A Simple Stochastic Rate Model for Rate Equity Hybrid Products

Ernst Eberlein
Department of Mathematical Stochastics
University of Freiburg
email: eberlein@stochastik.uni-freiburg.de

Dilip Madan
Robert H. Smith School of Business
University of Maryland
email: dbm@rhsmith.umd.edu

Martijn Pistorius
Department of Mathematics
Imperial College London
email: m.pistorius@imperial.ac.uk

Marc Yor
Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie
4, Place Jussieu F 75252 Paris Cedex

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Abstract

A positive spot rate model driven by a gamma process and correlated to equity is introduced and calibrated via closed forms for the joint characteristic function for the rate r, its integral y and the logarithm of the stock price s under the T*-forward measure. The law of the triple (r, y, s) is expressed as a nonlinear transform of three independent processes, a gamma process, a variance gamma process and a Wiener integral with respect to the Dirichlet process. The generalized Stieltjes transform of the Wiener integral with respect to the Dirichlet process is derived in closed form. Inversion of this transform using Schwarz (2005, The generalized Stieltjes transform and its inverse, Journal of Mathematical Physics, doi: 10.1063/1.1825077) makes large step simulations possible. Valuing functions are built and hedged using quantization and high dimensional interpolation methods. The hedging objective is taken to be capital minimization as described in Carr, Madan and Vicente Alvarez (2011, Markets, profits, capital, leverage and returns, Journal of Risk, 14, pp. 95-122).*

When structured investments offer yield enhancements in return for investors taking an exposure to some loss of coupon and/or principal linked to some equity and/or rate event we have a hybrid product combining stochastic features for movements in interest rates and equity prices. A number of these products have a random maturity as they may be auto-cancellable on an equity event. There is then an interest in valuing such products using stochastic models for the evolution of rates correlated with movements in equity prices. As an initial attempt to understand how values of such products relate to interest rate levels, realized money market returns and stock price levels, we consider here the formulation with a one dimensional Markovian interest rate model that we also correlate to stock price movements.

The literature contains numerous examples of interest rate models driven by an underlying Brownian motion and we may cite for example the models by Vasicek (1977), Hull and White (1990), Cox, Ingersoll and Ross (1985) and Black and Karasinski (1991). The underlying uncertainty in these models is a Brownian motion and the first two models can result in negative rates in a forward simulation, the third has positive rates provided the rate volatility is not too high and the fourth has positive rates with a lognormal distribution. For the fourth model the return on the money market account has an infinite expectation.

It has long been recognized in both the rate literature and the literature on equity options that Lévy processes provide considerable flexibility in calibrating models to market data. By way of example we cite Eberlein and Raible (1999) for rate modeling and Schoutens (2005), Cont and Tankov (2004) for equity modeling. Eberlein and Raible (1999) apply Lévy process drivers to rate modeling and generalize the Heath, Jarrow and Morton (1992) model of forward rates driven by Brownian motions. Eberlein and Özkan (2005) employ Lévy process drivers for a Libor based generalization of Brace, Gatarek and Musiela (1997).

Here we go back to one factor short rate models that are driven by a Lévy process. In fact, we take the Lévy process to be a subordinator i.e. an increasing Lévy process. This has the attractive feature of keeping rates positive, with no restriction on the rate volatility and yielding simultaneously a tractable model for the realized return on the money market account. We restrict ourselves even further and take the increasing process to be a gamma process. More precisely, we model the short rate r by an Ornstein-Uhlenbeck process driven by a gamma subordinator $g(t)$ and the logarithm of the stock price $s(t)$ by

$$s(t) = s(0) + y(t) - qt + X(t) - \beta g(t) + \omega(t)$$

where $y(t)$ is the integral of $r(s)$ over the interval $0 < s < t$, q is the dividend yield, $\omega(t)$ is the convexity correction and $X(t)$ is an independent variance gamma process.

For this model we derive large step analytical simulation procedures. This is an attractive property from the perspective of pricing and risk managing long dated structures with monthly, quarterly or longer observation periods.

Results of James (2010), Cifarelli and Regazzini (1990), Cifarelli and Melilli (2000) enable us to write the joint law of the triple constituted by i) the short rate, ii) the return on the money market account and iii) the logarithm of the model stock price at a future date as a nonlinear transform of three independent variables. The independent variables are a gamma variate, an independent variance gamma variate and the third is a stochastic integral of a deterministic integrand with respect to an independent Dirichlet process. We provide an analytical closed form for the generalized Stieltjes transform of the third independent variable and use the methods of Schwarz (2005) to get the density and the distribution function for simulation.

The model is calibrated to market data on equity options and the Treasury discount curve. This calibration requires the development of the characteristic function for the triple consisting of the rate r, its integral y and the log of the stock price s under the T-forward measure and this is developed in the article. After calibration we go on to analyse and risk manage a sample hybrid product. For valuation purposes we simulate in large steps the triple (r, y, s). For risk management purposes we wish to describe the value of the product as a function of the triple. We could construct a three dimensional grid for this purpose and evaluate the product at every point on the grid. Such a procedure is computationally expensive especially as one increases the dimension of the space to be discretized. We instead choose to simulate in large steps the triple of r, y, s at each reset date to generate a cloud of points. We then follow recommendations of Pagès, Pham and Printems (2003) and quantize this cloud into a smaller set of points. The product values are then computed on the much smaller set of quantized points and the values at the original set of points in the simulated cloud are obtained by applying high dimensional radial basis function interpolation or Tri Scattered interpolation procedures.

Once we have the product values on a simulated cloud of points in the future we consider risk managing by holding a portfolio of traded options as a hedge. We can easily simulate the payoffs to the hedge portfolio and then construct the residual cash flow. The portfolio position is chosen to minimize capital requirements for covering residual risk as defined in Carr, Madan and Vicente Alvarez (2011). Such a hedging criterion provides us with an objective function to be used in hedging that has a parameter expressing the degree of aggressiveness of the hedge. It also works simultaneously on lowering ask prices and maximizing bid prices in a two price economy as studied for example in Cherny and Madan (2010) and Madan and Schoutens (2011a). The results of the hedge are compared with the more classical criterion of variance minimization.

In summary the contributions of the paper constitute a new hybrid, relatively simple, rate-equity model, along with analytical calibration procedures. This is followed by analytical procedures for large step simulation involving an inversion of a generalized Stieltjes transform. We then develop quantization procedures for the three dimensional clouds of simulated outcomes at the reset dates along with applications of high dimensional interpolation methods. Finally we hedge the product value with a position in traded options with a view to minimizing capital required for residual risk exposure.

The outline of the rest of the paper is as follows. Section 1 introduces the rate-equity model. The joint characteristic function is derived in Section 2 and the calibration is carried out in Section 3. Section 4 presents large step simulation procedures. In Section 5 we simulate and quantize clouds of points at 20 quarters going out 5 years. Section 6 employs radial basis function interpolation to construct target remaining value functions to be hedged. Hedging procedures are described and implemented in Section 7. Section 8 concludes. An appendix contains all proofs.

1 A gamma driven rate model correlated with equity

The instantaneous short rate of interest $r(t)$ is modeled by an Ornstein-Uhlenbeck process driven by the gamma process $g(t)$, which is a one factor mean-reverting Markov process. The gamma process is defined in terms of the unit time gamma density with scale parameter $c > 0$, shape parameter $\gamma > 0$ and density

$$f(x) = \frac{c^{\gamma}}{\Gamma(\gamma)} x^{\gamma-1} e^{-cx}, \ x \geq 0.$$

The expectation of the gamma variate is γ/c while its variance is γ/c^2. The characteristic function of a gamma distributed random variable g is

$$E\left[e^{iug}\right] = \left(\frac{c}{c-iu}\right)^{\gamma}.$$

The gamma distribution is infinitely divisible and gives rise to the gamma Lévy process $g(t)$ with characteristic function

$$E\left[e^{iug(t)}\right] = \left(\frac{c}{c-iu}\right)^{\gamma t}.$$

This gamma process is an increasing purely discontinuous process with Lévy density

$$k(x) = \gamma \frac{e^{-cx}}{x}, \ x > 0.$$

Since the Lévy density has an infinite integral, the process has infinitely many jumps in any interval with all but a finite set being arbitrarily small. For a mean reversion rate of κ we suppose that the short rate with an initial spot rate of r_0 satisfies the equation

$$dr = -\kappa r dt + dg. \tag{1}$$

The continuously compounded realized return on the money market is then given by

$$y(t) = \int_0^t r(s)ds.$$

The spot rate of equation (1) is an exponentially weighted integral of the past jumps in the gamma process that are all positive, so that the spot rate will never be negative for any driving gamma process. We are therefore always assured of a positive spot rate in this model with no constraints on the parameters of the process for this purpose. The resulting model for the evolution of the spot rate is an example of the class of non-Gaussian Ornstein-Uhlenbeck (OU) models introduced by Barndorff-Nielsen and Shepard (2001).

In addition to the gamma process driving the instantaneous short rate, the stock or stock index is also driven by an independent variance gamma process $X(t)$, (Madan and Seneta (1990), Madan, Carr and Chang (1998)) obtained as a Brownian motion with drift θ, and volatility σ time changed by an independent gamma process with unit mean rate and variance rate ν. The characteristic function for $X(t)$ is given by

$$\begin{aligned} & E[e^{iuX(t)}] \stackrel{def}{=} \phi_{VG}(u,t;\sigma,\nu,\theta) \\ = & \left(\frac{1}{1-iu\theta\nu+\frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}} \\ = & {}^{def}\phi_X(u)^t \end{aligned} \tag{2}$$

We also write for later use

$$\begin{aligned} \phi_X(u) &= \exp\left(\psi_X(u)\right) \\ \psi_X(u) &= -\frac{1}{\nu}\ln\left(1-iu\theta\nu+\frac{\sigma^2\nu}{2}u^2\right). \end{aligned} \tag{3}$$

The unit time variance gamma density is infinitely divisible and $X(t)$ is a Lévy process with Lévy measure identified in Madan, Carr and Chang (1998) where it is also shown that the process $X(t)$ can be written as the difference of two independent gamma processes. The Lévy measure takes the form of the gamma process Lévy measure already identified above via its density.

The risk neutral process for the stock, in the absence of dividends, when discounted by the accumulation in the money market account is a martingale. This property leads to a specification for the logarithm of the stock price for a continuously compounded dividend yield of q in the form

$$\begin{aligned} & \ln S(t) \stackrel{def}{=} s(t) \\ = & \; s(0)+y(t)-qt+X(t)-\beta g(t)+\omega t \end{aligned}$$

where we have induced a correlation between rates and the stock via the response of stock prices to rate shocks in the term $-\beta g(t)$. The constant ω is a convexity correction ensuring the required martingale condition via the definition

$$\omega = -\ln\left(E\left[\exp\left(X(1)-\beta g(1)\right)\right]\right).$$

We term the model given by the short rate $r(t)$ and the stock price process $S(t)$, $VGGDSR$ for VG log stock prices enhanced by gamma driven stochastic rates.

For purposes of pricing hybrid claims we focus attention on stochastically evolving in large steps the triple of the instantaneous spot rate $r = (r(t), t \geq 0)$, its time integral or the money market accumulation factor $y = (y(t), t \geq 0)$ and the logarithm of the stock price $s = (s(t), t \geq 0)$. The value function at time t, $V(t)$, for the value of cash flows remaining after time t, of a prospective claim discounted back to time 0 is of the form

$$V(t) = \exp\left(-y(t)\right)\Phi\left(r(t), s(t), t\right), \tag{4}$$

as the future evolution of uncertainty after time t is Markov in the state variables $r(t), s(t)$. The objective is not just to value the product at time zero but to exhibit $V(t)$ the discounted to time zero value function Φ in equation (4) at each of the contract reset dates as a function of the state variables given by the triple $(r(t), y(t), s(t))$. The first step is to use market information to calibrate the parameters of the process.

2 The joint law of the spot rate, the cumulated return on the money market account and the logarithm of the stock under the T-forward measure.

For the calibration of this model to market data it is useful to evaluate the joint characteristic function

$$\phi_{VGGDSR}(u, v, w, t) = E\left[\exp\left(iur(t) + ivy(t) + iws(t)\right)\right].$$

With stochastic rates it is advantageous to work under the t-forward measure as explained in Jamshidian (1989) and Geman, El Karoui and Rochet (1995) for example. For a call option on the stock with strike K and maturity t we wish to evaluate

$$\begin{aligned} w(K,t) &= E\left[\exp\left(-\int_0^t r(u)du\right)(S(t)-K)^+\right] \\ &= E\left[\exp\left(-\int_0^t r(u)du\right)\right]\frac{E\left[\exp\left(-\int_0^t r(u)du\right)(S(t)-K)^+\right]}{E\left[\exp\left(-\int_0^t r(u)du\right)\right]} \\ &=^{def} P(0,t)E^t\left[(S(t)-K)^+\right] \end{aligned} \tag{5}$$

where $P(0, t)$ is the price of a zero coupon bond of maturity t and E^t denotes expectation under the t-forward measure Q^t where

$$\frac{dQ^t}{dQ} = \frac{\exp\left(-\int_0^t r(u)du\right)}{E\left[\exp\left(-\int_0^t r(u)du\right)\right]}.$$

We therefore focus attention more generally on the joint characteristic of the triple under the T-forward measure for $T \geq t$.

Proposition 1 *The joint characteristic function under the T-forward measure $\phi^T_{VGGDSR}(u,v,w,t)$ is given by*

$$
\begin{aligned}
\phi^T_{VGGDSR}(u,v,w,t) &= F_1(t,u,v,w)\ F_2(t,u,v,w)\ F_3(t,u,v,w) \\
F_1(t,u,v,w) &= \exp\begin{pmatrix} \left(iue^{-\kappa t} + (iv+iw)\frac{1-e^{-\kappa t}}{\kappa}\right) r(0) \\ +iw\begin{pmatrix} \ln S(0) - qt - t\ln\phi_X(-i) \\ -\gamma t\left(\ln(c) - \ln(c+\beta)\right)\end{pmatrix}\end{pmatrix} \\
F_2(t,w) &= \exp\left(t\psi_X(w)\right) \\
F_3(t,u,v,w) &= \exp\left(\left(G(1) - G(e^{-\kappa t})\right) - \left(H(1) - H(e^{-\kappa t})\right)\right) \\
G(x) &= \frac{\gamma}{\kappa}\begin{pmatrix} Li_2\left(-\frac{bx}{a+c}\right) - \ln(x)\times \\ \left((a+bx+c) - \ln\left(\frac{bx}{b+c}+1\right)\right) + \\ \ln(c)\ln(x)\end{pmatrix} \\
H(x) &= \frac{\gamma}{\kappa}\begin{pmatrix} Li_2\left(-\frac{b'x}{a'+c}\right) - \ln(x)\times \\ \left((a'+b'x+c) - \ln\left(\frac{b'x}{b'+c}+1\right)\right) + \\ \ln(c)\ln(x)\end{pmatrix} \\
a &= iw\beta - i\frac{v+w}{\kappa} + \frac{1}{\kappa} \\
b &= \frac{i(v+w)}{\kappa} - iu - \frac{e^{-\kappa(T-t)}}{\kappa} \\
a' &= \frac{1}{\kappa} \\
b' &= -\frac{e^{-\kappa(T-t)}}{\kappa}
\end{aligned}
$$

where Li_2 is the dilogarithmic function.

The proof of Proposition 1 is provided in the Appendix. The function Li_2 is also defined by the integral representation

$$Li_2(x) = -\int_0^x \frac{\ln(1-t)}{t}dt.$$

The prices of zero coupon bonds may be directly obtained from the joint characteristic function. For prices of options we follow the procedures of Carr and Madan (1999).

3 The model calibrated to S&P 500 index options and the discount curve

We calibrated the $VGGDSR$ model to market data on 520 options on the $S\&P$ 500 index and 11 maturities of the pure discount curve on August 15, 2011.

The estimated parameters of the Gamma driven stochastic rate model with VG driven stock price yielded the following result:

$$\begin{aligned} \kappa &= 0.1868 \\ c &= 570.3251 \\ \gamma &= 4.7936 \\ r_0 &= 0 \\ \beta &= .0529 \end{aligned}$$

The mean jump in the rates is 84.05 basis points with a volatility of 38.39 basis points. The long term mean interest rate is estimated as $.008406/.1868 = 4.5\%$. The root mean square error (RMSE) on the discount curve was 89.75 basis points. There were 11 maturities and we report the RMSE, the average absolute error (AAE), and the average percentage error (APE) by maturity in Table 1.

TABLE 1

Maturity	RMSE	AAE	APE
.1260	0.5131	0.4427	0.0358
.1836	0.6295	0.5747	0.0405
.2603	0.7195	0.6422	0.0392
.3753	0.8592	0.7410	0.0353
.5863	1.0134	0.8843	0.0301
.6247	0.9829	0.8415	0.0340
.8356	1.0706	0.8908	0.0249
.8740	1.1996	0.9889	0.0234
1.3534	1.4711	1.0643	0.0192
1.8521	1.9180	1.4534	0.0217
2.3507	2.5903	1.9728	0.0234

We present in Figure 1 a graph of the fit of this model to these options.

We used separate VG parameters at each maturity and the estimated values are presented in TABLE 2.

TABLE 2

Maturity	σ	ν	θ
.1260	0.2305	0.1300	−0.5920
.1836	0.2231	0.2159	−0.4557
.2603	0.2207	0.3118	−0.3793
.3753	0.2145	0.4172	−0.3301
.5863	0.2054	0.5748	−0.2789
.6247	0.2049	0.6421	−0.2654
.8356	0.1962	0.7421	−0.2482
.8740	0.1904	0.7029	−0.2557
1.3534	0.1766	0.8680	−0.2316
1.8521	0.1510	0.8410	−0.2380
2.3507	0.0302	0.6512	−0.3097

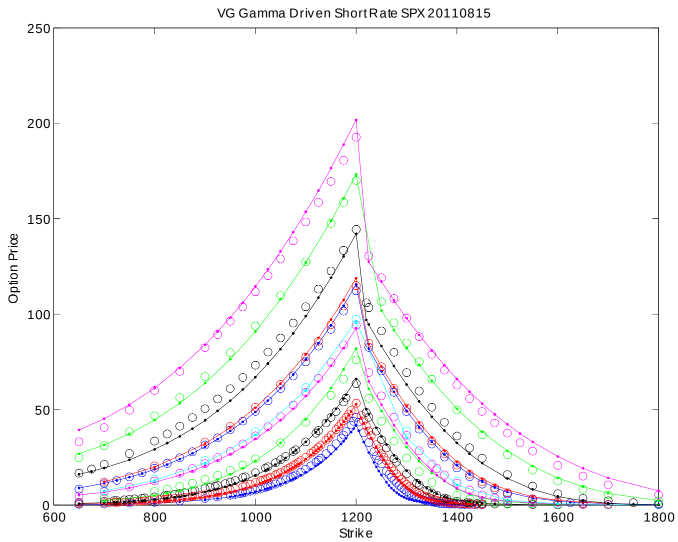

Figure 1: Fit of Gamma driven short rate model to SPX option surface on August 15, 2011.

3.1 Parameter identification

Parameters once estimated are not necessarily identified. Identification requires an evaluation of the sensitivity of the estimation criterion with respect to the parameters. With regard to determining the identification of stochastic rate parameters from the option surface we present the average absolute value of the derivative of the root mean square error across strikes and maturities with respect to the five rate parameters by maturity in Table 3.

TABLE 3

Maturity	κ	γ/c	$\sqrt{\gamma}/c$	r_0	β
0.1260	0.0009	2.4607	2845	38.97	0
0.1836	0.0038	7.3681	3326	79.96	0.0001
0.2603	0.0121	16.7099	3364	127.62	0.0004
0.3753	0.0365	35.0460	3419	185.27	0.0011
0.5863	0.1555	96.3318	3528	324.82	0.0032
0.6247	0.1623	94.4650	3302	298.22	0.0032
0.8356	0.4510	197.518	3156	463.04	0.0058
0.8740	0.5462	228.902	3199	513.07	0.0075
1.3534	2.2983	631.562	3130	899.95	0.0156
1.8521	7.0313	1433.62	3637	1473.9	0.0378
2.3507	16.7069	2723.68	4296	2180.8	0.7768

The optimization criterion is quite insensitive to β, increases with maturity for the mean reversion κ, the long term mean γ/c, and the initial spot rate r_0, and is fairly stable across maturity for the volatility of rates $\sqrt{\gamma}/c$.

Another way to investigate parameter identification is to evaluate the eigenvalues of the matrix of second order derivatives of the estimation criterion with respect to the estimated parameters. We could look here at the second derivatives of the root mean square error estimation criterion by the eleven maturities. This would give us 11 matrices of dimension 5 by 5. We already know that the longer maturities are doing most of the work. So we compute just one matrix by averaging across all 11 maturities.

We expect identification with respect to r_0 and investigate the identification of κ, m, v, β. In this computation we recognize the transformations

$$\begin{aligned} c &= \frac{m}{v^2} \\ \gamma &= \frac{m^2}{v^2}. \end{aligned}$$

For the four parameters κ, m, v, β the matrix of second order derivatives is

$$\begin{bmatrix} 13.1436 & -2183.97 & -2.9369 & 0.0447 \\ -2183.97 & 381759 & 474.33 & -7.6451 \\ -2.9369 & 474.33 & 8.9202 & 0.0103 \\ 0.0447 & -7.6451 & 0.0103 & -0.0012 \end{bmatrix}$$

The four eigenvalues are 38177, 8.337, 0.6543, and -0.0013 corresponding to m, v, κ, and β. Hence we conclude that the option surface and the discount curve help to identify the long term mean m, the volatility of rates v, the rate of mean reversion κ and the correlation β in that order. The identification of correlation is weak.

4 Large step simulation procedures

We may write the stochastic components of the triple (r, y, s) at a fixed large time step t as $(\widetilde{r}, \widetilde{y}, \widetilde{s})$ where

$$\widetilde{r}(t) = r(t) - r(0)e^{-\kappa t} = \int_0^t e^{-\kappa(t-u)} dg(u),$$

$$\widetilde{y}(t) = y(t) - r(0)\frac{1-e^{-\kappa t}}{\kappa} = \int_0^t \frac{1-e^{-\kappa(t-u)}}{\kappa} dg(u)$$

and

$$\begin{aligned} \widetilde{s}(t) &= s(t) - s(0) + qt - t\ln\phi_X(-i) + \gamma t\left(\ln(c) - \ln(c+\beta)\right) - r(0)\frac{1-e^{-\kappa t}}{\kappa} \\ &= \widetilde{y}(t) + X(t) - \beta g(t) \end{aligned}$$

Employing time reversal for the gamma process, the following equalities in law hold jointly

$$\begin{aligned}
\widetilde{r}(t) &= \int_0^t e^{-\kappa u} dg(u) \\
\widetilde{y}(t) &= \int_0^t \frac{1-e^{-\kappa u}}{\kappa} dg(u) \\
\widetilde{s}(t) &= \widetilde{y}(t) + X(t) - \beta \int_0^t dg(u)
\end{aligned}$$

The Dirichlet process with parameter t, $(D(s), s \leq t)$ is defined in terms of the gamma process $(g(u), u \geq 0)$ by

$$D(s) = \frac{g(s)}{g(t)}, s \leq t.$$

The Dirichlet process is independent of $g(t)$. We now express the three processes in terms of the Dirichlet process $D(s)$ and $g(t)$ as

$$\begin{aligned}
\widetilde{r}(t) &= g(t) \int_0^t e^{-\kappa u} dD(u) \\
\widetilde{y}(t) &= \frac{1}{\kappa} g(t) \left(1 - \int_0^t e^{-\kappa u} dD(u)\right) \\
\widetilde{s}(t) &= \widetilde{y}(t) + X(t) - \beta g(t)
\end{aligned}$$

Defining $\rho(t)$ as

$$\rho(t) = \int_0^t e^{-\kappa u} dD(u)$$

we have

$$\begin{aligned}
\widetilde{r}(t) &= g(t)\rho(t) \\
\widetilde{y}(t) &= \frac{g(t)}{\kappa} (1 - \rho(t)) \\
\widetilde{s}(t) &= \frac{g(t)}{\kappa} (1 - \rho(t)) + X(t) - \beta g(t)
\end{aligned}$$

We have now expressed the stochastic triple as a simple nonlinear transformation of three independent variables $\rho(t), g(t), X(t)$. We may easily simulate at a large time step the gamma variate $g(t)$ and the variance gamma variate $X(t)$ and we need to learn how to simulate $\rho(t)$.

4.1 The density of $\rho(t)$.

We investigated an evaluation of the characteristic function of $\rho(t)$ and this turned out to be intractable. However we were able to evaluate analytically the generalized Stieltjes transform of $\rho(t)$. More generally define the random variable

$$W = \int_0^\eta \psi(y) dD(y)$$

as a stochastic integral with respect to a deterministic integrand $\psi(y)$. We integrate here with respect to the Dirichlet process ($D = D(y)$, $0 < y < \eta$) where without loss of generality we may take $D(y) = \gamma(y)/\gamma(\eta)$ for a standard gamma process $\gamma(y)$ with unit scale and shape parameter in the interval $0 < y < \eta$. We distinguish the gamma process $g(t)$ from this standard gamma process $\gamma(t)$. A shape parameter that differs from unity may be incorporated by adjusting η, i.e. the domain of integration, while the result is independent of the scale parameter. The generalized Stieltjes transform of W is defined for a transform parameter s and a generalization parameter η by

$$\zeta(s) = E\left[\left(\frac{1}{s+W}\right)^\eta\right]$$

Proposition 2 *The generalized Stieltjes transform of* W, $\zeta(s)$, *is*

$$\zeta(s) = \exp\left(-\int_0^\eta \log(s+\psi(y))dy\right). \tag{6}$$

The formula (6) is sometimes called the Markov-Krein identity; see Tsilevich and Vershik (1999). We give a direct proof of it in the Appendix. The specific function $\psi(y)$ of interest in the construction of $\rho(t)$ is $\psi(y) = e^{-\kappa y}$ and we wish to invert the generalized Stieltjes transform of

$$\int_0^\eta e^{-\kappa y} dD(y).$$

for the density of $\rho(t)$ in the unit interval. For this inversion we follow Schwarz (2005) whereby the density is given in terms of ζ by

$$f(y) = -\frac{1}{2\pi i} y^\eta \int_{\mathcal{C}} (1+w)^{\eta-1} \zeta'(yw) dw$$

and the integration is performed counterclockwise on the unit circle C, starting and ending at -1, where for the explicit case considered here

$$\zeta(s) = \exp\left(\frac{1}{\kappa}\left(Li_2\left(-\frac{1}{s}\right) - Li_2\left(-\frac{e^{-\kappa\gamma\eta}}{s}\right) - \kappa\gamma\eta\log(s)\right)\right)$$

when the gamma process employed has shape parameter γ and we integrate to time η with mean reversion κ.

For the computation of $\zeta'(s)$ we note that

$$\zeta'(s) = \zeta(s)\left(\frac{1}{\kappa}\left(\frac{1}{s}\ln\left(1+\frac{1}{s}\right) - \frac{1}{s}\ln\left(1+\frac{e^{-\kappa\gamma\eta}}{s}\right) - \frac{\kappa\gamma\eta}{s}\right)\right).$$

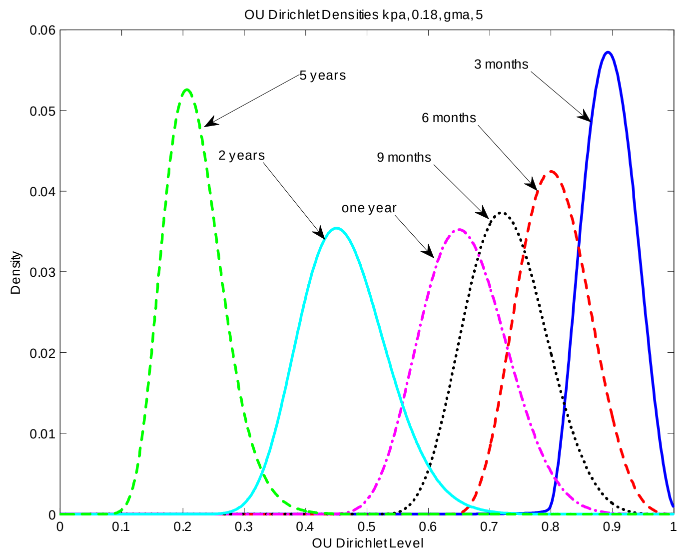

Figure 2: Densities for $\rho(\eta)$ using generalized Stieltjes transform inversion by Schwarz (2005).

4.2 Sample densities

We take $\kappa = 0.18$ and $\gamma = 5$ as the estimated values for a variety of values for η to obtain densities for $\rho(\eta)$ by inverse generalized Stieltjes transforms as per Schwarz (2005). The results are presented in Figure 2. For the simulation of the Dirichlet process we obtain the density by an inverse generalized Stieltjes transform on a fine grid in the unit interval. The density is converted to a probability on normalization from which we obtain the distribution function and use the inverse distribution function applied to uniform variates to simulate the Dirichlet process.

5 Five year quarterly simulation and quantization of the triple r,y,s

We employ the large step simulation procedure outlined in Section 4 to generate three dimensional clouds of $10,000$ points for the triple $(r(t), y(t), s(t))$ at $t = ih$ for $h = .25$ and $i = 1, \cdots, 20$ to construct a sample of possible realizations at each of twenty quarters going out five years. We shall use these points to value the remaining uncertainty in a prospective hybrid structured product at each quarter end. The valuation could be constructed on a grid of points but recognizing that grids can be large, expensive and wasteful as we increase the dimension we work instead with a simulated sample.

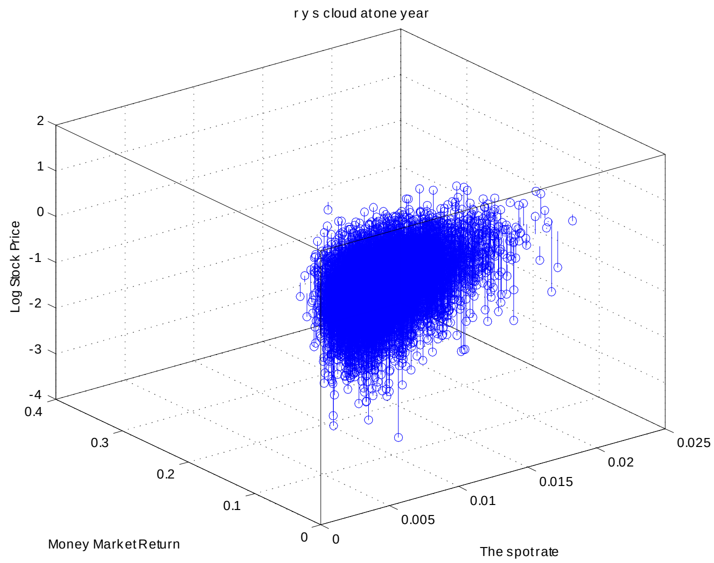

Figure 3: Simulated points for the spot rate, the money market return and the logarithm of the stock at the one year point.

For the parameter setting $c = 571.3251$, $\gamma = 4.7936$, $\kappa = 0.1867$, $q = .02$, $\sigma = 0.1991$, $\nu = 0.6615$, $\theta = -0.2554$, $\beta = 0.25$ we present in Figure 3 a graph of the cloud at this one year point. The parameters are obtained by a calibration of this stochastic rate-equity option model to S&P 500 option data and the discount curve on August 15, 2011, where we now restrict the VG parameters to be the same across 11 maturities.

We observe as expected that the three variables are on different scales. There is considerable movement in the log of the stock price relative to the money market account followed by the level of the spot rate. The variables are also correlated and dependent.

In building the time zero remaining value of a structured product at any time point one could start the simulation at a potential point at a particular time to generate future cash flows that are then discounted back to time zero and averaged to construct the product value. However, it is not necessary to build this value at each of the $10,000$ points in the simulation. One could reduce the $10,000$ points to a smaller set of representative points. The objective then becomes one of summarizing the cloud using a much smaller set of representative points. This activity is called quantization and much has been written in the computer science, mathematics and engineering literature towards accomplishing this task optimally for a variety of optimization criteria. The initial and classical approaches quantize a cloud of M points by a prespecified set of $N < M$ points with each point in the space allocated to its nearest neighbour among the N points with a view to minimizing the total distance between the points and their nearest neighbours. One of the widely used algorithms accomplishing this is Lloyd's algorithm (Lloyd (1982)) also known as Voronoi iteration or relaxation. The typical metric employed is the Euclidean distance. Pagès, Pham and Printems (2003) consider applications to finance and address the problem of quantization of a Markov process by a Markov chain. They illustrate their methods on problems of pricing European and American options and a variety of hedging, filtering and stochastic control problems. For the quantization of a Markov process the appropriate metric for measuring the distance between a random variable and a quantized representation of it is the L^P norm.

We report here the results of two quantizations. First we apply Lloyd's algorithm directly to the data on the (r, y, s) clouds and second we transform the data using marginal distributions to uniformly distributed random variables. The variables are then on the same scale. We find that once this has been done, Lloyd's algorithm is capable of coping with all kinds of dependence structures. This includes detecting submanifolds in which the data may reside. Hence we quantize the cloud of uniformly distributed triples with Lloyd's algorithm and then we map the quantized points back into the original cloud using the inverse uniform mapping applied to the marginal distribution functions.

We first perform the quantization of the raw data of simulated points into 128 quantized points. To observe the quality of this quantization we plot the original data and the associated quantized points in the three subspaces of (r, y), (y, s) and (r, s). This is presented in Figures 4, 5 and 6.

We see from these graphs that the raw quantization procedure essentially ignores the subspace of the spot rate and the money market return.

We now consider the second quantization that quantizes the data after transformation into the unit cube of three dimensions. The quantized points are then transformed back into the original space using inverse marginal distribution functions. First we present in Figure 7 a three dimensional plot of the cloud and the quantized points and then we present the three slices in Figures 8, 9 and 10. We employ 2048 points in this quantization.

We can observe the considerable improvement generated by the transformation in the three slices. We shall construct valuations at each of 2048 points in this quantization at each date.

6 Remaining value of a structured product on a quantized cloud

We take by way of an example a particular structured product that is a coupon paying auto cancellable note taking the equity risk of the stock. There are interest payment times t_i, $i = 1, \cdots, n$ at which the payment is

$$Nk_i(t_{i+1} - t_i)\mathbf{1}_{S(t_i) > I_i} \prod_{u_j \leq t_i} \mathbf{1}_{S(u_j) \leq V_j}$$

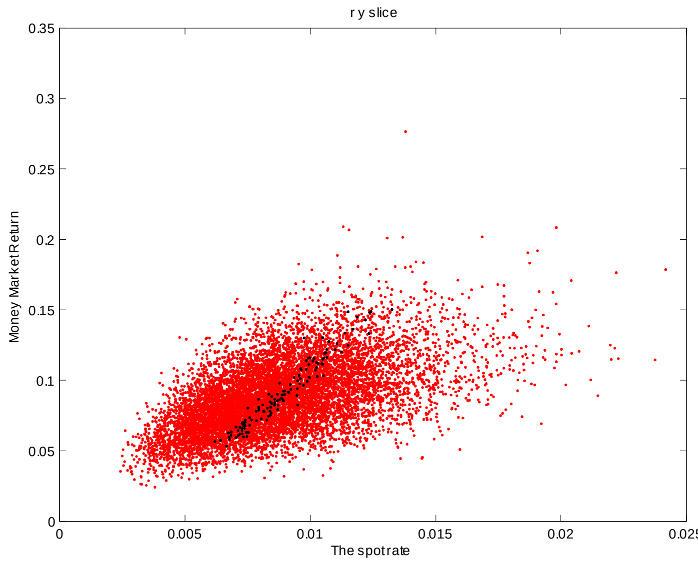

Figure 4: Spot rate, money market return slice of points in red and 128 quantized points in black.

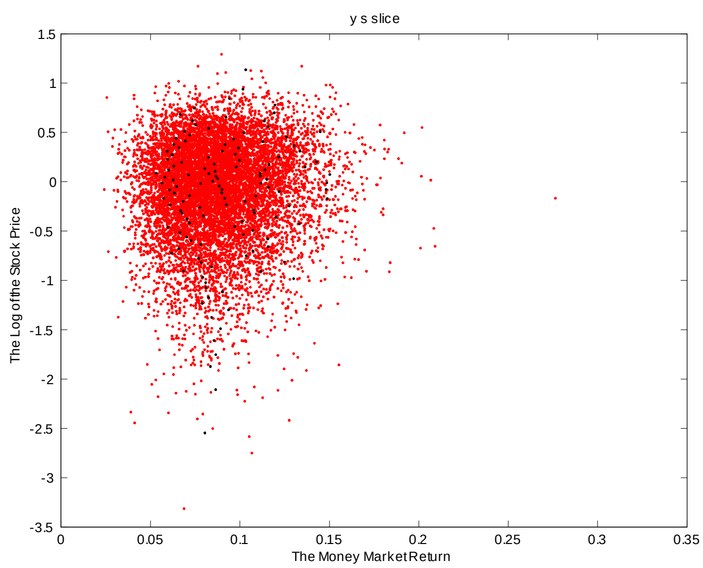

Figure 5: Money market return and log stock price slice of points in red and 128 quantized points in black.

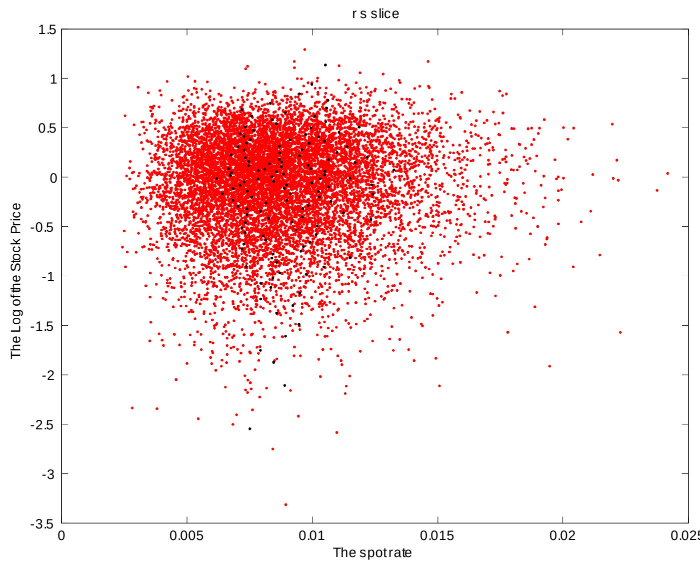

Figure 6: Spot rate and log stock price slice of points in red and 128 quantized points in black.

In addition there is a final payment time T at which the payment is

$$\left(N\mathbf{1}_{S(T)>B} + \mathbf{1}_{K\leq S(T)\leq B} N\left(1 + \left(1 - \frac{S(T)}{B}\right)\right) + \mathbf{1}_{S(T)<K}\frac{N}{B}\right)\left(\prod_{u_j\leq T}\mathbf{1}_{S(u_j)\leq V_j}\right)$$

and early redemption times u_j, $j = 1, \cdots, m$ at which the payment is

$$N\prod_{v_k\leq u_j}\mathbf{1}_{S(v_k)\leq V_k}\mathbf{1}_{S(u_j)>V_j}$$

The product is specified by listing the sequence of interest payment dates t_i, barriers for the stock price I_i triggering nonpayment of interest, interest coupons k_i, redemption dates u_j, redemption barriers V_j, terminal date T, strike B and knockout strike K.

We take interest payments at each quarter end so $t_i = .25i$, for $i = 1, \cdots, 20$ for a five year product. The coupon is 10% per annum and we start the stock price at 100. The interest rate strikes are uniformly 70 and the redemption barriers are uniformly set at 120. The notional is set at 10000 dollars. Early redemption is at par and for the final payment the upper strike B is 80 while the lower strike K is 50.

At any date t_i in the future if redemption has not occurred, one may value as a function of the level of r, s at time t_i the value at time t_i of all future cash flows discounted back to time t_i. Furthermore, given the level y of returns to

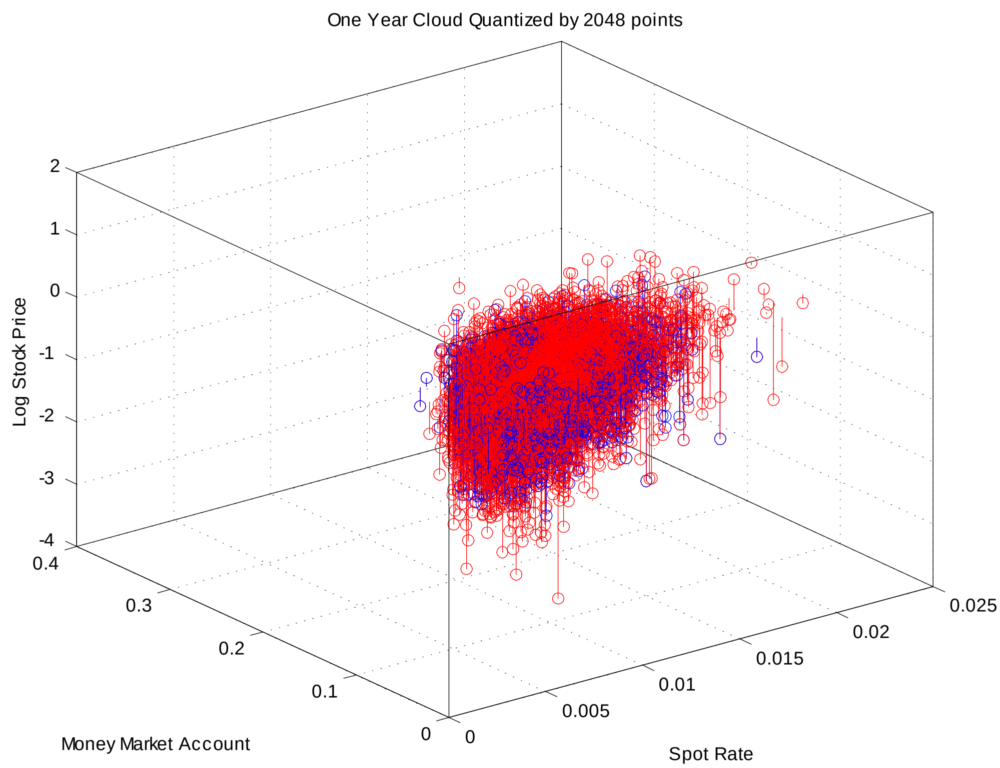

Figure 7: Quantization of one year cloud by 2048 points after transformation to unit cube.

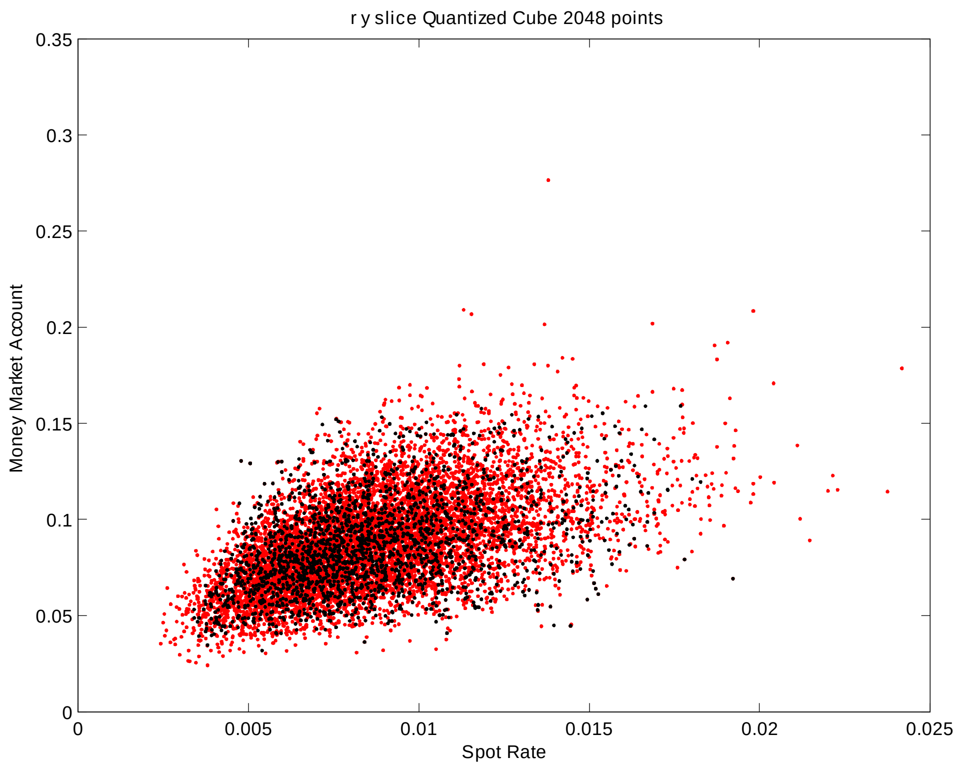

Figure 8: The spot rate and money market account slice after a unit cube quantization

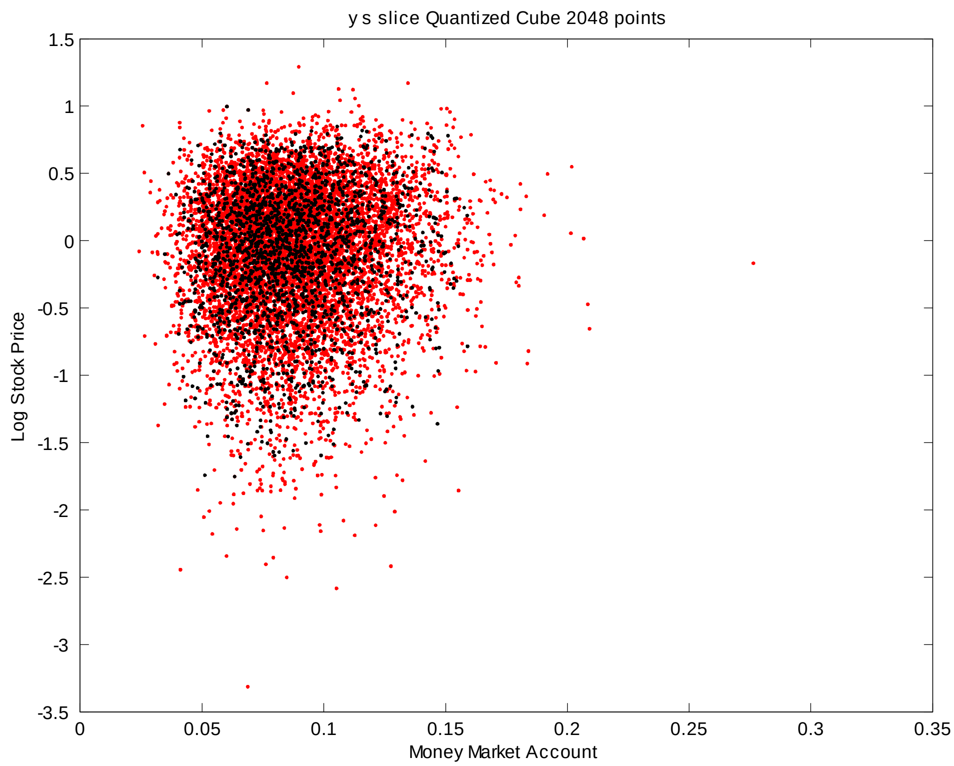

Figure 9: The money market account and log stock price slice after unit cube quantization.

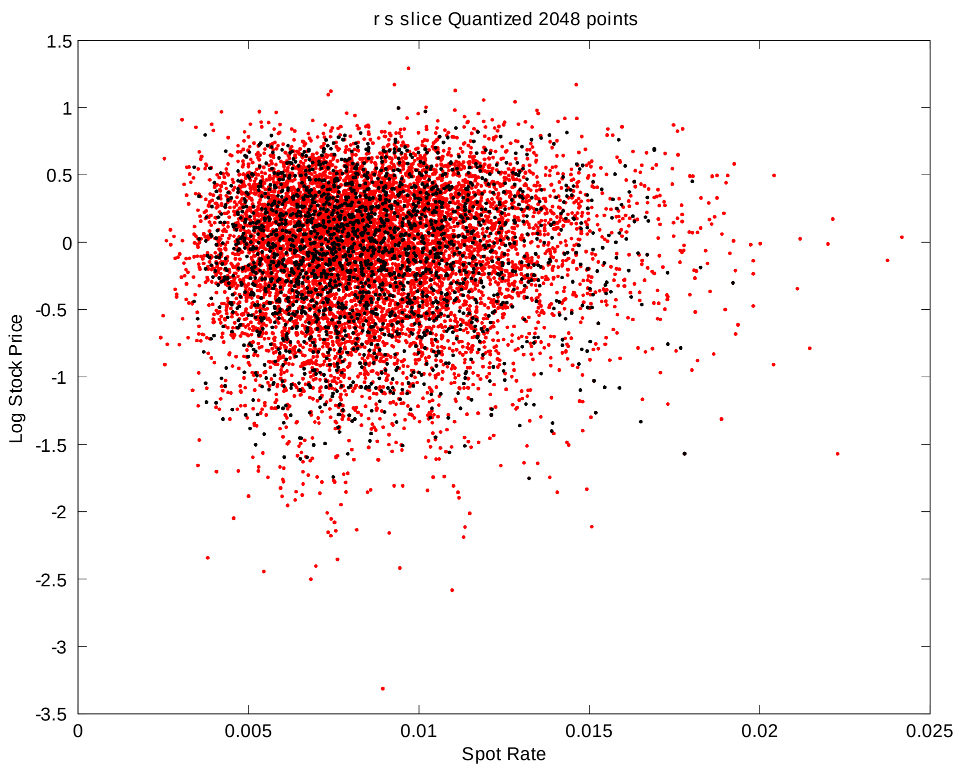

Figure 10: The spot rate and log stock price slice after unit cube quantization

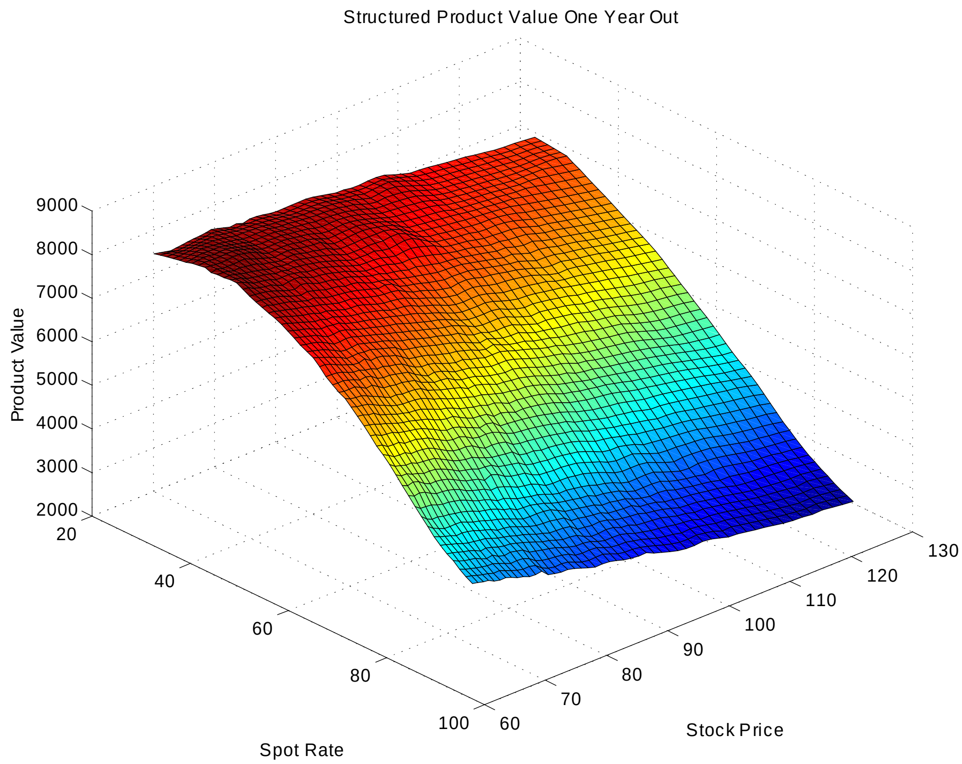

Figure 11: Remaining value of the structured product at the end of year one as a function of the spot rate and the stock price.

the money market account at time t_i, we may value the discounted future cash flows back to time 0. We begin with time 20 and work back these valuations to time 0. The valuations are conducted on the 2056 quantization points where we have added eight points to the 2048 quantized points to create a cube to which all the data belong. We can then use tri scattered interpolation to construct the value function at each date as function of the state space (r, y, s). For graphing purposes we take the money market accumulation factor to be at its i^{th} level when the interest rate is at its i^{th} level. We present in Figures 11 and 12 a graph of the remaining value for the structured product as a function of the spot rate and the stock price at the one and two year points.

7 Hedging exercises

This section reports on hedging the value function at the one-year point using a static position in options on the rate at the one year point, two-year bond

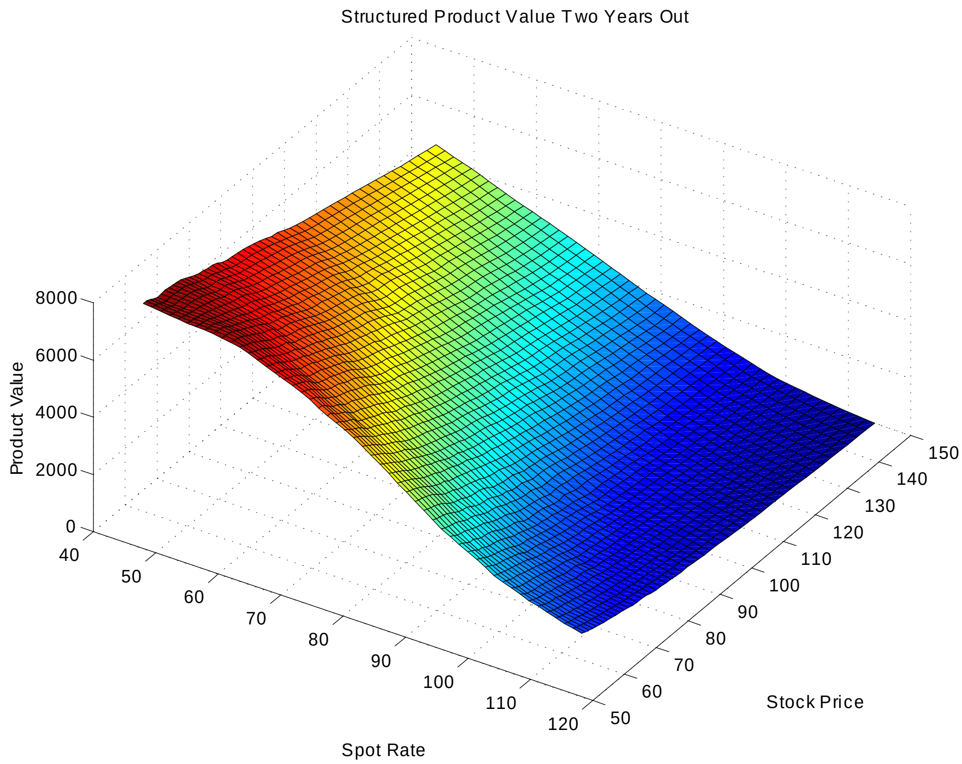

Figure 12: Remaining value of the structured product at the end of year two as a function of the spot rate and the stock price.

options at the one year point and stock options with a one year maturity. We use two puts and two calls on the rate struck at 40 and 50 basis points for the puts and 60 and 70 basis points for the calls. For the unit face two year bond options we have strikes of 0.9725, 0.975, for calls and 0.9675, 0.97 for puts. The stock option puts are struck at 80, 90 and the calls are struck at 110, 120. In addition we take a position in the stock itself. In all we have 13 hedging assets.

The target cash flow vector denoted tcf is given by the remaining value function evaluated at the $10,000$ simulated points obtained by interpolation across the 2056 quantized points at the one year point. We construct a matrix H of dimension 13 by $10,000$ that gives the zero cost value of the cash flow to the 13 hedging assets in each of the $10,000$ simulated realizations of the triple (r, y, s) at year end. For any prospective position α in the hedging assets one may define the residual cash flow denoted rcf by

$$rcf = tcf + \alpha' H.$$

The objective function for the choice of the hedge portfolio is the minimization of capital required as defined in Carr, Madan and Vicente Alvarez (2011). This is the difference between the ask and bid prices of a two price economy computing these prices using the concave distortion *minmaxvar* at the stress level 0.25. This stress level was also used in Madan and Schoutens (2011b) to construct comonotonicity indices for the US economy for the three year period ending December 2011. With no hedge the required capital was 1434. This is reduced to 509 by the hedge. The positions taken in the 13 hedging assets are given in Table 4.

TABLE 4

Hedge Positions Minimizing Capital

	Strike	Type	Position
	40	P	-1.17
Rates	50	P	1.85
	60	C	-28.55
	70	C	27.65
	80	P	240.81
Stock	90	P	-173.74
	110	C	155.79
	120	C	-116.46
	0	C	16.47
	0.9675	P	-1574.42
Bond	0.97	P	2170.49
	0.9725	C	-2702.72
	0.975	C	1194.54

8 Conclusion

This paper introduces a spot rate model evolving in accordance with an OU equation driven by a gamma process which is in the class studied by Barndorff-Nielsen and Shepard (2001). The gamma process driving the rate also simultaneously affects stock prices thereby correlating stock prices with rates. The stock has an additional uncertainty modeled by an independent variance gamma process. For this model we derive in closed form the joint characteristic function for the triple of the spot rate r, the accumulation on the money market y and the logarithm of the stock price s under the T-forward measure. This model is termed VGGDSR for the VG process enhanced by a gamma driven stochastic rate. The VGGDSR T-forward characteristic function is used to calibrate the model to data on equity options and the discount curve.

It is shown that at a large time step the triple (r, y, s) can be expressed as a nonlinear transform of three independent processes, a gamma process, a variance gamma process and a stochastic integral with respect to the Dirichlet process. The generalized Stieltjes transform of the stochastic integral with respect to the Dirichlet process is derived in closed form and inverted using Schwarz (2005). One thereby has access to large step simulations.

Large step simulations are conducted for 20 quarters out to five years for the calibrated model. With a view to building value functions the simulated clouds of triples for (r, y, s) are quantized after transformation into the unit cube of three dimensions. It is shown that such a quantization adequately covers the two dimensional subspaces, as well as synthesizing the three dimensional cloud.

A sample structured product is valued for its remaining cash flows on the quantized points at quarter end. High dimensional interpolation methods are used to obtain the value function at arbitrary points of the state space. This value function is then hedged using options on rates, bonds and the stock with a view to minimizing capital as a hedging objective as described in Carr et al. (2011). The hedge with 13 hedging assets more than halves the required capital for the structured position.

Appendix

Proof of Proposition 1.

We now develop the joint characteristic function of

$$(r(t), y(t), s(t))$$

under the T-forward measure for $t < T$ where

$$\begin{aligned}
\phi_{VGGDSR}(t,u,v,w) &= E^T\left[\exp\left(iur(t) + iv\int_0^t r(s)ds + iw\ln(S(t))\right)\right] \\
&= E\left[\Lambda_t^T \exp\left(iur(t) + iv\int_0^t r(s)ds + iw\ln(S(t))\right)\right] \\
\Lambda_t^T &= \frac{1}{P(0,T)}E_t\left[\exp\left(-\int_0^T r(s)ds\right)\right],
\end{aligned}$$

where $P(0,T)$ is as defined in equation (5) and E_t denotes conditional expectation with respect to information at time t. Under the original risk neutral specification we have that

$$r(t) = r(0)e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}dg(u)$$

with

$$\begin{aligned}
y(t) &= \int_0^t r(u)du \\
&= r(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \frac{1-e^{-\kappa(t-u)}}{\kappa}dg(u)
\end{aligned}$$

and

$$\begin{aligned}
s(t) &= s(0) + \int_0^t r(u)du - qt + X(t) - \beta g(t) - t\ln\phi_{VG}(-i) \\
&\quad -\gamma t\left(\ln(c) - \ln(c+\beta)\right).
\end{aligned}$$

The martingale Λ_t^T is the compensated jump exponential martingale

$$\Lambda_t^T = \mathcal{E}\left[\left(\exp\left(-e^{-\frac{1-e^{-\kappa(T-s)}}{\kappa}x}\right) - 1\right) * \left(\mu_g(dx,ds) - k_g(x)dxds\right)\right]_t,$$

where μ_g is the random measure of jumps associated with the gamma process g and k_g denotes the Lévy density of g.

Since

$$N_t = \mathcal{E}\left[\left(e^{iux} - 1\right) * \left(\mu_g(dx,ds) - k_g(x)dxds\right)\right]_t$$

is also a compensated jump exponential martingale and as

$$E\left[e^{iug(t)}\right] = \exp\left(t\gamma\left(\ln(c) - \ln(c - iu)\right)\right)$$

it follows that

$$\int_0^t \int_0^\infty \left(e^{iux} - 1\right) k_g(x) dx ds = t\gamma\left(\ln(c) - \ln(c - iu)\right).$$

We are interested in the expectation of the exponential of

$$\begin{array}{l} iu\left(r(0)e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)} dg(u)\right) \\ +iv\left(r(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \frac{1-e^{-\kappa(t-u)}}{\kappa} dg(u)\right) \\ +iw\left(\begin{array}{c} \ln S(0) + \int_0^t r(u)du - qt + X(t) - \beta g(t) - t\ln\phi_X(-i) \\ -\gamma t\left(\ln(c) - \ln(c+\beta)\right) \end{array}\right) \end{array}$$

under the T-forward measure.
This is made of a number of terms that we now identify. The first term is

$$\begin{array}{rcl} F_1(t,u,v,w) & = & \exp\left(\begin{array}{c} iur(0)e^{-\kappa t} \\ +ivr(0)\frac{1-e^{-\kappa t}}{\kappa} \\ +iwr(0)\frac{1-e^{-\kappa t}}{\kappa} \\ +iw\left(\begin{array}{c} \ln S(0) - qt - t\ln\phi_X(-i) \\ -\gamma t\left(\ln(c) - \ln(c+\beta)\right) \end{array}\right) \end{array}\right) \\ & = & \exp\left(\begin{array}{c} \left(iue^{-\kappa t} + (iv+iw)\frac{1-e^{-\kappa t}}{\kappa}\right) r(0) \\ +iw\left(\begin{array}{c} \ln S(0) - qt - t\ln\phi_X(-i) \\ -\gamma t\left(\ln(c) - \ln(c+\beta)\right) \end{array}\right) \end{array}\right) \end{array}$$

The second term is

$$F_2(t,w) = \exp\left(t\psi_X(w)\right)$$

where ψ_X is the characteristic exponent of the variance gamma process defined in equation (3)

The third term is the expectation of the exponential of

$$\begin{array}{rl} & iu\int_0^t e^{-\kappa(t-u)} dg(u) \\ & +iv\int_0^t \frac{1-e^{-\kappa(t-u)}}{\kappa} dg(u) \\ & +iw\int_0^t \left(\frac{1-e^{-\kappa(t-u)}}{\kappa} - \beta\right) dg(u) \\ = & \int_0^t \left(iue^{-\kappa(t-u)} + (iv+iw)\frac{1-e^{-\kappa(t-u)}}{\kappa} - iw\beta\right) dg(u) \end{array}$$

The logarithm of this expectation is given by

$$
\begin{aligned}
& \int_0^t \int_0^\infty \left[\exp\left(\left(iue^{-\kappa(t-s)} + (iv+iw)\frac{1-e^{-\kappa(t-s)}}{\kappa} - iw\beta\right)x\right) - 1\right] e^{-\frac{1-e^{-\kappa(T-s)}}{\kappa}x} k_g(x)dxds \\
= & \int_0^t \int_0^\infty \left[\exp\left(\left(iue^{-\kappa(t-s)} + (iv+iw)\frac{1-e^{-\kappa(t-s)}}{\kappa} - iw\beta - \frac{1-e^{-\kappa(T-s)}}{\kappa}\right)x\right) - 1\right] k_g(x)dxds \\
& -\int_0^t \int_0^\infty \left[\exp\left(\left(-\frac{1-e^{-\kappa(T-s)}}{\kappa}\right)x\right) - 1\right] k_g(x)dxds \\
= & \int_0^t \psi_U\left(ue^{-\kappa(t-s)} + (v+w)\frac{1-e^{-\kappa(t-s)}}{\kappa} - w\beta + i\frac{1-e^{-\kappa(T-s)}}{\kappa}\right)ds \\
& -\int_0^t \psi_U\left(i\frac{1-e^{-\kappa(T-s)}}{\kappa}\right)ds \\
= & \int_0^t \psi_U\left(ue^{-\kappa(t-s)} + (v+w)\frac{1-e^{-\kappa(t-s)}}{\kappa} - w\beta + i\frac{1-e^{-\kappa(T-t)}e^{-\kappa(t-s)}}{\kappa}\right)ds \\
& -\int_0^t \psi_U\left(i\frac{1-e^{-\kappa(T-t)}e^{-\kappa(t-s)}}{\kappa}\right)ds.
\end{aligned}
$$

We now make the substitution

$$
\begin{aligned}
y &= e^{-\kappa(t-s)} \\
dy &= \kappa y ds
\end{aligned}
$$

to get

$$
\begin{aligned}
& \frac{1}{\kappa}\int_{e^{-\kappa t}}^1 \frac{\psi_U\left(uy + (v+w)\frac{1-y}{\kappa} - w\beta + i\frac{1-e^{-\kappa(T-t)}y}{\kappa}\right)}{y}dy \\
& -\frac{1}{\kappa}\int_{e^{-\kappa t}}^1 \frac{\psi_U\left(i\frac{1-e^{-\kappa(T-t)}y}{\kappa}\right)}{y}dy
\end{aligned}
$$

We may rewrite as

$$
\begin{aligned}
& \frac{\gamma}{\kappa}\int_{e^{-\kappa t}}^1 \frac{\ln(c) - \ln(c+a+by)}{y}dy \\
& -\frac{\gamma}{\kappa}\int_{e^{-\kappa t}}^1 \frac{\ln(c) - \ln(c+a'+b'y)}{y}dy
\end{aligned}
$$

where

$$
\begin{aligned}
a+by &= -i\left(uy+(v+w)\frac{1-y}{\kappa}-w\beta+i\frac{1-e^{-\kappa(T-t)}y}{\kappa}\right)\\
&= \left(iw\beta-i\frac{v+w}{\kappa}+\frac{1}{\kappa}\right)+\left(\frac{i(v+w)}{\kappa}-iu-\frac{e^{-\kappa(T-t)}}{\kappa}\right)y\\
a'+b'y &= -i\left(i\frac{1-e^{-\kappa(T-t)}y}{\kappa}\right)\\
&= \frac{1}{\kappa}-\frac{e^{-\kappa(T-t)}}{\kappa}y
\end{aligned}
$$

So

$$
\begin{aligned}
a &= iw\beta-i\frac{v+w}{\kappa}+\frac{1}{\kappa}\\
b &= \frac{i(v+w)}{\kappa}-iu-\frac{e^{-\kappa(T-t)}}{\kappa}\\
a' &= \frac{1}{\kappa}\\
b' &= -\frac{e^{-\kappa(T-t)}}{\kappa}
\end{aligned}
$$

and

$$F_3(t,u,v,w)=\exp\left(\left(G(1)-G(e^{-\kappa t})\right)-\left(H(1)-H(e^{-\kappa t})\right)\right)$$

where

$$
\begin{aligned}
G(x) &= \frac{\gamma}{\kappa}\left(\begin{array}{c} Li_2\left(-\frac{bx}{a+c}\right)-\ln(x)\times \\ \left((a+bx+c)-\ln\left(\frac{bx}{b+c}+1\right)\right)+ \\ \ln(c)\ln(x)\end{array}\right)\\
H(x) &= \frac{\gamma}{\kappa}\left(\begin{array}{c} Li_2\left(-\frac{b'x}{a'+c}\right)-\ln(x)\times \\ \left((a'+b'x+c)-\ln\left(\frac{b'x}{b'+c}+1\right)\right)+ \\ \ln(c)\ln(x)\end{array}\right)
\end{aligned}
$$

Proof of Proposition 2, (The Markov Krein identity).
We are interested in evaluating

$$E\left[\frac{1}{\left(s+\int_0^\eta \psi(y)\frac{d\gamma(y)}{\gamma(\eta)}\right)^\eta}\right]$$

We proceed as follows

$$
\begin{aligned}
& E\left[\frac{1}{\left(s+\int_0^\eta \psi(y)\frac{d\gamma(y)}{\gamma(\eta)}\right)^\eta}\right] \\
= \; & E\left[\frac{1}{s^\eta\left(1+\frac{1}{s}\int_0^\eta \psi(y)\frac{d\gamma(y)}{\gamma(\eta)}\right)^\eta}\right] \\
= \; & \frac{1}{s^\eta}E\left[\exp\left(-\frac{1}{s}\int_0^\eta \psi(y)\frac{d\gamma(y)}{\gamma(\eta)}\gamma'(\eta)\right)\right]
\end{aligned}
$$

where $\gamma'(\eta)$ is a variable distributed according to $\gamma(\eta)$ which is independent of $\left(\frac{\gamma(y)}{\gamma(\eta)}, y \leq \eta\right)$.

We next observe that

$$
\begin{aligned}
& \frac{1}{s^\eta}E\left[\exp\left(-\frac{1}{s}\int_0^\eta \psi(y)\frac{d\gamma(y)}{\gamma(\eta)}\gamma'(\eta)\right)\right] \\
= \; & \frac{1}{s^\eta}E\left[\exp\left(-\frac{1}{s}\int_0^\eta \psi(y)d\gamma(y)\right)\right] \\
= \; & \frac{1}{s^\eta}\exp\left(-\int_0^\eta dy \;\, \log\left(1+\frac{\psi(y)}{s}\right)\right) \\
= \; & \exp\left(-\int_0^\eta dy \;\, \log(s+\psi(y))\right)
\end{aligned}
$$

where the third expectation follows from the second on employing the Laplace transform of a gamma variate.

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