C})$:
- (C1) $g \in L^1_{bc}(\mathbb{R})$
- (C2) $M_{X_T}(R)$ is finite
- (C3) $\widehat{g} \in L^1(\mathbb{R})$

Assumptions $(\mathbb{C}')$:
- (C1′) $g \in L^1(\mathbb{R})$
- (C3′) $(e^{Rx} P_{X_T})^{\wedge} \in L^1(\mathbb{R})$

We will present the formula and the proof under $(\mathbb{C}')$ since the analogous result under Assumptions $(\mathbb{C})$ has been given in detail in Eberlein et al. (2010).

Theorem 1. *Assume* $(\mathbb{C})$ *or alternatively* $(\mathbb{C}')$, *where the asset price is modeled by an exponential Lévy model* $S = (S_t)_{t \geq 0}$ *as given in* (1.22) *which satisfies* $(\mathbb{EM})$. *Then the time-0-price of an option on S with payoff* $f(X_T - s)$ *at maturity can be represented as*

$$\mathbb{V}_f(X;s) = \frac{e^{-Rs}}{2\pi} \int_{\mathbb{R}} e^{-ius} \varphi_{X_T}(u - iR) \widehat{f}(-u + iR) du. \tag{1.33}$$

Proof. First observe that

$$\mathbb{V}_f(X;s) = \int_{\Omega} f(X_T - s) dP = e^{-Rs} \int_{\mathbb{R}} e^{Rx} g(x - s) P_{X_T}(dx). \tag{1.34}$$

Now we merge e^{Rx} in (1.34) as a density with P_{X_T}. Then (C3′) implies that the distribution $e^{Rx}P_{X_T}$ has a continuous, bounded Lebesgue density, say $\rho(x)$. By Fourier inversion

$$\rho(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)du. \tag{1.35}$$

Now we get from (1.34)

$$\begin{aligned}
\mathbb{V}_f(X;s) &= e^{-Rs}\int_{\mathbb{R}} g(x-s)\rho(x)dx \\
&= \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} g(x-s)\left(\int_{\mathbb{R}} e^{-ixu}\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)du\right)dx \\
&= \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x-s)e^{-ixu}dx\right)\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)du \\
&= \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\left(\int_{\mathbb{R}} g(x-s)e^{i(x-s)(-u)}dx\right)\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)du \\
&= \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\widehat{g}(-u)\varphi_{X_T}(u-iR)du \\
&= \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\varphi_{X_T}(u-iR)\widehat{f}(iR-u)du.
\end{aligned}$$

The use of Fubini's theorem is justified here since by (C1′) and (C3′)

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} g(x-s)|e^{-ixu}|\left|\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)\right|dudx \\
&\qquad \leq \int_{\mathbb{R}} g(x-s)\left(\int_{\mathbb{R}}\left|\left(e^{R\cdot}P_{X_T}\right)^{\wedge}(u)\right|du\right)dx \leq K\int_{\mathbb{R}} g(x)dx < \infty. \qquad \square
\end{aligned}$$

Assumptions ($\mathbb{C}$) are appropriate if the payoff function is continuous as is the case for example for call and put options. This continuity is not required under Assumptions ($\mathbb{C}'$), but note that (C3′) implies absolute continuity of the distribution of $e^{Rx}P_{X_T}$ with respect to Lebesgue measure. Consequently one can say that the representation of the price in Theorem 1 can be achieved under some continuity assumption. This can be the continuity of the payoff function or the absolute continuity of the distribution. The representation (1.33) can still be achieved if non of the two is guaranteed, but in this case one has to check the variation and the continuity of $\mathbb{V}_f(X;s)$ as a function of x. For details see Theorem 2.7 in Eberlein et al. (2010).

As far as the verification of the Assumptions ($\mathbb{C}$) or ($\mathbb{C}'$) is concerned, the non-trivial one is (C3) resp. (C3′). As a side result an elegant sufficient condition for (C3) was obtained in Eberlein et al. (2010, Lemma 2.5). (C3) holds true if g is in the Sobolev space $H^1(\mathbb{R})$.

The two ingredients which are necessary for the integral representation (1.33) are $\widehat{f}$ and φ_{X_T}. $\widehat{f}$ is obtained via an elementary integration. Let us consider some examples. For a call option with $f(x) = (e^x - K)^+$ one gets

$$\widehat{f}(u+iR) = \frac{K^{1+iu-R}}{(iu-R)(1+iu-R)} \qquad \text{where } R \in I_1 = (1,\infty). \tag{1.36}$$

The put with $f(x) = (K-e^x)^+$ has exactly the same transform $\widehat{f}$, but R has to be chosen differently, namely $R \in I_1 = (-\infty, 0)$. For a digital call with payoff $f(x) = \mathbb{1}_{\{e^x > B\}}$ for some $B > 0$ one gets

$$\widehat{f}(u+iR) = -B^{iu-R}\frac{1}{iu-R} \qquad \text{where } R \in I_1 = (0,\infty). \tag{1.37}$$

If the payoff is $f(x) = \mathbb{1}_{\{e^x < B\}}$, the minus sign in front of the right side of (1.37) becomes a plus sign and R has to be chosen from $I_1 = (-\infty, 0)$.

For a double digital call option with $f(x) = \mathbb{1}_{\{\underline{B} < e^x < \overline{B}\}}$ one gets

$$\widehat{f}(u+iR) = \frac{1}{iu-R}\left(\overline{B}^{iu-R} - \underline{B}^{iu-R}\right) \qquad \text{where } R \in I_1 = \mathbb{R} \setminus \{0\}. \tag{1.38}$$

Another example is an asset-or-nothing digital call with $f(x) = e^x \mathbb{1}_{\{e^x > B\}}$. The corresponding Fourier transform is

$$\widehat{f}(u+iR) = -\frac{B^{1+iu-R}}{1+iu-R} \qquad \text{for } R \in I_1 = (1,\infty). \tag{1.39}$$

Finally we mention self-quanto calls with $f(x) = e^x(e^x - K)^+$. Here we get

$$\widehat{f}(u+iR) = \frac{K^{2+iu-R}}{(1+iu-R)(2+iu-R)} \qquad \text{where } R \in I_1 = (2,\infty). \tag{1.40}$$

Now let us turn to the second ingredient, the characteristic function φ_{X_T}. For non-path-dependent European options with underlying price process $S_t = S_0 \exp(L_t)$, $(X_t)_{t\geq 0}$ is just the driving process $(L_t)_{t\geq 0}$. Furthermore as mentioned earlier in (1.13)

$$\varphi_{L_T}(u) = (\varphi_{L_1}(u))^T. \tag{1.41}$$

Consequently we need only φ_{L_1} in explicit form whatever the maturity T of the option is. For the generalized hyperbolic Lévy motion (see e.g. Eberlein and Prause (2002)) with five parameters $0 \leq |\beta| < \alpha$, $\mu \in \mathbb{R}$, $\delta > 0$, and $\lambda \in \mathbb{R}$ one can easily derive

$$\varphi_{L_1}(u) = e^{iu\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \tag{1.42}$$

where K_λ denotes the modified Bessel function of the third kind with index λ. In order to demonstrate how easy one gets these characteristic functions in most cases let us consider a gamma process $(L_t)_{t\geq 0}$. The moment generating function is

$$
\begin{aligned}
E[e^{uL_1}] &= \int e^{ux} \frac{c^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-cx} dx \\
&= \int \frac{c^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-(c-u)x} dx \\
&= \frac{c^\gamma}{(c-u)^\gamma} \int \frac{(c-u)^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-(c-u)x} dx \\
&= \left(\frac{c}{c-u}\right)^\gamma \text{ for } u < c
\end{aligned}
\tag{1.43}
$$

since the last integral is just 1. The corresponding characteristic function is then

$$
\varphi_{L_1}(u) = E[\exp(iuL_1)] = \left(\frac{c}{c-iu}\right)^\gamma. \tag{1.44}
$$

The same argument can be used for all distributions whose density has a linear term of the form cx in the exponent.

Stochastic volatility models can be handled in this context as well. Let us briefly discuss the stochastic volatility Lévy model introduced by Carr, Geman, Madan, and Yor (2003). First a stochastic clock $Y_t = \int_0^t y_s ds$ is defined where the integrand is given by a CIR process which satisfies the stochastic differential equation

$$
dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{\frac{1}{2}} dW_t \tag{1.45}
$$

for parameters κ, η, and λ. The characteristic function for Y_t is well known from the work of CIR. Now consider a pure jump Lévy process $X = (X_t)_{t\geq 0}$ which is independent of $Y = (Y_t)_{t\geq 0}$. The stochastic volatility Lévy process is then defined by

$$
H_t = X_{Y_t}. \tag{1.46}
$$

Its characteristic function depends on the characteristic functions of X and Y in the following way

$$
\varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu\varphi_{X_t}(-i)))^{iu}}. \tag{1.47}
$$

Before we turn to the more sophisticated situation where the driving process depends on the whole path in the next section, let us mention that options on multiple assets can be treated along the same lines. Typical examples are basket options such as options on the minimum of assets with price processes $S^1, \dots, S^d$. The payoff is given by the functional $(S_T^1 \wedge \cdots \wedge S_T^d - K)^+$. Other examples where several processes have to be considered are multiple functionals of one asset such as barrier options of the type $(S_T - K)^+ \mathbb{1}_{\{\bar{S}_T > B\}}$ or slide-in or corridor options with payoff $(S_T - K)^+ \sum_{i=1}^N \mathbb{1}_{\{L < S_{T_i} < H\}}$. In all multiple asset cases one models the price processes by $S_t^i = S_0^i \exp(L_t^i)$ $(1 \leq i \leq d)$ as before and the function f is now a function $f : \mathbb{R}^d \to \mathbb{R}_+$ with a dampened payoff $g(x) = e^{-\langle R,x\rangle} f(x)$ $(x \in \mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^d$. There is a d-dimensional version of assumptions (see $(\mathbb{C})$ resp. $(\mathbb{C}')$) which allow an integral representation analogous to (1.33). For details see Eberlein et al. (2010).

Another issue which is discussed in Eberlein et al. (2010) are the *sensitivities* or *Greeks*. Given the integral formula (1.33) – where it is preferable to write it as a function of $S_0 = e^{-s}$ instead of s – one can easily take the first and the second derivative with respect to S_0 in order to get an explicit form for the delta and the gamma of the option. Whenever integration with respect to u and taking the derivative with respect to S_0 can be interchanged, one gets a formula similar to (1.33) for the delta and the gamma.

1.5 Path dependent options

For a fixed strike lookback call with payoff $(\overline{S}_T - K)^+$ the function f in the general valuation formula (1.33) is the same as for a standard call and thus $\widehat{f}$ is given by (1.36). The quantity which is nontrivial in this case is φ_{X_T} since $(X_t)_{t\geq 0}$ is the running supremum $(\overline{L}_t)_{t\geq 0}$ of the Lévy process L. This section is devoted to the study of the characteristic function of the running supremum $\overline{L}$ and the running infimum $\underline{L}$ of a Lévy process L. Remember that we always assume $(\mathbb{EM})$ (see (1.25)) in order to secure enough integrability for the process. From Sato's result (see (1.14) and (1.15)) it follows that Assumption $(\mathbb{EM})$ implies $E[\exp(uL_t)] < \infty$ for all $u \in [-M, M]$. It has been shown in Eberlein et al. (2011a, Lemma 8.4) that $(\mathbb{EM})$ implies even more, namely that for $u \leq M$ also

$$E[\exp(u\overline{L}_t)] < \infty \qquad \text{and} \qquad E[\exp(-u\underline{L}_t)] < \infty. \tag{1.48}$$

The key result which we need in order to get the characteristic functions of $\overline{L}_t$ and $\underline{L}_t$ is the *Wiener–Hopf factorization*. Let θ denote an exponentially distributed random variable with parameter $q > 0$ which is independent of L. Then for all $u \in \mathbb{R}$

$$E[\exp(iuL_\theta)] = E[\exp(iu\overline{L}_\theta)]E[\exp(iu\underline{L}_\theta)]. \tag{1.49}$$

As sophisticated as this celebrated factorization looks, the basic idea behind its proof is simple. Write $\overline{L}_t = (\overline{L}_t - L_t) + L_t$. From the fluctuation theory for Lévy processes it is well know that $\overline{L}_t - L_t$ has the same distribution as $-\underline{L}_t$ (see e.g. Kyprianou (2006, Lemma 3.5)). Using in addition independence in the distributional equality $L_t \overset{\mathscr{L}}{=} \overline{L}_t + \underline{L}_t$ one can derive (1.49).

Since the characteristic function on the left side of (1.49) can be easily evaluated as $q(q - \psi(u))^{-1}$ where $\psi(u)$ is the characteristic exponents of L (see (1.11)), we can express (1.49) equivalently in the form which appears in many books under the name Wiener–Hopf factorization

$$\frac{q}{q - \psi(u)} = \varphi_q^+(u)\varphi_q^-(u) \quad (u \in \mathbb{R}). \tag{1.50}$$

Here φ_q^+ resp. φ_q^- denotes the characteristic function of $\overline{L}_\theta$ resp. $\underline{L}_\theta$. These so-called *Wiener–Hopf factors* have the following integral representation

$$\varphi_q^+(u) = \int_0^\infty E[e^{iu\overline{L}_t}]qe^{-qt}dt, \tag{1.51}$$

$$\varphi_q^-(u) = \int_0^\infty E[e^{iu\underline{L}_t}]qe^{-qt}dt. \tag{1.52}$$

Another representation (see Sato (1999, Theorems 45.2, 45.7, and Corollary 45.8)) is

$$\varphi_q^+(u) = \exp\left[\int_0^\infty t^{-1}e^{-qt}\int_0^\infty (e^{iux}-1)P_{L_t}(dx)dt\right], \tag{1.53}$$

$$\varphi_q^-(u) = \exp\left[\int_0^\infty t^{-1}e^{-qt}\int_{-\infty}^0 (e^{iux}-1)P_{L_t}(dx)dt\right]. \tag{1.54}$$

Since the characteristic function φ_{X_T} appears in formula (1.33) with a complex argument, it is necessary to extend the representations for φ_q^+ and φ_q^- to the complex plane as far as possible. For this purpose let us first define a constant $\alpha^*(M)$. Recall the triplet of local characteristics (b,c,F) as given in (1.10) and (1.11). Define

$$\overline{\alpha}(M) = M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} |e^{Mx}-1-Mx|F(dx), \tag{1.55}$$

$$\underline{\alpha}(M) = M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} |e^{-Mx}-1+Mx|F(dx), \tag{1.56}$$

and

$$\alpha^*(M) = \max\{\overline{\alpha}(M), \underline{\alpha}(M), \psi(-iM)\}. \tag{1.57}$$

Now let L be a Lévy process which is not a compound Poisson process. Suppose the parameter q of the exponentially distributed random variable θ satisfies $q > \alpha^*(M)$. Then the Wiener–Hopf factors φ_q^+ resp. φ_q^- can be extended analytically to the half planes $\{z \in \mathbb{C} \mid -M < \mathrm{Im}(z) < \infty\}$ resp. $\{z \in \mathbb{C} \mid -\infty < \mathrm{Im}(z) < M\}$ (see Eberlein et al. (2011a, Lemma 8.7)). Formulas (1.51) and (1.52) continue to hold in this domain. Furthermore for $\xi \in \{z \in \mathbb{C} \mid -M < \mathrm{Im}(z) < \infty\}$ also the maps $q \to \varphi_q^+(\xi)$ and $q \to \varphi_q^-(-\xi)$ have an analytic extension to the half plane $\{z \in \mathbb{C} \mid \alpha^*(M) < \mathrm{Re}(z) < \infty\}$. Now we are ready to invert the Wiener–Hopf factors in order to get the characteristic function of $\overline{L}_t$ resp. $\underline{L}_t$, i.e. of $\overline{L}$ and $\underline{L}$ considered at a *fixed* time point t.

Theorem 2. *Let L be a Lévy process that satisfies Assumption* $(\mathbb{EM})$ *and is not a compound Poisson process. Then the analytically extended characteristic functions of $\overline{L}_t$ and $\underline{L}_t$ are given by*

$$\varphi_{\overline{L}_t}(\xi) = \lim_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} \frac{e^{t(Y+iv)}}{Y+iv}\varphi_{Y+iv}^+(\xi)dv \tag{1.58}$$

resp.

$$\varphi_{\underline{L}_t}(-\xi) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{t(\widetilde{Y}+iv)}}{\widetilde{Y}+iv} \varphi^{-}_{\widetilde{Y}+iv}(-\xi)dv \tag{1.59}$$

for $\xi \in \{z \in \mathbb{C} \mid -M < \mathrm{Im}(z) < \infty\}$ *and* $Y, \widetilde{Y} > \alpha^*(M)$.

The proof is given in Eberlein et al. (2011a, Theorem 8.13).

At this point we can also explain why the constant M in Assumption $(\mathbb{EM})$ has to have a minimum size $M > 1$. In the valuation formula (1.33) $\varphi_{\overline{L}_t}$ appears with the argument $u - iR$. According to Theorem 2 $\varphi_{\overline{L}_t}$ is available on the half plane $\{z \in \mathbb{C} \mid -M < \mathrm{Im}(z) < \infty\}$. This requires $R < M$. On the other side not only the assumptions on the distribution of $\overline{L}_t$ but also the assumptions on f ((C1) and (C3)) restrict the domain from which R can be chosen. According to (1.36) R has to be larger than 1. Consequently only for $M > 1$ there is a nonempty intersection of these two domains namely the interval $(1, M)$.

As the theory which we exposed in this section shows, there are at least four integrations necessary in order to compute the price of an option whose payoff depends on the running supremum or the running infimum of a Lévy process. This is the integration in the valuation formula (1.33) itself, then there is an integration to get the characteristic function of the underlying process (see (1.58) and (1.59)), and finally the Wiener–Hopf factors φ^+ and φ^- are represented as double integrals (see (1.51)–(1.54)). From the numerical point of view four integrations take too much time for practical purposes.

Fortunately under slight additional regularity assumptions the double integral in the representation of the Wiener–Hopf factors can be reduced to a single integration. The following discussion is motivated by a similar result in Boyarchenko and Levendorskiĭ (2002), but the assumptions as well as the proofs are different. Almost all of the Lévy processes which we use in financial models are within the class to be defined now.

Definition 1. Let $\lambda_- < 0 < \lambda_+$ and $\nu \in (0, 2]$. A Lévy process L is called a *regular Lévy process of exponential type* $[-M, M]$ *and order* $\nu > 0$ *(RLPE)* if the following holds

(1) There exist constants $c > 0$ and $\nu_1 \in [0, \nu]$ as well as a function $\phi : \mathbb{C} \to \mathbb{C}$ such that

   (a) ϕ is analytic on the strip $S = \{z \in \mathbb{C} \mid -M < \mathrm{Im}(z) < M\}$
   (b) ϕ is continuous on $\overline{S} = \{z \in \mathbb{C} \mid -M \leq \mathrm{Im}(z) \leq M\}$
   (c) $-\phi(\xi) = -c|\xi|^{\nu} + O(|\xi|^{\nu_1})$ for $|\xi| \to \infty$ where $\xi \in \overline{S}$ and $c > 0$
   (d) for $\xi \in \overline{S}$ the characteristic exponent is given in the form

$$\psi(\xi) = i\mu\xi - \phi(\xi).$$

(2) There exist constants $\widetilde{C} > 0$ and $\nu_2 \in [0, \nu)$ such that the derivative of ϕ satisfies for $\xi \in \overline{S}$

$$|\phi'(\xi)| \leq \widetilde{C}(1 + |\xi|)^{\nu_2}.$$

Brownian motion is an RLPE of order 2, generalized hyperbolic (GH) Lévy motions are RLPE $[-M,M]$ of order 1 provided $[-M,M] \subset (-\alpha+\beta, \alpha+\beta)$ where α resp. β denotes the shape resp. the skewness parameter of the generating GH distribution. Only variance gamma processes pose a problem in this context since they are of order 0. Now we are ready to state a significantly simplified formula for the Wiener–Hopf factors.

Theorem 3. *Suppose $q > \alpha^*(M)$ and L is an RLPE of order $\nu > 0$ which satisfies* ($\mathbb{EM}$). *Furthermore choose ω_+, ω_-, and $d > 0$ such that*

$$q - \operatorname{Re}(\psi(\xi)) \geq d(1+|\xi|)^\nu > 0$$

holds for $\xi \in \{z \in \mathbb{C} \mid \omega_- \leq \operatorname{Im}(z) \leq \omega_+\}$. Then

$$\begin{aligned} \varphi_q^+(\xi) &= \exp\left[-\frac{1}{2\pi i}\int_{-\infty+i\omega_-}^{\infty+i\omega_-} \frac{\psi'(\eta)}{q-\psi(\eta)} \ln\left(\frac{\eta-\xi}{\eta}\right) d\eta\right] \\ &= \exp\left[\frac{1}{2\pi i}\int_{-\infty+i\omega_-}^{\infty+i\omega_-} \frac{\xi \ln(q-\psi(\eta))}{(\xi-\eta)\eta} d\eta\right] \end{aligned} \tag{1.60}$$

for $\xi \in \{z \in \mathbb{C} \mid \omega_- < \operatorname{Im}(z) < \infty\}$ and

$$\begin{aligned} \varphi_q^-(\xi) &= \exp\left[\frac{1}{2\pi i}\int_{-\infty+i\omega_+}^{\infty+i\omega_+} \frac{\psi'(\eta)}{q-\psi(\eta)} \ln\left(\frac{\eta-\xi}{\eta}\right) d\eta\right] \\ &= \exp\left[-\frac{1}{2\pi i}\int_{-\infty+i\omega_+}^{\infty+i\omega_+} \frac{\xi \ln(q-\psi(\eta))}{(\xi-\eta)\eta} d\eta\right] \end{aligned} \tag{1.61}$$

for $\xi \in \{z \in \mathbb{C} \mid -\infty < \operatorname{Im}(z) < \omega_+\}$.

These integral representations have been achieved and proved in Maier (2011, Section 4).

The speed of the numerical evaluation of formulas (1.60) and (1.61) can be further increased by making use of the following symmetry properties of the Wiener–Hopf factors. Suppose $Y > \alpha^*(M)$ and a, $v \in \mathbb{R}$, then

$$\operatorname{Re}(\varphi^\pm_{Y+iv}(a+ib)) = \operatorname{Re}(\varphi^\pm_{Y-iv}(-a+ib)) \tag{1.62}$$

and

$$\operatorname{Im}(\varphi^\pm_{Y+iv}(a+ib)) = -\operatorname{Im}(\varphi^\pm_{Y-iv}(-a+ib)), \tag{1.63}$$

where for φ^+_{Y+iv} the value of b has to be chosen from (ω_-,∞) and for φ^-_{Y+iv} from $(-\infty,\omega_+)$. Some numerical results for normal inverse Gaussian (NIG) processes based on these symmetries which have been obtained in Maier (2011) will be presented. For a NIG process the characteristic exponent is given by

$$\psi(u) = iu\mu + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right) \quad (u \in \mathbb{R}) \tag{1.64}$$

where $0 \leq |\beta| < \alpha$, $\mu \in \mathbb{R}$ and $\delta > 0$.

NIG Lévy processes are the subclass of generalized hyperbolic Lévy processes with class parameter $\lambda = -0.5$. We choose parameter values which were estimated from daily DAX returns for the period June 1, 1997 to June 1, 1999 (see Raible (2000)). They are $\alpha = 85.312$, $\beta = -27.566$, and $\delta = 0.0234$. μ is determined by the martingale condition (1.26) and has the value 0.00783. We show graphs for the real and the imaginary part of $\varphi^{+}_{100+iv}(a+25i)$ where a varies between -900 and $+900$ and v between -1000 and $+1000$. The step size for both variables is 1.

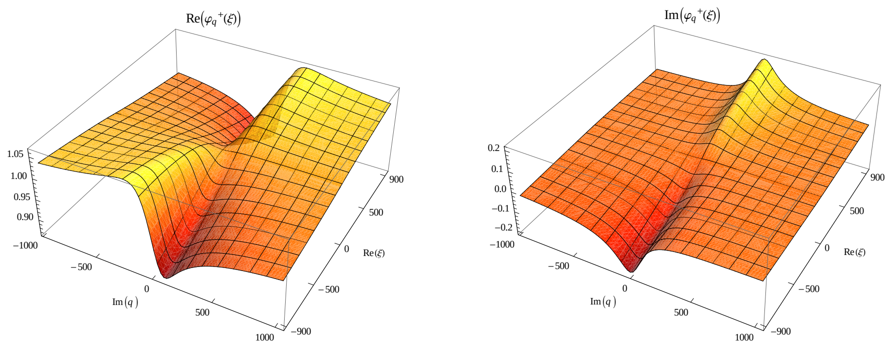

Fig. 1.1: Real and imaginary part of $\varphi^{+}_{q}(\xi)$. Source: A. Maier (2011)

The next two graphs show the real and the imaginary part of $\varphi^{-}_{100-iv}(a-20i)$ with a and v varying in the same intervals.

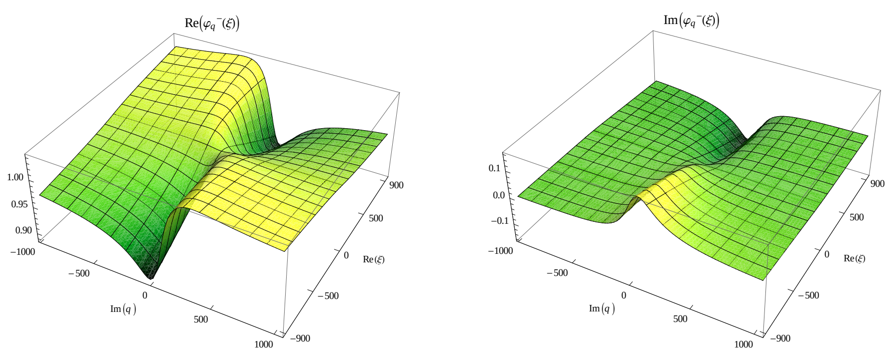

Fig. 1.2: Real and imaginary part of $\varphi^{-}_{q}(\xi)$. Source: A. Maier (2011)

Once one has $\varphi_{\overline{L}_t}$ in the form as given in (1.58) one gets for a fixed strike lookback call with payoff $(\overline{S}_T - K)^{+}$ taking (1.36) into account the explicit time-0 pricing formula

$$\mathbb{C}_T(\overline{S};K) = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R+iu}\varphi_{\overline{L}_T}(u-iR)\frac{K^{1-iu-R}}{(-iu-R)(1-iu-R)}du, \tag{1.65}$$

where $R \in (1,M)$. There is an analogous formula for the fixed strike lookback put option. Prices for floating strike lookback options with payoff $(\overline{S}_T - S_T)^+$ can be derived via a duality formula. For details of duality theory see Eberlein and Papapantoleon (2005) and Eberlein, Papapantoleon, and Shiryaev (2008).

An option which is of particular interest in this context is the one-touch call with payoff given by $\mathbb{1}_{\{\overline{S}_T>B\}}$. Since one takes the expectation of an indicator function (see (1.37)), the formula for the call price

$$\mathbb{C}_T(\overline{S};B) = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A} S_0^{R+iu}\varphi_{\overline{L}_T}(u-iR)\frac{B^{-R-iu}}{R+iu}du \tag{1.66}$$

for $R \in (0,M)$ provides at the same time an explicit formula for the distribution function of the running supremum of the Lévy process L. One has just to realize that

$$\mathbb{C}_T(\overline{S};B) = P[\overline{L}_T > \log(B/S_0)]. \tag{1.67}$$

One touch call options represent the case with discontinuous payoff function and not necessarily absolute continuous distribution of $\overline{L}_T$. Therefore besides of the standard assumption $(\mathbb{EM})$ for (1.66) to hold one has to assume that the Lévy process has infinite variation or has infinite activity and is regular upwards.

1.6 Interest rate term structure modeling

Contrary to the situation in equity markets where it is a priori clear which quantity is basic and has to be modeled as a stochastic process, in fixed income markets one has some freedom to choose which quantity is considered to be basic and is modeled and consequently which quantities are derived from the basic one. The quantities one has to consider are zero coupon bond prices $B(t,T)$, instantaneous forward rates $f(t,T)$, forward Libor rates $L(t,T)$, forward price processes $F_B(t,T,U)$ and short rates $r(t)$. To be more precise, by $B(t,T)$ we denote the price at time $t \in [0,T]$ of a *default-free zero coupon bond* which matures at time T. One refers to $B(t,T)$ also as a *discount factor*. The *instantaneous forward rate* $f(t,T)$ is closely related to it by the equation

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \tag{1.68}$$

Therefore if one starts by modeling $f(t,T)$ as is done in the classical Heath–Jarrow–Morton (HJM) approach (Heath, Jarrow, and Morton (1992)), one immediately gets the dynamics of $B(t,T)$ as well. The *short rate* $r(t)$ is given implicitly by modeling $f(t,T)$ since $r(t) = f(t,t)$. Another quantity which can be taken as the starting point is the *forward price process* defined for two maturities T and U as the quotient of the corresponding discount factors

$$F_B(t,T,U) = \frac{B(t,T)}{B(t,U)}. \tag{1.69}$$

The default-free *forward Libor rate* $L(t,T)$ is the discretely compounded annualized interest rate which can be earned for a future interval starting at T and ending at $T+\delta$ considered at the time point $t<T$

$$L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) \tag{1.70}$$

Note that the following master equation clarifies the relations between these quantities

$$1+\delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta). \tag{1.71}$$

The basic difference between models for stock markets and fixed income markets is that in the former one considers one single security or a finite collection of them whereas in the latter typically a *continuum of financial securities* is modeled, one for each maturity $T \in [0,T^*]$. This fact and in particular the stochastic dependence structure between these instruments makes interest rate models a priori more demanding from the mathematical point of view. Although in the case of the Libor model only a finite number of successive discrete rates along a certain tenor structure is considered, the mathematical challenge comes from the fact that each single rate has to be modeled as a martingale. To illustrate the continuum of quantities to be considered in the fixed income world we show in the following graph (Figure 1.3) the term structure of interest rates for four currencies for the maturity time span from three months to 10 years. The curves were fitted on data observed on February 17, 2004, by using the Svensson parametrization. This six parameter family is used nowadays by most of the national reserve banks. The highest line represents EURO interest rates, below is the US dollar term structure, then the Swiss franc follows, and the lowest curve represents interest rates for default-free investments in Japanese yen.

An interest rate model should be able to reproduce the observable term structure of interest rates as well as the market prices of interest rate derivatives such as caps, floors, swaptions, and more exotic instruments. The model should also be analytically tractable. There is a substantial collection of short rate models, which are driven by Brownian motions, starting with the models by Merton and Vasiček in the 70ies. Because of their relative analytic simplicity short rate models of this type are still used in the industry although in these models all rates depend on a single one in a deterministic way. As a consequence short rate models cannot describe the sophisticated movements of continuous time curves such as twists and changes in curvature. Nevertheless the sophistication of a problem at hand can force one to use a short rate model. As an example we mention a recent paper (Eberlein, Madan, Pistorius, and Yor (2011b)) where interest rates and correlated equity prices

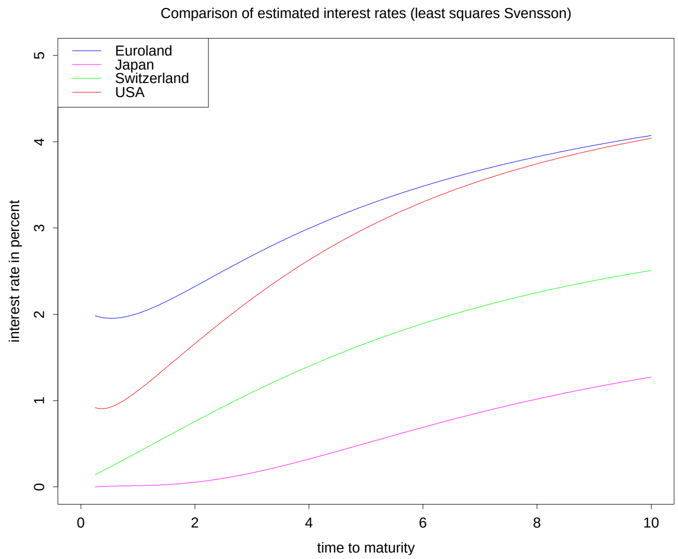

Fig. 1.3: Interest rate term structures, February 17, 2004

are modeled jointly in order to be able to price hybrid products. Since joint laws in such a case are not easily derived, the fixed income side is modeled in this reference by a short rate process driven by a Lévy motion. For the models which we will discuss in the following it is natural to use a wider class of driving processes, namely time-inhomogeneous Lévy processes. One of the reasons for using a larger class is that due to the measure changes which are typical for modeling interest rates, one drops out of the class of Lévy processes anyhow. The wider class does not harm the analytical tractability. At the same time one gains considerable statistical flexibility. In the implementations one considers usually a mild form of time-inhomogeneity, namely Lévy processes where the Lévy parameters are kept constant for a while. See the next graph (Figure 1.4).

A d-dimensional *time-inhomogeneous Lévy process* is a process $L = (L^1, \dots, L^d)$ which has independent increments and the law of L_t is given by the characteristic function

$$E[\exp(i\langle u, L_t\rangle)] = \exp\left(\int_0^t \theta_s(iu)ds\right) \tag{1.72}$$

with cumulant function

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d}\left(e^{\langle z,x\rangle} - 1 - \langle z,x\rangle\right) F_s(dx). \tag{1.73}$$

Here $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative-definite $d\times d$-matrix and F_t is a Lévy measure. Implicitly we make two integrability assumptions, namely that for some

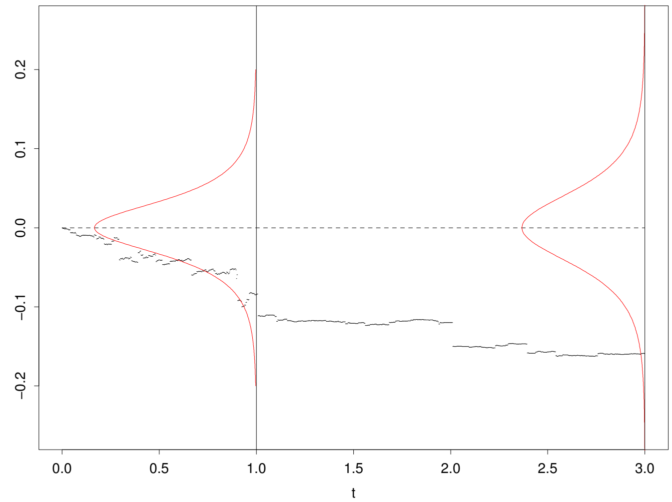

Fig. 1.4: Simulation of a Lévy process with generating distributions, NIG(10,0,0.100,0) on [0,1], NIG(10,0,0.025,0) on [1,3].

time horizon $T^* > 0$ which includes all maturities T to be considered in the model, we have

$$\int_0^{T^*}\left(|b_s| + \|c_s\| + \int_{\mathbb{R}^d}(|x|^2 \wedge 1)F_s(dx)\right)ds < \infty \tag{1.74}$$

and for some $M > 1$

$$\int_0^{T^*}\int_{\{|x|>1\}} \exp(\langle u, x\rangle)F_s(dx)ds < \infty \qquad \text{for } |u| \leq M. \tag{1.75}$$

The triplet $(b,c,F) = (b_s, c_s, F_s)_{0\leq s\leq T^*}$ is again called the triplet of local characteristics of the process L. Let us mention that such a process is called a *process with independent increments and absolutely continuous characteristics (PIIAC)* in Jacod and Shiryaev (1987). Note that we do not need a truncation function of the type $\mathbb{1}_{\{|x|\leq 1\}}$ in (1.73) because of the moment assumption (1.75). We do not repeat the arguments from Chapter 1.2 for this simplification (see formulas (1.11)–(1.17)). For the same reason one can immediately use the simpler canonical representation of the special semimartingale $L = (L_t)_{t\geq 0}$ given by

$$L_t = \int_0^t b_s ds + \int_0^t c_s^{1/2} dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(ds, dx) \tag{1.76}$$

with characteristics

$$B_t = \int_0^t b_s ds, \qquad C_t = \int_0^t c_s ds, \qquad \nu(ds,dx) = F_s(dx)ds. \tag{1.77}$$

Here $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, μ^L the random measure of jumps of L and ν is the compensator of μ^L.

Now we are ready to introduce the *Lévy forward rate approach* which was developed in a series of papers (Eberlein and Raible (1999), Eberlein and Özkan (2005), Eberlein, Jacod, and Raible (2005), Eberlein and Kluge (2006a)) starting in 1999. It generalizes the HJM-framework by using more powerful driving processes. Assume that for every fixed maturity $T \in [0,T^*]$ the dynamics of the instantaneous forward rates is given by

$$df(t,T) = \alpha(t,T)dt - \sigma(t,T)dL_t \quad (0 \leq t \leq T). \tag{1.78}$$

The initial values $f(0,T)$ are deterministic, bounded, and measurable in T. The drift and volatility coefficients $\alpha(t,T)$ and $\sigma(t,T)$ satisfy the usual measurability assumptions which are necessary for integration. For $t > T$ we set $\alpha(t,T) = \sigma(t,T) = 0$ and we assume $\sup_{t,T\leq T^*}(|\alpha(\omega,t,T)+|\sigma(\omega,t,T)|) < \infty$. In the implementations one takes usually *deterministic* (one-dimensional) volatilities $\sigma(t,T)$, where the popular ones are

$$\begin{aligned}
&\text{(a)}\ \ \sigma(t,T) = \widehat{\sigma} && \text{(Ho–Lee)} \\
&\text{(b)}\ \ \sigma(t,T) = \widehat{\sigma}\exp(-a(T-t)) && \text{(Vasiček)} \\
&\text{(c)}\ \ \sigma(t,T) = \widehat{\sigma}\tfrac{1+\gamma T}{1+\gamma t}\exp(-a(T-t)) && \text{(Moraleda–Vorst)}
\end{aligned} \tag{1.79}$$

Using equation (1.68) and Fubini's theorem one can easily derive an equation for the corresponding zero coupon prices

$$B(t,T) = B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))ds + \int_0^t \Sigma(s,T)dL_s\right), \tag{1.80}$$

where $A(s,T) = \int_{s\wedge T}^T \alpha(s,u)du$ and $\Sigma(s,T) = \int_{s\wedge T}^T \sigma(s,u)du$.

Remember that the short rate $r(t)$ is given by $f(t,t)$. Therefore the *money market* or *savings account* given by $B_t = \exp\left(\int_0^t f(u,u)du\right)$ can be represented in the form

$$B_t = \frac{1}{B(0,t)}\exp\left(\int_0^t A(s,t)ds - \int_0^t \Sigma(s,t)dL_s\right). \tag{1.81}$$

In the following we shall always assume that the volatility coefficient is deterministic and bounded in the following sense

Assumption $(\mathbb{DET})$**:** The volatility structure $\sigma(t,T)$ is a deterministic and bounded function such that for all $0 \leq s, T \leq T^*$

$$0 \leq \Sigma^i(s,T) \leq M \quad (i \in \{1,\dots,d\}),$$

where M is the constant from Assumption $(\mathbb{EM})$.

The following theorem is a key tool in developing the Lévy interest rate theory. It was proved in Eberlein and Raible (1999) and then generalized in Eberlein and Kluge (2006a).

Theorem 4. *Suppose* $f : \mathbb{R}_+ \to \mathbb{C}^d$ *is a continuous function such that* $|\mathrm{Re}(f^i(x))| \leq M$ *for all* $i \in \{1,\ldots,d\}$ *and* $x \in \mathbb{R}_+$, *then*

$$E\left[\exp\left(\int_t^T f(s)dL_s\right)\right] = \exp\left(\int_t^T \theta_s(f(s))ds\right).$$

Taking $f(s) = \Sigma(s,T)$ for some $T \in [0,T^*]$ one gets from this theorem

$$E\left[\exp\left(\int_0^t \Sigma(s,T)dL_s\right)\right] = \exp\left(\int_0^t \theta_s(\Sigma(s,T))ds\right). \tag{1.82}$$

Now we can see how we have to choose the drift coefficient $\alpha(t,T)$ such that the discounted zero coupon bond price processes are martingales. It is easy to show that for processes $(X_t)_{t\geq 0}$ with independent increments – this is the case for $\int_0^t \Sigma(s,T)dL_s$ – the process $\big(\exp(X_t)/E[\exp(X_t)]\big)_{t\geq 0}$ is a martingale. Of course $E[\exp(X_t)]$ has to be finite. The first part of the exponential in (1.80), $\exp\big(\int_0^t r(s)ds\big)$, is nothing but the discount factor B_t, the money market account. Therefore if we choose $\exp\big(\int_0^t A(s,T)ds\big)$ such that it equals $E\left[\exp\big(\int_0^t \Sigma(s,T)dL_s\big)\right]$ then we have martingality of the discounted bond prices $(B_t^{-1}B(t,T))_{t\geq 0}$. By (1.82) this is the case if the drift is chosen as

$$A(s,T) = \theta_s(\Sigma(s,T)). \tag{1.83}$$

This relation is the proper generalization of the famous *HJM drift condition*. Note that with (1.83) the coefficients $\alpha(t,T)$ and $A(t,T)$ are eliminated from our analysis. We assume from now on that forward rates are always given by (1.78) such that the drift condition (1.83) is satisfied. This means that the derived bond prices are given in the more specific form

$$B(t,T) = B(0,T)B_t \exp\left(-\int_0^t \theta_s(\Sigma(s,T))ds + \int_0^t \Sigma(s,T)dL_s\right). \tag{1.84}$$

The first integral in the exponent is also called the *exponential compensator* of the second integral. In Eberlein et al. (2005) it has been shown that the underlying martingale measure is unique in this one-dimensional setting. Thus for the Lévy forward rate model we are in a Black–Scholes situation with a unique pricing operator. A priori one could have expected a whole set of competing equivalent martingale measures as in exponential Lévy models for equity (see Eberlein and Jacod (1997)). The deeper reason for the uniqueness of the martingale measure in this model is that the number of instruments in the market – the continuum of bonds with maturities $T \in [0,T^*]$ – matches the number of degrees of freedom given by the jump sizes of the driving Lévy process. The two degrees of infinity coincide in this case. In order to apply directly the valuation formula derived in Theorem 1 we will write the bond price (1.84) in a different form. Replace first $A(s,T)$ in (1.81) by the drift condition (1.83), then (1.84) takes the form

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(\int_0^t (\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T)))\,ds + \int_0^t (\Sigma(s,T) - \Sigma(s,t))\,dL_s\right). \tag{1.85}$$

If we write the deterministic part as

$$D(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(\int_0^t (\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T)))\,ds\right) \tag{1.86}$$

and the stochastic part as

$$X_t = \int_0^t (\Sigma(s,T) - \Sigma(s,t))\,dL_s, \tag{1.87}$$

we can write bond prices in the simple form

$$B(t,T) = D(t,T)\exp(X_t). \tag{1.88}$$

Thus bond prices turn out to have the form of an exponential model of the type as studied in Chapter 1.4 where the driving process is $(X_t)_{t\geq 0}$ as given in (1.87). For any European option with maturity t on a zero-coupon bond with maturity T we can express its time-0 value formally as a function of X_t and $s = -\ln D(t,T)$, namely

$$\mathbb{V}_0(t,T) = E[B_t^{-1} f(X_t - s)], \tag{1.89}$$

where f is a function of the payoff of the option. In order to calculate this expectation one needs the joint distribution of B_t and X_t or of B_t and $B(t,T)$. In principle one can proceed this way, but from the numerical point of view such a straightforward approach is very inefficient and time consuming. Fortunately there is an elegant way to avoid joint distributions by making a measure change which is also called a change of numeraire. One switches from the *spot martingale measure* used so far to the *forward martingale measure* for the settlement date t denoted by P_t. We define

$$\frac{dP_t}{dP} = \frac{1}{B_t B(0,t)}. \tag{1.90}$$

From (1.81) and (1.83) one derives the explicit form of this density process as

$$\frac{dP_t}{dP} = \exp\left(\int_0^t \Sigma(s,t)dL_s - \int_0^t \theta_s(\Sigma(s,t))ds\right). \tag{1.91}$$

By using Girsanov's theorem, under P_t the compensator of the random measure of jumps μ^L becomes

$$\nu^t(ds,dx) = \exp(\langle \Sigma(s,t),x\rangle)\nu(ds,dx) \tag{1.92}$$

and

$$W_s^t = W_s - \int_0^s c_u^{1/2}\Sigma(u,t)du \tag{1.93}$$

is a standard Brownian motion. Since the change of the characteristics is done by deterministic functions in these equations, one can conclude that under P_t, L is still a process with independent increments. Thus with respect to the forward martingale measure, L is still a time-inhomogeneous Lévy process. Using the forward martingale measure P_t, the option value formula (1.89) simplifies to

$$\mathbb{V}_0(t,T) = B(0,t)E_{P_t}[f(X_t - s)] \tag{1.94}$$

since $\int_\Omega B_t^{-1} f(X_t - s)dP = \int_\Omega B_t^{-1} f(X_t - s)B_t B(0,t)dP_t$.

Joint distributions are no longer needed to evaluate (1.94). Assume now Assumptions $(\mathbb{C})$ hold for f and X, then we get as in Theorem 1 the integral representation

$$\mathbb{V}_0(t,T) = B(0,t)\frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\varphi_{X_t}(u-iR)\widehat{f}(-u+iR)du. \tag{1.95}$$

The main difference to the original results which were proved in Eberlein and Kluge (2006a) is in the assumptions. The resulting formulas are the same and differ only in the notation. In Eberlein and Kluge (2006a) the integral formulas were derived from a convolution representation. Above we use instead in the spirit of Eberlein et al. (2010) Fubini's theorem which is possible by assumptions (C1)–(C3).

The standard assumption in Eberlein and Kluge (2006a) about the existence of a Lebesgue density for the distribution of X_t can actually be weakened. Instead of convoluting two functions (see the proof of Theorem 12 in Eberlein and Kluge (2006a)) one could as well convolute a function and a distribution.

An explicit expression for the term φ_{X_t} in (1.95) or equivalently M_{X_t} (both under P_t) is available by using Theorem 4. More precisely one gets the following result (Eberlein and Kluge (2006a, Lemma 13)). Suppose that for all $s,T \in [0,T^*]$, $\Sigma(s,T) < M'$ for some $M' < M$ where M is from Assumption $(\mathbb{EM})$, then $M_{X_t}(R) < \infty$ for every $R \in (1, 1+\frac{M-M'}{M'}]$. Then the following explicit expression holds for $z \in \mathbb{C}$ with $\mathrm{Re}(z) = R$,

$$M_{X_t}(z) = \exp\left(\int_0^t (\theta_s(z\Sigma(s,T) + (1-z)\Sigma(s,t)) - \theta_s(\Sigma(s,t)))\,ds\right). \tag{1.96}$$

Note that this equation provides the moment generating function at the right argument $z = R + iu$ for (1.95) since $\varphi_{X_t}(u - iR) = M_{X_t}(R+iu)$.

Contrary to equity derivatives interest rate derivatives typically generate cash flows along a discrete tenor structure $T_0 < T_1 < \cdots < T_{n-1} < T_n$. According to the day count convention in the contract specification the time intervals $\delta_i = T_i - T_{i-1}$ $(1 \le i \le n)$ can depend on i. For simplicity we assume a constant δ which usually is 3 or 6 months. The most important interest rate derivatives are caps, floors and swaptions. A *(forward start) cap* is a sequence of call options on the Libor rate. Each option in this sequence with time-T_j-payoff $N\delta(L(T_{j-1},T_{j-1})-K)^+$ is called a *caplet*. N is the notional amount, which we set $N=1$. K is the strike rate. A *(forward start) floor* is a sequence of put options on the Libor rate – each one called a *floorlet* – with time-T_j-payoff $N\delta(K-L(T_{j-1},T_{j-1}))^+$.

A payoff $\delta(L(T,T)-K)^+$ which is made at time $T+\delta$ is equivalent to a discounted payoff $B(T,T+\delta)\delta(L(T,T)-K)^+$ at time T. Since

$$B(T,T+\delta)\delta(L(T,T)-K)^+ = (1+\delta K)\left((1+\delta K)^{-1}-B(T,T+\delta)\right)^+ \tag{1.97}$$

one can interpret a caplet as a put option with strike $(1+\delta K)^{-1}$ and notional amount $(1+\delta K)$ on a zero coupon bond with maturity $T+\delta$. Analogously a floorlet is a call option on a zero coupon bond. There is also a put-call-parity relation between caps and floors. Caps and floors are used as an insurance against rising or falling interest rates in contracts with variable interest rates. Given the interpretation in (1.97) in order to price caps and floors we have to price put and call options on zero coupon bonds as underlying quantity. More specifically, the price of a call with maturity t and strike K on a zero coupon bond with maturity T where $t<T$ is given by (1.95) where $f(x)=(e^x-K)^+$. In the case of a put one has to choose $f(x)=(K-e^x)^+$. Since $\widehat{f}$ for these functions f has been computed in (1.36), we get the following explicit form for the time-0-price of calls and puts on zero coupon bonds. Suppose $R\in(1,\infty)$ such that $M_{X_t}(R)<\infty$. Then the call price has the representation

$$\mathbb{C}_0(t,T,K) = B(0,t)\frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{-ius}\frac{K^{1-iu-R}}{(R+iu)(R-1+iu)}M_{X_t}(R+iu)du. \tag{1.98}$$

The formula for the put price $\mathbb{P}_0(t,T,K)$ is exactly the same, only the assumptions differ. One has to choose $R\in(-\infty,0)$ such that $M_{X_t}(R)<\infty$ (see (1.36)). In order to price a caplet with strike rate K, according to (1.97) one has to choose K in (1.98) as $(1+\delta K)^{-1}$ and furthermore one has to multiply the notional amount with the factor $(1+\delta K)$. The sum over all the caplets along the tenor structure gives the price of the cap.

Pricing swaptions is equivalent to pricing call respectively put options on a coupon bearing bond (see Musiela and Rutkowski (1997, Section 16.2.3.)). The general representation (1.95) applies here as well with f chosen appropriately. For details see Section 5 in Eberlein and Kluge (2006a). Again the assumption on the existence of a Lebesgue density for the distribution of X is not necessary. Instead one can assume (C1)–(C3). These assumptions are easy to verify in the case of swaption as well as of cap and floor valuation. In particular (C3) follows always as an application of Lemma 2.5 in Eberlein et al. (2010). Let us demonstrate this for the case of a call, i.e. $f(x)=(e^x-K)^+$. Then g is bounded since $g(x)=e^{-Rx}(e^x-K)\le K^{1-R}$ for $x\ge \ln K$ and $g(x)=0$ for $x<\ln K$. Note that $1-R<0$ for the call function. Furthermore

$$\int_{\mathbb{R}}|g(x)|dx = -\left(\frac{1}{1-R}+\frac{1}{R}\right)K^{1-R}<\infty$$

implies $g \in L^1_{bc}(\mathbb{R})$. To verify (C3) according to Eberlein et al. (2010, Lemma 2.5) it is sufficient to prove $g \in H^1(\mathbb{R})$. First we show $g \in L^2(\mathbb{R})$ since

$$\int_{\mathbb{R}} |g(x)|^2 dx = \left(\frac{2}{1-2R} + \frac{1}{2R} - \frac{1}{2(1-R)} \right) K^{2(1-R)} < \infty.$$

The weak derivative of g is

$$\partial g(x) = \begin{cases} 0 & x < \ln K \\ e^{-Rx}(e^x - \mathrm{Re}^x + RK) & x > \ln K \end{cases}$$

and

$$\int_{\mathbb{R}} |\partial g(x)|^2 dx = -\left(\frac{1-R}{2} + \frac{2(1-R)R}{1-2R} - \frac{R}{2} \right) K^{2(1-R)} < \infty,$$

which implies $\partial g \in H^1(\mathbb{R})$.

1.7 Valuation in the Lévy Libor model

Instantaneous forward rates which represent the basic quantity in the modeling approach in the previous section are an infinitesimal quantity since (see (1.68))

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T).$$

These rates are not observable in the market. What is observable instead are forward Libor rates $L(t,T)$ as defined in (1.70). Therefore Brace, Gątarek, and Musiela (1997) (BGM) chose these rates as the basic quantities and introduced the Libor or market model. The Lévy Libor model as a generalization was introduced in Eberlein and Özkan (2005). We sketch the model briefly in the following – for the detailed construction see Eberlein and Özkan (2005) – and show then how Fourier based valuation formulas can be derived.

The model is constructed by backward induction and driven by a time-inhomogeneous Lévy process L^{T^*} as given in (1.76). T^* denotes here the end point of a tenor structure $0 = T_0 < T_1 < \cdots < T_{n-1} < T_n = T^*$. Write again $\delta = T_{k+1} - T_k$. Since because of measure changes the indices T_k become important now, we repeat (1.76) in the form

$$L_t^{T^*} = \int_0^t b_s^{T^*} ds + \int_0^t c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^{T^*} - \nu^{T^*})(ds,dx).$$

The meaning of the quantities with upper T^* is the same as in (1.76). L^{T^*} is defined on a complete stochastic basis $(\Omega, \mathscr{F} = \mathscr{F}_{T^*}, \mathbb{F}, P_{T^*})$ where P_{T^*} should be regarded as the forward martingale measure for the settlement date T^*. A spot martingale measure P is not needed in this approach. L^{T^*} is required to satisfy Assumption $(\mathbb{EM})$. Two ingredients are needed:

Assumption $(\mathbb{LR}.1)$**:** For any maturity T_k there is a deterministic function $\lambda(\cdot,T_k): [0,T^*] \to \mathbb{R}^d$ which represents the volatility of the forward Libor rate process $L(\cdot,T_k)$. This function satisfies

$$\sum_{k=1}^{n-1} |\lambda^i(s,T_k)| \leq M' \qquad \text{for all } s \in [0,T^*] \text{ and } i \in \{1,\dots,d\}$$

for some $M' < \frac{M}{2}$, where M is the constant from Assumption $(\mathbb{EM})$. For $s > T_k$ we assume $\lambda(s,T_k) = 0$.

Assumption $(\mathbb{LR}.2)$**:** The initial term structure $B(0,T_k) \quad (1 \leq k \leq n)$ is strictly positive and strictly decreasing in k.

The backward induction starts by setting the most distant Libor rate $L(t,T_{n-1})$ under P_{T^*} as

$$L(t,T_{n-1}) = L(0,T_{n-1}) \exp\left(\int_0^t \lambda(s,T_{n-1}) dL_s^{T^*}\right). \tag{1.99}$$

Now one forces this to become a P_{T^*}-martingale by choosing b^{T^*} such that

$$\begin{aligned}\int_0^t \langle \lambda(s,T_{n-1}), b_s^{T^*} \rangle ds = -\frac{1}{2}\int_0^t \langle \lambda(s,T_{n-1}), c_s \lambda(s,T_{n-1}) \rangle ds \\ - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s,T_{n-1}),x \rangle} - 1 - \langle \lambda(s,T_{n-1}),x \rangle \right) \nu^{T^*}(ds,dx).\end{aligned}$$

This at the same time eliminates the drift coefficient b^{T^*}. Define – where $L(t-,\cdot)$ denotes left limits –

$$\begin{aligned}\ell(t-,T_{n-1}) &= \frac{\delta L(t-,T_{n-1})}{1+\delta L(t-,T_{n-1})}, \\ \alpha(t,T_{n-1}) &= \ell(t-,T_{n-1}) \lambda(t,T_{n-1})\end{aligned}$$

and

$$\beta(t,x,T_{n-1}) = \ell(t-,T_{n-1}) \left(e^{\langle \lambda(t,T_{n-1}),x \rangle} - 1 \right) + 1$$

then the forward process $F(\cdot,T_{n-1},T^*)$ is given as a stochastic exponential

$$F(t,T_{n-1},T^*) = F(0,T_{n-1},T^*) \mathscr{E}_t(M^1)$$

with

$$M_t^1 = \int_0^t c_s^{1/2} \alpha(s,T_{n-1}) dW_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} (\beta(s,x,T_{n-1}) - 1)(\mu^{T^*} - \nu^{T^*})(ds,dx)$$

and is consequently a P_{T^*}-martingale. We use this forward process as a density process and define the forward measure $P_{T_{n-1}}$ via

$$\frac{dP_{T_{n-1}}}{dP_{T^*}} = \frac{F(T_{n-1}, T_{n-1}, T^*)}{F(0, T_{n-1}, T^*)} = \mathscr{E}_{T_{n-1}}(M^1).$$

By the semimartingale version of Girsanov's theorem (see Jacod and Shiryaev (1987))

$$W_t^{T_{n-1}} := W_t^{T^*} - \int_0^t c_s^{1/2} \alpha(s, T_{n-1}) ds$$

is a $P_{T_{n-1}}$-standard Brownian motion and

$$\nu^{T_{n-1}}(dt, dx) := \beta(t, x, T_{n-1}) \nu^{T^*}(dt, dx)$$

is the $P_{T_{n-1}}$-compensator of μ^{T^*}. Now one defines the forward Libor rate $L(\cdot, T_{n-2})$ under $P_{T_{n-1}}$ as

$$L(t, T_{n-2}) = L(0, T_{n-2}) \exp\left(\int_0^t \lambda(s, T_{n-2}) dL_s^{T_{n-1}} \right)$$

where

$$L_t^{T_{n-1}} = \int_0^t b_s^{T_{n-1}} ds + \int_0^t c_s^{1/2} dW_s^{T_{n-1}} + \int_0^t \int_{\mathbb{R}^d} x \left(\mu^{T_{n-1}} - \nu^{T_{n-1}} \right) (ds, dx).$$

$b^{T_{n-1}}$ is again eliminated in such a way that $L(t, T_{n-2})$ becomes a $P_{T_{n-1}}$-martingale. Continuing this way one gets forward Libor rates $L(t, T_k)$ and forward measures $P_{T_{k+1}}$ such that for $k \in \{1, \ldots, n-1\}$

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t \lambda(s, T_k) dL_s^{T_{k+1}} \right) \tag{1.100}$$

is a $P_{T_{k+1}}$-martingale. The driving process has the form

$$L_t^{T_{k+1}} = \int_0^t b_s^{T_{k+1}} ds + \int_0^t c_s^{1/2} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x \left(\mu^{T_{k+1}} - \nu^{T_{k+1}} \right) (ds, dx),$$

where $\nu^{T_{k+1}}(ds, dx) = F_s^{T_{k+1}}(dx) ds$ is the $P_{T_{k+1}}$-compensator of $\mu^{T_{k+1}}$ and the drift coefficient $b^{T_{k+1}}$ is chosen analogously to the first induction step replacing T_{n-1} by T_k. The other quantities are

$$\begin{aligned} \ell(s-, T_k) &= \frac{\delta L(s-, T_k)}{1 + \delta L(s-, T_k)} \\ \alpha(s, T_k) &= \ell(s-, T_k) \lambda(s, T_k) \\ \beta(s, x, T_k) &= \ell(s-, T_k) \left(e^{\langle \lambda(s, T_k), x \rangle} - 1 \right) + 1 \end{aligned}$$

and we have the recursive relations

$$W_t^{T_k} = W_t^{T_{k+1}} - \int_0^t c_s^{1/2} \alpha(s, T_k) ds$$

and

$$F_s^{T_k}(dx) = \beta(s, x, T_k) F_s^{T_{k+1}}(dx).$$

Furthermore the successive densities can be written as

$$\frac{dP_{T_k}}{dP_{T_{k+1}}} = \frac{1 + \delta L(T_k, T_k)}{1 + \delta L(0, T_k)}. \tag{1.101}$$

Since $L(t, T_k)$ is a $P_{T_{k+1}}$-martingale, so is

$$\frac{B(t, T_k)}{B(t, T_{k+1})} = 1 + \delta L(t, T_k) \tag{1.102}$$

which is up to the constant $(1 + \delta L(0, T_k))^{-1}$ the density process

$$\frac{dP_{T_k}}{dP_{T_{k+1}}}\Big|_{\mathscr{F}_t} = \frac{1 + \delta L(t, T_k)}{1 + \delta L(0, T_k)}. \tag{1.103}$$

By iterating this we get

$$\begin{aligned} \frac{dP_{T_{k+1}}}{dP_{T_n}} &= \prod_{\ell=k+1}^{n-1} \frac{1 + \delta L(T_{k+1}, T_\ell)}{1 + \delta L(0, T_\ell)} \\ &= \frac{B(0, T_n)}{B(0, T_{k+1})} \prod_{\ell=k+1}^{n-1} (1 + \delta L(T_{k+1}, T_\ell)). \end{aligned}$$

Applying Proposition III.3.8 of Jacod and Shiryaev (1987) – which is a fundamental result for interest rate modeling – we see that its restriction to $\mathscr{F}_t$

$$\frac{dP_{T_{k+1}}}{dP_{T_n}}\Big|_{\mathscr{F}_t} = \frac{B(0, T_n)}{B(0, T_{k+1})} \prod_{\ell=k+1}^{n-1} (1 + \delta L(t, T_\ell)) \qquad (t \in [0, T_{k+1}]) \tag{1.104}$$

is a P_{T_n}-martingale.

As a consequence of representations of the type (1.104) of arbitrary quotients $B(t, T_j)/B(t, T_k)$ as products of quotients with successive maturities T_k and T_{k+1}, Proposition III.3.8 of Jacod and Shiryaev (1987) guarantees also that properly discounted zero coupon bond prices $B(t, T_j)/B(t, T_k)$ are P_{T_k}-martingales. This means that the Libor approach as developed above creates an arbitrage-free model.

With respect to numerical aspects it is important to note that already with the first measure change one looses the property that the driving processes $L^{T_{k+1}}$ are time-inhomogeneous Lévy processes. This is because the coefficients $\alpha(s, T_k)$ and $\beta(s, x, T_k)$ contain the random quantity $L(s-, T_k)$ via $\ell(s-, T_k)$. The simplest approach to preserve this property for numerical purposes is to replace $\ell(s-, T_k)$ by its deterministic starting value $\ell(0,T_k) = \frac{\delta L(0,T_k)}{1+\delta L(0,T_k)}$. This is called the *frozen drift approximation*. A number of more sophisticated approximations has been studied in recent years.

The approach which we present here is exposed in Eberlein and Kluge (2006a) and is based on the following approximation for exponential terms:

($\mathbb{A}$): For small values $|x|$ and $\varepsilon > 0$ we have

$$1 + \varepsilon \exp(x) \approx (1+\varepsilon)\exp\left(\frac{\varepsilon}{1+\varepsilon}x\right).$$

We want to price standard interest rate derivatives such as caps, floors, and swaptions in the Lévy Libor model by numerically efficient methods. Since floor prices can be derived from the corresponding put-call-parity relation we concentrate on caps. The payoff of a caplet with strike rate K and maturity T_k is

$$\delta(L(T_k,T_k) - K)^+$$

where the payment is made at time point T_{k+1}. Consequently its time-0-price is given by

$$\mathbb{C}_0(T_k,K) = \delta B(0,T_{k+1}) E_{P_{T_{k+1}}}[(L(T_k,T_k)-K)^+]. \tag{1.105}$$

For a convenient representation of this expectation we introduce for $0 \le t \le T_{k+1}$ two processes which turn out to be P_{T_n}-martingales (see (1.104))

$$M_t^1 := \prod_{\ell=k+1}^{n-1} (1+\delta L(t,T_\ell)) \frac{L(T_k,T_k)}{K} \tag{1.106}$$

and

$$M_t^2 := \prod_{\ell=k+1}^{n-1} (1+\delta L(t,T_\ell)). \tag{1.107}$$

Then

$$K\left(M^1_{T_{k+1}} - M^2_{T_{k+1}}\right)^+ = (L(T_k,T_k)-K)^+ \prod_{\ell=k+1}^{n-1}(1+\delta L(T_{k+1},T_\ell))$$

which implies by (1.104)

$$\mathbb{C}_0(T_k,K) = \delta B(0,T_n) K E_{P_{T_n}}\left[\left(M^1_{T_{k+1}} - M^2_{T_{k+1}}\right)^+\right]. \tag{1.108}$$

Substituting $L(t,T_\ell)$ in (1.106) and (1.107) by its explicit form (1.100) and using the fact that $L^{T_{k+1}}$ and L^{T_n} differ only by a drift term, we get the representation

$$M_t^1 = \prod_{\ell=k+1}^{n-1}\left[1+\delta L(0,T_\ell)\exp\left(\int_0^t \lambda(s,T_\ell)dL_s^{T_n} + \text{ drift}\right)\right] \\ \times \frac{L(0,T_k)}{K}\exp\left(\int_0^{T_k}\lambda(s,T_k)dL_s^{T_n} + \text{ drift}\right)$$

and similarly for M_t^2 without the factor in the second line. Now we approximate each factor in the product above using $(\mathbb{A})$, i.e. we replace

$$1+\delta L(0,T_\ell)\exp\left(\int_0^t \lambda(s,T_\ell)dL_s^{T_n} + \text{ drift}\right)$$

by

$$(1+\delta L(0,T_\ell))\exp\left(\int_0^t \ell(0,T_\ell)\lambda(s,T_\ell)dL_s^{T_n} + \text{ new drift}\right).$$

The result are approximations $\widetilde{M}_t^1$ and $\widetilde{M}_t^2$ of M_t^1 and M_t^2 which can be written in the form

$$\widetilde{M}_t^1 = \frac{L(0,T_k)}{K}\frac{B(0,T_{k+1})}{B(0,T_n)}\exp\left(\int_0^t f^k(s)dL_s^{T_n} + \int_0^{T_k}\lambda(s,T_k)dL_s^{T_n} + D_t^1\right)$$

and

$$\widetilde{M}_t^2 = \frac{B(0,T_{k+1})}{B(0,T_n)}\exp\left(\int_0^t f^k(s)dL_s^{T_n} + D_t^2\right),$$

where

$$f^k(s) = \sum_{\ell=k+1}^{n-1}\ell(0,T_\ell)\lambda(s,T_\ell),$$

$$D_t^1 = \ln\left(\frac{E_{P_{T_n}}\left[\exp\left(\int_0^{T_k}\lambda(s,T_k)dL_s^{T_n}\right)\right]}{E_{P_{T_n}}\left[\exp\left(\int_0^t f^k(s)dL_s^{T_n} + \int_0^{T_k}\lambda(s,T_k)dL_s^{T_n}\right)\right]}\right),$$

and

$$D_t^2 = \ln\left(E_{P_{T_n}}\left[\exp\left(\int_0^t f^k(s)dL_s^{T_n}\right)\right]^{-1}\right).$$

We can replace now (1.108) by the approximative formula

$$\mathbb{C}_0(T_k,K) \approx \delta B(0,T_n)KE_{P_{T_n}}\left[\left(\widetilde{M}_{T_{k+1}}^1 - \widetilde{M}_{T_{k+1}}^2\right)^+\right]. \tag{1.109}$$

Implicitly it is assumed that $\widetilde{M}^1$ and $\widetilde{M}^2$ are P_{T_n}-martingales. This allows to introduce a $\widetilde{P}_{T_{k+1}}$-forward measure by setting

$$\frac{d\widetilde{P}_{T_{k+1}}}{dP_{T_n}} = \frac{\widetilde{M}^2_{T_{k+1}}}{\widetilde{M}^2_0} = \exp\left(\int_0^{T_{k+1}} f^k(s)dL_s^{T_n} + D^2_{T_{k+1}}\right).$$

Expressing (1.109) in terms of the new measure we get

$$\mathbb{C}_0(T_k,K) \approx \delta B(0,T_{k+1})KE_{\widetilde{P}_{T_{k+1}}}\left(\left(\exp(X_{T_{k+1}})-1\right)^+\right],$$

where X is defined as the process

$$X_t = \ln\frac{\widetilde{M}^1_t}{\widetilde{M}^2_t} = \ln\left(\frac{L(0,T_k)}{K}\right) + \int_0^{T_k}\lambda(s,T_k)dL_s^{T_n} + D^1_t - D^2_t.$$

We finally reached the form

$$\mathbb{C}_0(T_k,K) \approx \delta B(0,T_{k+1})KE_{\widetilde{P}_{T_{k+1}}}[f(X_{T_{k+1}})] \tag{1.110}$$

for $f(x) = (e^x-1)^+$. This means Theorem 1 can be applied with the payoff of a call option with strike 1. The corresponding Fourier transform $\widehat{f}$ is given in (1.36) and s equals 0. Therefore we get the following explicit integral representation for the formula (1.110). Suppose $R \in (1, 1+\varepsilon)$ such that the moment generating function of $X_{T_{k+1}}$ with respect to $\widetilde{P}_{T_{k+1}}$ is finite at R, i.e. $\widetilde{M}_{X_{T_{k+1}}}(R) < \infty$, then

$$C_0(T_k,K) \approx \delta B(0,T_{k+1})\frac{K}{2\pi}\int_{\mathbb{R}}\widetilde{M}_{X_{T_{k+1}}}(R+iu)\frac{1}{(-iu-R)(1-iu-R)}du. \tag{1.111}$$

An explicit form for the moment generating function $\widetilde{M}_{X_{T_{k+1}}}$ can be obtained again using Theorem 4. Suppose $R \in (1, 1+\varepsilon)$ such that $\widetilde{M}_{X_{T_{k+1}}}(R) < \infty$. Then for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) = R$

$$\begin{aligned}\widetilde{M}_{X_{T_{k+1}}}(z) = &\left(\frac{L(0,T_k)}{K}\right)^z \qquad (1.112)\\ &\times\exp\left(\int_0^{T_k}\Big[\theta_s(f^k(s)+z\lambda(s,T_k)) - z\theta_s(f^k(s)+\lambda(s,T_k))\right.\\ &\left. +(z-1)\theta_s(f^k(s)) + z\theta_s(\lambda(s,T_k))\Big]ds\right).\end{aligned}$$

A detailed proof of this formula is given in Rudmann (2011, Satz 4.2.5).

As mentioned earlier pricing swaptions is equivalent to pricing calls and puts on a coupon bearing bond. Therefore by choosing the appropriate payoff function f swaptions can be priced in the Lévy Libor model as well. The corresponding numerically efficient Fourier based integral representation formula has been derived in Kluge (2005, Section 3.2.2).

References

O. E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2:41–68, 1998.

K. Borovkov and A. Novikov. On a new approach to calculating expectations for option pricing. *Journal of Applied Probability*, 39:889–895, 2002.

S. I. Boyarchenko and S. Z. Levendorskiĭ. *Non-Gaussian Merton–Black–Scholes Theory*. World Scientific, 2002.

A. Brace, D. Gątarek, and M. Musiela. The market model of interest rate dynamics. *Mathematical Finance*, 7:127–155, 1997.

P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2(4):61–73, 1999.

P. Carr, H. Geman, D. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75:305–332, 2002.

P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.

E. Eberlein and K. Glau. PIDEs for pricing European options in Lévy models – a Fourier approach, 2011.

E. Eberlein and J. Jacod. On the range of options prices. *Finance and Stochastics*, 1:131–140, 1997.

E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1:281–299, 1995.

E. Eberlein and W. Kluge. Exact pricing formulae for caps and swaptions in a Lévy term structure model. *Journal of Computational Finance*, 9(2):99–125, 2006a.

E. Eberlein and W. Kluge. Valuation of floating range notes in Lévy term structure models. *Mathematical Finance*, 16:237–254, 2006b.

E. Eberlein and W. Kluge. Calibration of Lévy term structure models. In M. Fu, R. A. Jarrow, J.-Y. Yen, and R. J. Elliot, editors, *Advances in Mathematical Finance: In Honor of Dilip B. Madan*, pages 147–172. Birkhäuser, 2007.

E. Eberlein and N. Koval. A cross-currency Lévy market model. *Quantitative Finance*, 6:465–480, 2006.

E. Eberlein and F. Özkan. The Lévy Libor model. *Finance and Stochastics*, 9: 327–348, 2005.

E. Eberlein and A. Papapantoleon. Symmetries and pricing of exotic options in Lévy models. In A. Kyprianou, W. Schoutens, and P. Wilmott, editors, *Exotic Option Pricing and Advanced Lévy Models*, pages 99–128. Wiley, 2005.

E. Eberlein and K. Prause. The generalized hyperbolic model: Financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, and T. Vorst, editors, *Mathematical Finance – Bachelier Congress 2000*, pages 245–267. Springer, 2002.

E. Eberlein and S. Raible. Term structure models driven by general Lévy processes. *Mathematical Finance*, 9:31–53, 1999.

E. Eberlein, J. Jacod, and S. Raible. Lévy term structure models: No-arbitrage and completeness. *Finance and Stochastics*, 9:67–88, 2005.

E. Eberlein, W. Kluge, and P. J. Schönbucher. The Lévy Libor model with default risk. *J. Credit Risk*, 2:3–42, 2006.

E. Eberlein, A. Papapantoleon, and A. N. Shiryaev. On the duality principle in option pricing: Semimartingale setting. *Finance and Stochastics*, 12:265–292, 2008.

E. Eberlein, K. Glau, and A. Papapantoleon. Analysis of Fourier transform valuation formulas and applications. *Applied Mathematical Finance*, 17:211–240, 2010.

E. Eberlein, K. Glau, and A. Papapantoleon. Analyticity of the Wiener–Hopf factors and valuation of exotic options in Lévy models. In G. di Nunno and B. Øksendal, editors, *Advanced Mathematical Methods for Finance*, pages 223–245. Springer, 2011a.

E. Eberlein, D. B. Madan, M. R. Pistorius, and M. Yor. A simple stochastic rate model for rate equity hybrid products, 2011b. Preprint, University of Freiburg.

D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60: 77–105, 1992.

F. Hubalek, J. Kallsen, and L. Krawczyk. Variance-optimal hedging for processes with stationary independent increments. *Annals of Applied Probability*, 16:853–885, 2006.

T. R. Hurd and Z. Zhou. A Fourier transform method for spread option pricing. *SIAM Journal of Financial Mathematics*, 1:142–157, 2009.

J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, 1987.

W. Kluge. *Time-Inhomogeneous Lévy Processes in Interest Rate and Credit Risk Models*. PhD thesis, University of Freiburg, 2005.

A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, 2006.

D. B. Madan and E. Seneta. The variance gamma (vg) model for share market returns. *Journal of Business*, 63:511–524, 1990.

A. Maier. *Vereinfachung und numerische Berechnung von Wiener-Hopf-Faktoren*. Diplomarbeit, University of Freiburg, 2011.

M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer, 1997.

S. Raible. *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts.* PhD thesis, University of Freiburg, 2000.

M. Rudmann. *Fourier-basierte Derivatbewertung in Lévy-Zinsstrukturmodellen ohne Faltungsannahmen*. Diplomarbeit, University of Freiburg, 2011.

K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

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