ANALYTICITY OF THE WIENER–HOPF FACTORS AND VALUATION OF EXOTIC OPTIONS IN LÉVY MODELS

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ABSTRACT. This paper considers the valuation of exotic path-dependent options in Lévy models, in particular options on the supremum and the infimum of the asset price process. Using the Wiener–Hopf factorization, we derive expressions for the analytically extended characteristic function of the supremum and the infimum of a Lévy process. Combined with general results on Fourier methods for option pricing, we provide formulas for the valuation of one-touch options, lookback options and equity default swaps in Lévy models.

1. INTRODUCTION

The ever-increasing sophistication of derivative products offered by financial institutions, together with the failure of traditional Gaussian models to describe the dynamics in the markets, has lead to a quest for more realistic and flexible models. In fact one of the lessons from the current financial crisis is the following: the Gaussian copula model is inappropriate to describe the interdependence between the tails of asset returns because, among other pitfalls, the tail dependence coefficient is always zero; hence, this model cannot capture systemic risk.

In the search for appropriate alternatives, *Lévy processes* are playing a leading role, either as models for financial assets themselves, or as building blocks for models, e.g. in Lévy-driven stochastic volatility models or in affine models. The field of Lévy processes has become popular in modern mathematical finance, and the interest from academics and practitioners has led to inspiring and challenging questions.

Lévy processes are attractive for applications in mathematical finance because they can describe some of the observed phenomena in the markets in a rather adequate way. This is due to the fact that their sample paths may have jumps and the generated distributions can be heavy-tailed and skewed. Another important improvement concerns the famous smile effect. See Eberlein and Keller (1995) and Eberlein and Prause (2002) for an extensive empirical justification of the non-Gaussianity of asset returns and the appropriateness of (generalized hyperbolic) Lévy processes. For an overview of the application of Lévy processes in finance the interested reader is referred to the textbooks of Cont and Tankov (2004), Schoutens (2003) as well as the collection

edited by Kyprianou et al. (2005). There are, of course, several textbooks dealing with the theory of Lévy processes; we mention Bertoin (1996), Sato (1999), Applebaum (2004) and Kyprianou (2006), while the collection by Barndorff-Nielsen et al. (2001) contains an overview of the application of Lévy processes in different areas of research, such as quantum field theory and turbulence.

The application of Lévy processes in financial modeling, in particular for the pricing and hedging of derivatives, has led to new challenges of both *analytical* and *numerical* nature. In Lévy models simple closed form valuation formulas are typically not available even for plain vanilla European options, let alone for exotic path-dependent options. The numerical methods which have been developed in the classical Gaussian framework lead to completely new challenges in the context of Lévy driven models. These numerical methods can be classified roughly in three areas: probabilistic numerical methods (Monte Carlo methods), deterministic numerical methods (PIDE methods), and Fourier transform methods; for an excellent survey of these methods, their applicability and limitations, we refer to Hilber et al. (2009).

This paper focuses on the application of Fourier transform methods for the valuation of exotic path-dependent options, in particular options depending on the supremum and the infimum of Lévy processes. The bulk of the literature on this latter topic focuses on the numerical aspects. Our focus is on the analytical aspects. More specifically, we show first that the Wiener–Hopf factorization of a Lévy process possesses an analytic extension, and then we prove that the Wiener–Hopf factorization (viewed as a Laplace transform in time) can be inverted. These results allow us to derive expressions for the extended characteristic function of the supremum and the infimum of a Lévy process. This latter result, combined with general results on option pricing by Fourier methods (cf. Eberlein et al. 2010), allows us to derive pricing formulas for lookback options, one-touch options and equity default swaps in Lévy models.

Let us briefly comment on some papers where the Wiener–Hopf factorization is used to price exotic options in Lévy models. Boyarchenko and Levendorskiĭ (2002a) derive valuation formulas for barrier and one-touch options for driving Lévy processes that belong to the class of so-called "regular Lévy processes of exponential type" (RLPE); cf. also the book by Boyarchenko and Levendorskiĭ (2002b). The results of these authors are based on the theory of pseudodifferential operators. The numerics of this approach is pushed further in Kudryavtsev and Levendorskiĭ (2006, 2009). Avram et al. (2004), Asmussen et al. (2004), Kyprianou and Pistorius (2003), Alili and Kyprianou (2005), and Levendorskiĭ et al. (2005) consider the valuation of American and Russian options, either on a finite or an infinite time horizon. Jeannin and Pistorius (2009) develop methods for the computation of prices and Greeks for various Lévy models. Central in their argumentation is the approximation of different Lévy models by the class of "generalized hyper-exponential Lévy models", which have a tractable Wiener–Hopf factorization. The same approach is also applied in Asmussen et al. (2007) for the pricing of equity default swaps in Lévy models.

The major open challenge in this field is the development of analytical expressions for the Wiener–Hopf factors for general Lévy processes. In a remarkable recent development, Hubalek and Kyprianou (2010) generate a family of spectrally negative Lévy processes with tractable Wiener–Hopf factors, using results from potential theory for subordinators. These results were later extended in Kyprianou and Rivero (2008) and applied to problems in actuarial mathematics in Kyprianou et al. (2009). Moreover, in two very recent papers Kuznetsov (2010, 2009) introduces special families of Lévy processes such that the Wiener–Hopf factors can be computed as infinite products over the roots of certain transcendental equations. These families include processes with behavior similar to the CGMY process, while the author shows that the numerical computation of the infinite products can be performed quite efficiently.

This paper is structured as follows: in section 2, we briefly review Lévy processes and prove the analyticity of the characteristic function of the supremum. In section 3, we review the Wiener–Hopf factorization, prove its analytic extension and invert it in time. In section 4, we present some examples of popular Lévy models and comment on the continuity of their laws. Finally, in section 5, we derive valuation formulas for lookback and one-touch options as well as for equity default swaps.

Important Remark. This paper is intimately tied to, and intended to be read together with, the companion paper Eberlein, Glau, and Papapantoleon (2010), which will be abbreviated EGP in the sequel. In particular, we will make heavy use of the notation and results from that paper.

2. Lévy processes

We start by fixing the notation that will be used throughout the paper and providing some estimates on the exponential moments of a Lévy process. Then, we prove the analytic extension of the characteristic function of the supremum and the infimum of a Lévy process, sampled either at a fixed time or at an independent, exponentially distributed time.

2.1. **Notation.** Let $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P)$ be a complete stochastic basis in the sense of Jacod and Shiryaev (2003, I.1.3), where $\mathcal{F} = \mathcal{F}_T$, $0 < T \leq \infty$ and $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. Let $L = (L_t)_{0 \leq t \leq T}$ be a *Lévy process* on this stochastic basis, i.e. L is a semimartingale with *independent* and *stationary increments* (PIIS), and $L_0 = 0$ a.s. We denote the *triplet of predictable characteristics* of L by (B, C, ν) and the *triplet of local characteristics* by (b, c, λ); using Jacod and Shiryaev (2003, II.4.20) the two triplets are related via

$$B_t(\omega) = bt, \quad C_t(\omega) = ct, \quad \nu(\omega; \mathrm{d}t, \mathrm{d}x) = \lambda(\mathrm{d}x)\,\mathrm{d}t.$$

We assume that the following condition is in force.

Assumption ($\mathbb{EM}$). There exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} \mathrm{e}^{ux}\lambda(\mathrm{d}x) < \infty, \qquad \forall u \in [-M, M].$$

The triplet of predictable characteristics of a PIIS determines the law of the random variables; more specifically, for a Lévy process we know from the Lévy–Khintchine formula that

$$E\left[e^{iuL_t}\right] = \exp\left(t \cdot \kappa(iu)\right), \tag{2.1}$$

for all $t \in [0, T]$ and all $u \in \mathbb{R}$, where the cumulant generating function is

$$\kappa(u) = ub + \frac{u^2}{2}c + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\lambda(\mathrm{d}x). \tag{2.2}$$

Assumption $(\mathbb{EM})$ entails that the Lévy process L is a *special* and *exponentially special* semimartingale, hence the use of a truncation function can be and has been omitted. Applying Theorem 25.3 in Sato (1999) we get that

$$E\left[e^{uL_t}\right] < \infty, \qquad \forall u \in [-M, M], \ \ \forall t \in [0, T].$$

Recall that for any stochastic process X we denote by $\overline{X}$ the supremum and by $\underline{X}$ the infimum process of X respectively.

In the sequel, we will provide the proofs of the results for the supremum process. The proofs for the infimum process can be derived analogously or using the duality between the supremum and the infimum process; see the following remark.

Remark 2.1. Let L be a Lévy process with local characteristics (b, c, λ). The *dual* of the Lévy process L defined by $L' := -L$, has the triplet of local characteristics (b', c', λ') where $b' = -b$, $c' = c$ and $1_A(x) * \lambda' = 1_A(-x) * \lambda$, $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$. Moreover, we have that

$$\underline{L}_t = \inf_{0 \le s \le t} L_s = -\sup_{0 \le s \le t}(-L_s) = -\overline{L'}_t.$$

2.2. **Analytic extension, fixed time case.** In this section, we establish the existence of an analytic extension of the characteristic function of the *supremum* and the *infimum* of a Lévy process, and derive explicit bounds for the exponential moments of the supremum and infimum process.

The next lemma endows us with a link between the existence of exponential moments of a measure ϱ and the analytic extension of the characteristic function $\widehat{\varrho}$.

Lemma 2.2. *Let ϱ be a measure on the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $\int e^{ux}\varrho(\mathrm{d}x) < \infty$ for all $u \in [-a, b]$ with $a, b \ge 0$, then the characteristic function $\widehat{\varrho}$ has an extension that is continuous on $(-\infty, \infty) \times i[-b, a]$ and is analytic in the interior of the strip, $(-\infty, \infty) \times i(-b, a)$. Moreover $\widehat{\varrho}(u) = \int e^{iux}\varrho(\mathrm{d}x)$ for all $u \in \mathbb{C}$ with $\Im(u) \in [-b, a]$.*

Proof. The function $u \mapsto e^{iux}$ clearly extends to an entire function and the extension

$$\widehat{\varrho}(u) := \int e^{iux}\varrho(\mathrm{d}x) \quad \left(u \in \mathbb{C} \text{ with } \Im(u) \in [-b, a]\right)$$

is well-defined since

$$\left|e^{iux}\right| = e^{-\Im(u)x} \le e^{-ax}1_{\{x \le 0\}} + e^{bx}1_{\{x > 0\}} =: h(x),$$

for $u \in \mathbb{C}$ with $\Im(u) \in [-b, a]$, and we have that $h \in L^1(\varrho)$ by assumption. Moreover, Lebesgue's dominated convergence theorem yields that this extension is continuous.

We will prove the analyticity of $\widehat{\varrho}$ in $(-\infty, \infty) \times i(-b, a)$ using the theorem of Morera (cf. for example Theorem 10.17 in Rudin 1987). Let γ be a triangle in the open set $(-\infty, \infty) \times i(-b, a)$; the theorems of Fubini and Cauchy immediately yield

$$\int_{\partial\gamma} \widehat{\varrho}(u)\mathrm{d}u = \int_{\partial\gamma} \int e^{iux}\varrho(\mathrm{d}x)\mathrm{d}u = \int \int_{\partial\gamma} e^{iux}\mathrm{d}u\, \varrho(\mathrm{d}x) = 0,$$

as $u \mapsto e^{iux}$ is analytic for every fixed $x \in \mathbb{R}$. Then, the analyticity of $\widehat{\varrho}$ follows from Morera's theorem. For a justification of the application of Fubini's theorem it is enough to note that

$$\int \int_{\partial\gamma} \left|e^{iux}\right| \mathrm{d}u\, \varrho(\mathrm{d}x) \leq \int \int_{\partial\gamma} h(x)\mathrm{d}u\, \varrho(dx) = \ell(\gamma) \int h(x)\varrho(\mathrm{d}x) < \infty,$$

where $\ell(\gamma)$ denotes the length of the curve $\partial\gamma$. $\square$

Lemma 2.3. *Let Y be a Lévy process and a special semimartingale with $E[Y_t] = 0$ for some, and hence for every, $t > 0$. Then*

$$E\left[e^{Y_t^*}\right] \leq 8E\left[e^{|Y_t|}\right],$$

where $Y_t^* = \sup_{0 \leq s \leq t} |Y_s|$.

Proof. Using that $\frac{(Y_t^*)^n}{n!}$ is positive for every $n \geq 0$ and the monotone convergence theorem, we get

$$E\left[e^{Y_t^*}\right] = E\sum_{n=0}^{\infty} \frac{(Y_t^*)^n}{n!} = \sum_{n=0}^{\infty} E\frac{(Y_t^*)^n}{n!}.$$

Now, Remark 25.19 in Sato (1999) yields

$$E(Y_t^*)^n \leq 8E|Y_t|^n, \qquad \text{for every } n \geq 1,$$

while for $n = 0$ the inequality holds trivially. Hence, we get

$$\sum_{n=0}^{\infty} E\frac{(Y_t^*)^n}{n!} \leq 8\sum_{n=0}^{\infty} E\frac{|Y_t|^n}{n!} = 8E\sum_{n=0}^{\infty} \frac{|Y_t|^n}{n!} = 8E\left[e^{|Y_t|}\right]. \qquad \square$$

Next, notice that under assumption $(\mathbb{EM})$ we have that

$$\int_{\mathbb{R}} \left|e^{Mx} - 1 - Mx\right| \lambda(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left|e^{-Mx} - 1 + Mx\right| \lambda(\mathrm{d}x) < \infty.$$

Let us introduce the following notation:

$$\overline{\alpha}(M) := M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} \left|e^{Mx} - 1 - Mx\right| \lambda(\mathrm{d}x) \qquad (2.3)$$

and

$$\underline{\alpha}(M) := M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} \left|e^{-Mx} - 1 + Mx\right| \lambda(\mathrm{d}x). \qquad (2.4)$$

Lemma 2.4. *Let $L = (L_t)_{0 \le t \le T}$ be a Lévy process that satisfies assumption* $(\mathbb{EM})$. *Then we have the following estimates*

$$E\big[\mathrm{e}^{u\overline{L}_t}\big] \le E\big[\mathrm{e}^{M\overline{L}_t}\big] \le 8\mathcal{C}(t, M) < \infty \qquad (u \le M),$$

and

$$E\big[\mathrm{e}^{-u\underline{L}_t}\big] \le E\big[\mathrm{e}^{-M\underline{L}_t}\big] \le 8\mathcal{C}(t, M) < \infty \qquad (u \le M),$$

where $\mathcal{C}(t, M) := \mathrm{e}^{t\overline{\alpha}(M)} + \mathrm{e}^{t\underline{\alpha}(M)}$.

Proof. For $u \le M$ we have

$$\mathrm{e}^{u\overline{L}_t} \le \mathrm{e}^{M\overline{L}_t},$$

since $\overline{L}_t = \sup_{0 \le s \le t} L_s$ is nonnegative. Further notice that

$$\overline{L}_t = \sup_{0 \le s \le t} \big[bs + \sqrt{c}W_s + L_s^d\big] \le \sup_{0 \le s \le t} \big[\sqrt{c}W_s + L_s^d\big] + \sup_{0 \le s \le t} [bs],$$

where $L_t = bt + \sqrt{c}W_t + L_t^d$ denotes the canonical decomposition of L, with Brownian motion W and a purely discontinuous martingale $L^d = x * (\mu - \nu)$. Let us further denote by

$$Y_s := \sqrt{c}W_s + L_s^d.$$

The process Y is not only a martingale but also a Lévy process and a special semimartingale with local characteristics $(0, c, \lambda)$. We have

$$\overline{L}_t \le \sup_{0 \le s \le t} Y_s + |b|t \le Y_t^* + |b|t,$$

hence we get that

$$E\big[\mathrm{e}^{M\overline{L}_t}\big] \le E\big[\mathrm{e}^{M(Y_t^* + |b|t)}\big] = \mathrm{e}^{M|b|t} E\big[\mathrm{e}^{MY_t^*}\big] \le 8\mathrm{e}^{M|b|t} E\big[\mathrm{e}^{M|Y_t|}\big], \qquad (2.5)$$

using Lemma 2.3 for the special semimartingale $Z := MY$, which is a Lévy process satisfying $E[Z_t] = 0$ for every $0 \le t \le T$.

Now it is sufficient to notice that

$$E\big[\mathrm{e}^{M|Y_t|}\big] \le E\big[\mathrm{e}^{MY_t}\big] + E\big[\mathrm{e}^{-MY_t}\big], \qquad (2.6)$$

where Theorem 25.17 in Sato (1999) yields

$$E\big[\mathrm{e}^{MY_t}\big] = \exp\Big(t\frac{cM^2}{2} + t\int_{\mathbb{R}} \big(\mathrm{e}^{Mx} - 1 - Mx\big)\,\lambda(\mathrm{d}x)\Big)$$

$$\le \mathrm{e}^{(\overline{\alpha}(M) - M|b|)t}; \qquad (2.7)$$

similarly,

$$E\big[\mathrm{e}^{-MY_t}\big] \le \mathrm{e}^{(\underline{\alpha}(M) - M|b|)t}. \qquad (2.8)$$

Summarizing, we can conclude from (2.5)–(2.8) that

$$E\big[\mathrm{e}^{M\overline{L}_t}\big] \le 8\mathrm{e}^{M|b|t}\Big(\mathrm{e}^{(\overline{\alpha}(M) - M|b|)t} + \mathrm{e}^{(\underline{\alpha}(M) - M|b|)t}\Big)$$

$$= 8\Big(\mathrm{e}^{\overline{\alpha}(M)t} + \mathrm{e}^{\underline{\alpha}(M)t}\Big),$$

as well as

$$E\big[\mathrm{e}^{-M\underline{L}_t}\big] \le 8\Big(\mathrm{e}^{\overline{\alpha}(M)t} + \mathrm{e}^{\underline{\alpha}(M)t}\Big). \qquad \square$$

A corollary of these results is the existence of an analytic continuation for the characteristic function $\varphi_{\overline{L}_t}$ of the supremum, resp. $\varphi_{\underline{L}_t}$ of the infimum, of a Lévy process.

Corollary 2.5. *Let L be a Lévy process that satisfies assumption* $(\mathbb{EM})$. *Then, the characteristic function $\varphi_{\overline{L}_t}$ of $\overline{L}_t$, resp. $\varphi_{\underline{L}_t}$ of $\underline{L}_t$, possesses a continuous extension*

$$\varphi_{\overline{L}_t}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\overline{L}_t}(\mathrm{d}x), \quad resp. \quad \varphi_{\underline{L}_t}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\underline{L}_t}(\mathrm{d}x),$$

to the half-plane $z \in \{z \in \mathbb{C} : -M \le \Im z\}$, resp. $z \in \{z \in \mathbb{C} : \Im z \le M\}$, that is analytic in the interior of the half-plane $\{z \in \mathbb{C} : -M < \Im z\}$, resp. $\{z \in \mathbb{C} : \Im z < M\}$.

Proof. This is a direct consequence of Lemmata 2.2 and 2.4. $\square$

Remark 2.6. One could derive the statement of Corollary 2.5 using the submultiplicativity of the exponential function and Theorem 25.18 in Sato (1999), see Lemma 5 in Kyprianou and Surya (2005). However, we will need the estimates of Lemma 2.4 in the following sections.

2.3. **Analytic extension, exponential time case.** The next step is to establish a relationship between the (analytic extension of the) characteristic function of the supremum, resp. infimum, at a *fixed* time and at an *independent* and *exponentially distributed* time. Independent exponential times play a fundamental role in the fluctuation theory of Lévy processes, since they enjoy a property similar to infinity: the time left after an exponential time is again exponentially distributed.

Let θ denote an exponentially distributed random variable with parameter $q > 0$, independent of the Lévy process L. We denote by $\overline{L}_\theta$, resp. $\underline{L}_\theta$, the supremum, resp. infimum, process of L sampled at θ, that is

$$\overline{L}_\theta = \sup_{0 \le u \le \theta} L_u \quad \text{and} \quad \underline{L}_\theta = \inf_{0 \le u \le \theta} L_u.$$

Lemma 2.7. *Let $L = (L_t)_{0 \le t \le T}$ be a Lévy process that satisfies assumption* $(\mathbb{EM})$, *and let $\theta \sim \mathrm{Exp}(q)$ be independent of the process L.*

If $q > \overline{\alpha}(M) \vee \underline{\alpha}(M)$, then the characteristic function $\varphi_{\overline{L}_\theta}$ of $\overline{L}_\theta$ possesses a continuous extension

$$\varphi_{\overline{L}_\theta}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\overline{L}_\theta}(\mathrm{d}x) = q \int_0^\infty \mathrm{e}^{-qt} E\big[\mathrm{e}^{iz\overline{L}_t}\big] \mathrm{d}t \qquad (2.9)$$

to the half-plane $z \in \{z \in \mathbb{C} : -M \le \Im z\}$, that is analytic in the interior of the half-plane $\{z \in \mathbb{C} : -M < \Im z\}$.

If $q > \overline{\alpha}(M) \vee \underline{\alpha}(M)$, then the characteristic function $\varphi_{\underline{L}_\theta}$ of $\underline{L}_\theta$ possesses a continuous extension

$$\varphi_{\underline{L}_\theta}(z) = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\underline{L}_\theta}(\mathrm{d}x) = q \int_0^\infty \mathrm{e}^{-qt} E\big[\mathrm{e}^{iz\underline{L}_t}\big] \mathrm{d}t \qquad (2.10)$$

to the half-plane $z \in \{z \in \mathbb{C} : \Im z \le M\}$, that is analytic in the interior of the half-plane $\{z \in \mathbb{C} : \Im z < M\}$.

Proof. We have that

$$E\big[\mathrm{e}^{u\overline{L}_\theta}\big] = \int\limits_0^\infty \int\limits_0^\infty \mathrm{e}^{ux} q\mathrm{e}^{-qt} P_{\overline{L}_t}(\mathrm{d}x)\mathrm{d}t = \int\limits_0^\infty E\big[\mathrm{e}^{u\overline{L}_t}\big] q\mathrm{e}^{-qt}\mathrm{d}t,$$

and, for $q > \overline{\alpha}(M) \vee \underline{\alpha}(M)$, by Lemma 2.4 we get

$$\int\limits_0^\infty E\big[\mathrm{e}^{M\overline{L}_t}\big] q\mathrm{e}^{-qt}\mathrm{d}t \le 8\left(q\int\limits_0^\infty \mathrm{e}^{-t\big[q-\overline{\alpha}(M)\big]}\mathrm{d}t + q\int\limits_0^\infty \mathrm{e}^{-t\big[q-\underline{\alpha}(M)\big]}\mathrm{d}t \right) < \infty;$$

hence, for $u \le M$, we have

$$E\big[\mathrm{e}^{u\overline{L}_\theta}\big] \le E\big[\mathrm{e}^{M\overline{L}_\theta}\big] < \infty \qquad (q > \overline{\alpha}(M) \vee \underline{\alpha}(M)). \qquad (2.11)$$

Inequality (2.11), together with Lemma 2.2, implies that the characteristic function $\varphi_{\overline{L}_\theta}$ has a continuous extension to the half-plane $\{z \in \mathbb{C} : -M \le \Im z\}$, that is analytic in $\{z \in \mathbb{C} : -M < \Im z\}$, and is given by

$$\varphi_{\overline{L}_\theta}(z) = E\big[\mathrm{e}^{iz\overline{L}_\theta}\big],$$

for every $z \in \mathbb{C}$ with $\Im z \ge -M$. Furthermore Fubini's theorem yields

$$E\big[\mathrm{e}^{iz\overline{L}_\theta}\big] = \int\limits_0^\infty \int\limits_0^\infty \mathrm{e}^{izx} q\mathrm{e}^{-qt} P_{\overline{L}_t}(\mathrm{d}x)\mathrm{d}t = q\int\limits_0^\infty \mathrm{e}^{-qt} E\big[\mathrm{e}^{iz\overline{L}_t}\big]\mathrm{d}t.$$

The application of Fubini's theorem is justified since, for $\Im z \ge -M$ and $q > \overline{\alpha}(M) \vee \underline{\alpha}(M)$, we have

$$E\big[\big|\mathrm{e}^{iz\overline{L}_\theta}\big|\big] = E\big[\mathrm{e}^{-\Im(z)\overline{L}_\theta}\big] \le E\big[\mathrm{e}^{M\overline{L}_\theta}\big] < \infty$$

by inequality (2.11). Similarly, we prove the assertion for the infimum. $\square$

3. The Wiener–Hopf factorization

We first provide a statement and brief description of the Wiener–Hopf factorization of a Lévy process, and then show that the Wiener–Hopf factorization holds true for the analytically extended characteristic functions. Next, we invert the Wiener–Hopf factorization, and derive an expression for the (analytically extended) characteristic function of the supremum, resp. infimum, of a Lévy process in terms of the Wiener–Hopf factors.

3.1. **Analyticity.** Fluctuation identities for Lévy processes originate from analogous results for random walks, first derived using combinatorial methods, see e.g. Spitzer (1964) or Feller (1971). Bingham (1975) used this discrete-time skeleton to prove results for Lévy processes; the same approach is followed in the book of Sato (1999). Greenwood and Pitman (1980a,1980b) proved these results for random walks and Lévy processes using excursion theory; see also the books of Bertoin (1996) and Kyprianou (2006).

The *Wiener–Hopf factorization*[1] serves as a common reference to a multitude of statements in the fluctuation theory for Lévy processes, regarding the distributional decomposition of the excursions of a Lévy process sampled at an independent and exponentially distributed time. The following

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[1]The historical reasons leading to the adoption of the terminology "Wiener–Hopf" are outlined in section 6.6 in Kyprianou (2006).

statement relates the characteristic function of the supremum, the infimum, and the Lévy process itself. Let L be a Lévy process and θ an independent, exponentially distributed time with parameter q; then we have that

$$E\big[\mathrm{e}^{izL_\theta}\big] = E\big[\mathrm{e}^{iz\overline{L}_\theta}\big]E\big[\mathrm{e}^{iz\underline{L}_\theta}\big]$$

or equivalently,

$$\frac{q}{q - \kappa(iz)} = \varphi_q^+(z)\varphi_q^-(z), \qquad z \in \mathbb{R};$$

here κ denotes the cumulant generating function of L_1, cf. (2.2), and φ_q^+, φ_q^- denote the so-called Wiener–Hopf factors.

In the sequel, we will make use of the Wiener–Hopf factorization as stated in the beautiful book of Kyprianou (2006), and prove the analytic extension of the Wiener–Hopf factors to the open half-plane $\{z \in \mathbb{C} : \Im z > -M\}$.

Recall the definitions of (2.3) and (2.4), and let us denote by

$$\alpha^*(M) := \max\Big\{\overline{\alpha}(M), \underline{\alpha}(M)\Big\}.$$

Theorem 3.1 (Wiener–Hopf factorization)**.** *Let L be a Lévy process that satisfies assumption* ($\mathbb{EM}$) *(and is not a compound Poisson process). The Laplace transform of $\overline{L}_\theta$, resp. $\underline{L}_\theta$, at an independent and exponentially distributed time θ, $\theta \sim \mathrm{Exp}(q)$, with $q > \alpha^*(M)$, can be identified from the Wiener–Hopf factorization of L via*

$$E\big[\mathrm{e}^{-\beta\overline{L}_\theta}\big] = \int_0^\infty qE\big[\mathrm{e}^{-\beta\overline{L}_t}\big]\mathrm{e}^{-qt}\mathrm{d}t = \frac{\overline{\kappa}(q,0)}{\overline{\kappa}(q,\beta)} \tag{3.1}$$

and

$$E\big[\mathrm{e}^{\beta\underline{L}_\theta}\big] = \int_0^\infty qE\big[\mathrm{e}^{\beta\underline{L}_t}\big]\mathrm{e}^{-qt}\mathrm{d}t = \frac{\underline{\kappa}(q,0)}{\underline{\kappa}(q,\beta)} \tag{3.2}$$

for $\beta \in \{\beta \in \mathbb{C} : \Re(\beta) > -M\}$. The Laplace exponent of the ascending, resp. descending, ladder process $\overline{\kappa}(\alpha,\beta)$, resp. $\underline{\kappa}(\alpha,\beta)$, for $\alpha \geq \alpha^*(M)$ and $\overline{k}, \underline{k} > 0$, has an analytic extension to $\beta \in \{\beta \in \mathbb{C} : \Re(\beta) > -M\}$ and is given by

$$\overline{\kappa}(\alpha,\beta) = \overline{k}\exp\left(\int_0^\infty \int_{(0,\infty)} (\mathrm{e}^{-t} - \mathrm{e}^{-\alpha t - \beta x})\frac{1}{t}P_{L_t}(\mathrm{d}x)\mathrm{d}t\right), \tag{3.3}$$

and

$$\underline{\kappa}(\alpha,\beta) = \underline{k}\exp\left(\int_0^\infty \int_{(-\infty,0)} (\mathrm{e}^{-t} - \mathrm{e}^{-\alpha t + \beta x})\frac{1}{t}P_{L_t}(\mathrm{d}x)\mathrm{d}t\right). \tag{3.4}$$

Remark 3.2. Note that the Wiener–Hopf factors φ_q^+ and φ_q^- are related to the Laplace exponents of the ascending and descending ladder process $\overline{\kappa}$ and $\underline{\kappa}$ via

$$\varphi_q^+(i\beta) = \frac{\overline{\kappa}(q,0)}{\overline{\kappa}(q,\beta)} \quad \text{and} \quad \varphi_q^-(-i\beta) = \frac{\underline{\kappa}(q,0)}{\underline{\kappa}(q,\beta)}. \tag{3.5}$$

We will prepare the proof of this theorem with an intermediate lemma. Let us denote the positive part by $a_+ := \max\{a, 0\}$.

Lemma 3.3. *Let L be a Lévy process that satisfies assumption* $(\mathbb{EM})$. *For $q > \kappa(M)_+$ the maps*

$$z \mapsto \int_0^\infty \int_{(0,\infty)} \left(1 - e^{izx}\right) P_{L_t}(\mathrm{d}x) \frac{e^{-qt}}{t} \mathrm{d}t \qquad (3.6)$$

and

$$z \mapsto \int_0^\infty \int_{(0,\infty)} \left(e^{-t} - e^{-qt+izx}\right) P_{L_t}(\mathrm{d}x) \frac{1}{t} \mathrm{d}t \qquad (3.7)$$

are well defined and analytic in the open half plane $\left\{z \in \mathbb{C} : \Im(z) > -M\right\}$.

Proof. We will show that for every compact subset $K \subset \{z \in \mathbb{C} : \Im(z) > -M\}$, there is a constant $C = C(K) > 0$ such that

$$\int_0^\infty \int_{(0,\infty)} \left|e^{izx} - 1\right| P_{L_t}(\mathrm{d}x) \frac{e^{-qt}}{t} \mathrm{d}t < C(K), \qquad (3.8)$$

for every $z \in K$. Then, applying Lebesgue's dominated convergence theorem yields the continuity of the function

$$z \mapsto \int_0^\infty \int_{(0,\infty)} \left(e^{izx} - 1\right) P_{L_t}(\mathrm{d}x) \frac{e^{-qt}}{t} \mathrm{d}t$$

inside the half-plane $\{z \in \mathbb{C} : \Im(z) > -M\}$. Moreover, let γ be an arbitrary triangle inside $\{z \in \mathbb{C} : \Im(z) > -M\}$; the theorems of Fubini and Cauchy yield

$$\int_{\partial\gamma} \int_0^\infty \int_{(0,\infty)} \left(e^{izx} - 1\right) P_{L_t}(\mathrm{d}x) \frac{e^{-qt}}{t} \mathrm{d}t \, \mathrm{d}z$$

$$= \int_0^\infty \int_{(0,\infty)} \int_{\partial\gamma} \left(e^{izx} - 1\right) \mathrm{d}z \, P_{L_t}(\mathrm{d}x) \frac{e^{-qt}}{t} \mathrm{d}t = 0. \quad (3.9)$$

Hence, applying Morera's theorem yields the analyticity of (3.6) in the open half-plane $\{z \in \mathbb{C} : \Im(z) > -M\}$.

The assertion for the second map immediately follows from the identity

$$\left(e^{-t} - e^{-qt+izx}\right)t^{-1} = \left(1 - e^{izx}\right)e^{-qt}t^{-1} + \left(e^{-t} - e^{-qt}\right)t^{-1}$$

and the integrability of the second part, since

$$\int_\epsilon^\infty \left|e^{-t} - e^{-qt}\right|t^{-1}\mathrm{d}t < \infty$$

and

$$\int_0^\epsilon \left|e^{-t} - e^{-qt}\right| t^{-1} dt = \int_0^\epsilon \left|e^{t(q-1)} - 1\right| e^{-qt} t^{-1} dt \leq C|q-1| \int_0^\epsilon e^{-qt} dt < \infty,$$

with $C > 1$, for $\epsilon > 0$ small enough.

To show estimation (3.8), we choose a constant $k = k(K) > 0$ only depending on the compact set K, such that $|z| < k$ for every $z \in K$, and we write

$$\int_{(0,\infty)} \left|e^{izx} - 1\right| P_{L_t}(dx)$$

$$= \int_{(0,1/k]} \left|e^{izx} - 1\right| P_{L_t}(dx) + \int_{(1/k,\infty)} \left|e^{izx} - 1\right| P_{L_t}(dx)$$

$$\leq \int_{(0,1/k]} |zx| P_{L_t}(dx) + \int_{(1/k,\infty)} \left|e^{izx}\right| P_{L_t}(dx) + \int_{(1/k,\infty)} P_{L_t}(dx). \qquad (3.10)$$

Using inequality (30.13) of Lemma 30.3 in Sato (1999) we can deduce

$$\int_{(0,1/k]} |zx| P_{L_t}(dx) \leq k \int_{(0,1/k]} |x| P_{L_t}(dx) \leq k E\left[|L_t| 1_{\{|L_t| \leq 1/k\}}\right] \leq C_1(K) t^{1/2},$$

with a constant $C_1(K)$ that depends only on the compact set K. Similarly, using inequality (30.10) in Sato (1999), we can estimate the last term of (3.10)

$$\int_{(1/k,\infty)} P_{L_t}(dx) = P\left(\{L_t > 1/k\}\right) \leq P\left(\{|L_t| > 1/k\}\right) \leq C_2(K) t,$$

with a constant $C_2(K)$ that depends only on the compact set K. In order to estimate the second term of inequality (3.10), let us note that we may choose $\epsilon > 0$ small enough, such that for every $z \in K$, we have $-\Im(z) < M' < M$ with $M' := M(1 - \epsilon)$, and we get

$$\int_{(1/k,\infty)} \left|e^{izx}\right| P_{L_t}(dx) \leq E\left[e^{M'L_t} 1_{\{|L_t| > 1/k\}}\right].$$

Applying Hölder's inequality with $p := \frac{1}{1-\epsilon}$ and $q := \frac{1}{\epsilon}$, together with Lemma 30.3 in Sato (1999), yields

$$E\left[e^{M'L_t} 1_{\{|L_t| > 1/k\}}\right] \leq \left(E\left[e^{pM'L_t}\right]\right)^{1/p} \left(P\left(\{|L_t| > 1/k\}\right)\right)^{1/q}$$

$$\leq C_3(K) t^\epsilon e^{(1-\epsilon)\kappa(M)t}.$$

Altogether we have

$$\int_{(0,\infty)} \left|e^{izx} - 1\right| P_{L_t}(dx) \leq C_1(K) t^{1/2} + C_2(K) t + C_3(K) t^\epsilon e^{(1-\epsilon)\kappa(M)t},$$

with positive constants $C_1(K)$, $C_2(K)$ and $C_3(K)$ that only depend on the compact set K. As $q > (1 - \epsilon)(\kappa(M))_+$, we can conclude (3.8), which completes the proof. □

Proof of Theorem 3.1. For $\beta \in \mathbb{C}$ with $\Re\beta \geq 0$ the assertion follows directly from Theorem 6.16 (ii) and (iii) in Kyprianou (2006).

From Lemma 2.7 we know that for $q > \alpha^*(M)$ the function

$$\beta \mapsto \varphi_{\overline{L}_\theta}(i\beta) = E\big[\mathrm{e}^{-\beta \overline{L}_\theta}\big]$$

has an analytic extension to the half-plane

$$\{\beta \in \mathbb{C} : \Re(\beta) > -M\},$$

whereas Lemma 3.3 yields that if $q > \alpha^*(M)$, the mapping

$$\beta \mapsto \frac{\overline{\kappa}(q, 0)}{\overline{\kappa}(q, \beta)}$$

has an analytic extension to the half-plane

$$\{\beta \in \mathbb{C} : \Re(\beta) > -M\},$$

while identity (3.3) still holds for this extension. The identity theorem for holomorphic functions yields that equation (3.1) holds for every $\{\beta \in \mathbb{C} : \Re(\beta) > -M\}$ if $q > \alpha^*(M)$. The proof for equations (3.2) and (3.4) follows along the same lines. □

Remark 3.4. Note that, by analogous arguments, we can prove that the Laplace exponent of the ascending, resp. descending, ladder process $\overline{\kappa}(\alpha, \beta)$, resp. $\underline{\kappa}(\alpha, \beta)$, has an analytic extension to $\alpha \in \{\alpha \in \mathbb{C} : \Re(\alpha) > \alpha^*(M)\}$, which is given by (3.3), resp. (3.4).

3.2. **Inversion.** The next step is to invert the Laplace transform in the Wiener–Hopf factorization in order to recover the characteristic function of $\overline{L}_t$, at a *fixed* time t. Let us mention that although the Wiener–Hopf factorization and the characteristic function of $\overline{L}_\theta$ are discussed in several textbooks, the extended characteristic function of $\overline{L}_t$ at a fixed time has not been studied in the literature before.

The main result is Theorem 3.6, which will make use of the following auxiliary lemma.

Lemma 3.5. *The maps $t \mapsto E\big[\mathrm{e}^{-\beta \overline{L}_t}\big]$ and $t \mapsto E\big[\mathrm{e}^{\beta \underline{L}_t}\big]$ are continuous for all $\beta \in \mathbb{C}$ with $\Re\beta \in [-M, \infty)$.*

Proof. Since the Lévy process L is right continuous, stochastically continuous and $\overline{L}$ is an increasing process, we get that $\overline{L}_s \nearrow \overline{L}_t$ a.s. as $s \to t$.

As $\overline{L}_s \geq 0$ we have

$$\big|\mathrm{e}^{-\beta \overline{L}_s}\big| = \mathrm{e}^{-\Re(\beta)\overline{L}_s} \leq \mathrm{e}^{M\overline{L}_s} \leq \mathrm{e}^{M\overline{L}_t},$$

and we may apply the dominated convergence theorem to get

$$E\big[\mathrm{e}^{-\beta \overline{L}_s}\big] \to E\big[\mathrm{e}^{-\beta \overline{L}_t}\big] \qquad \text{as } s \to t,$$

for every $\beta \in \mathbb{C}$ with $\Re(\beta) \geq -M$. Analogously, taking into account that $\big|\mathrm{e}^{\beta \underline{L}_s}\big| \leq \mathrm{e}^{-M\underline{L}_s}$ for $\Re\beta \geq -M$, the dominated convergence theorem yields the continuity of the second map. □

Theorem 3.6. *Let L be a Lévy process that satisfies assumption* $(\mathbb{EM})$ *(and is not a compound Poisson process). The Laplace transform of $\overline{L}_t$ and $\underline{L}_t$ at a fixed time t, $t \in [0,T]$, is given by*

$$E\big[e^{-\beta \overline{L}_t}\big] = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{t(Y+iv)}}{Y+iv} \frac{\overline{\kappa}(Y+iv,0)}{\overline{\kappa}(Y+iv,\beta)} dv, \qquad (3.11)$$

and

$$E\big[e^{\beta \underline{L}_t}\big] = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{t(\widetilde{Y}+iv)}}{\widetilde{Y}+iv} \frac{\underline{\kappa}(\widetilde{Y}+iv,0)}{\underline{\kappa}(\widetilde{Y}+iv,-\beta)} dv, \qquad (3.12)$$

for $\beta \in \mathbb{C}$ with $\Re\beta \in (-M,\infty)$ and $Y, \widetilde{Y} > \alpha^*(M)$.

Proof. Theorem 3.1, together with equation (3.1), immediately yield

$$\int_0^{\infty} e^{-qt} E\big[e^{-\beta \overline{L}_t}\big] dt = \frac{1}{q} \frac{\overline{\kappa}(q,0)}{\overline{\kappa}(q,\beta)}, \qquad (3.13)$$

for $\beta \in \mathbb{C}$ with $\Re(\beta) > -M$ and $q > \alpha^*(M)$.

In order to deduce that we can invert this Laplace transform, we want to verify the assumptions of Satz 4.4.3 in Doetsch (1950) for the real and imaginary part of $t \mapsto E\big[e^{-\beta \overline{L}_t}\big]$. From the proof of Lemma 2.7 we get that

$$\int_0^{\infty} e^{-qt} \Big| E\big[e^{-\beta \overline{L}_t}\big] \Big| dt \leq \int_0^{\infty} e^{-qt} E\big[e^{-\Re(\beta)\overline{L}_t}\big] dt < \infty;$$

this yields the required integrability, i.e. absolute convergence, of

$$\int_0^{\infty} e^{-qt} \Big| \Im\big(E\big[e^{\beta \overline{L}_t}\big]\big) \Big| dt \quad \text{and} \quad \int_0^{\infty} e^{-qt} \Big| \Re\big(E\big[e^{\beta \overline{L}_t}\big]\big) \Big| dt,$$

for $q > \alpha^*(M)$. Further the real and imaginary part of $t \mapsto E\big[e^{-\beta \overline{L}_t}\big]$ are of bounded variation for $\beta \in \mathbb{C}$ with $\Re\beta \in (-M,\infty)$.

Let us verify this assertion for the imaginary part, for $-M < \Re(\beta) \leq 0$ and $\Im(\beta) \leq 0$. We have that

$$\Im\left(E\big[e^{-\beta \overline{L}_t}\big]\right) = iE\big[\sin\big(-\Im(\beta)\overline{L}_t\big)e^{-\Re(\beta)\overline{L}_t}\big].$$

We can decompose $\sin(x) = f(x) - g(x)$, where f and g are increasing functions with $f(0) = g(0) = 0$, and $|f(x)| \leq x$ and $|g(x)| \leq x$. It follows that

$$\sin\big(-\Im(\beta)\overline{L}_t\big)e^{-\Re(\beta)\overline{L}_t} = f\big(-\Im(\beta)\overline{L}_t\big)e^{-\Re(\beta)\overline{L}_t} - g\big(-\Im(\beta)\overline{L}_t\big)e^{-\Re(\beta)\overline{L}_t},$$

where both terms are increasing in time and are integrable, since

$$E\Big[\Big|h\big(-\Im(\beta)\overline{L}_t\big)e^{-\Re(\beta)\overline{L}_t}\Big|\Big] \leq \big|\Im(\beta)\big| E\Big[\big|\overline{L}_t\big|e^{-\Re(\beta)\overline{L}_t}\Big]$$

$$\leq \text{const} \cdot E\big[e^{M\overline{L}_t}\big] < \infty,$$

for $h = g$ and $h = f$. The assertion for the other parts follows similarly.

Now, using the continuity of the map $t \mapsto E\big[\mathrm{e}^{-\beta \overline{L}_t}\big]$, cf. Lemma 3.5, we may apply Satz 4.4.3 in Doetsch (1950), to invert this Laplace transform; that is, to conclude that

$$
E\big[\mathrm{e}^{-\beta \overline{L}_t}\big] = (\mathrm{p.v.}) \, \frac{1}{2\pi i} \int\limits_{Y-i\infty}^{Y+i\infty} \frac{\mathrm{e}^{tz}}{z} \frac{\overline{\kappa}(z,0)}{\overline{\kappa}(z,\beta)} \mathrm{d}z
$$

$$
= \lim_{A \to \infty} \frac{1}{2\pi} \int\limits_{-A}^{A} \frac{\mathrm{e}^{t(Y+iv)}}{Y+iv} \frac{\overline{\kappa}(Y+iv,0)}{\overline{\kappa}(Y+iv,\beta)} \mathrm{d}v, \qquad (3.14)
$$

for all $\beta \in \mathbb{C}$ with $\Re\beta \in (-M, \infty)$ and for every $Y > \alpha^*(M)$. The proof for the infimum follows along the same lines. $\square$

4. LÉVY PROCESSES: EXAMPLES AND PROPERTIES

We first state some conditions for the continuity of the law of a Lévy process, and the continuity of the law of the supremum of a Lévy process. Then, we describe the most popular Lévy models for financial applications, and comment on their path and moment properties which are relevant for the application of Fourier transform valuation formulas.

4.1. **Continuity properties.** The valuation theorem for discontinuous payoff functions (Theorem 2.7 in EGP), and the analysis of the properties of discontinuous payoff functions (Examples 5.2, 5.3 and 5.4 in EGP), show that if the measure of the underlying random variable does not have atoms, then the valuation formula is valid as a pointwise limit. Thus, we present sufficient conditions for the continuity of the law of a Lévy process and its supremum, and discuss these conditions for certain popular examples.

Statement 4.1. Let L be a Lévy process with triplet (b, c, λ). Then, Theorem 27.4 in Sato (1999) yields that the law P_{L_t}, $t \in [0,T]$, is *atomless* iff L is a process of *infinite variation* or *infinite activity*. In other words, if one of the following conditions holds true:

> **(a):** $c \neq 0$ or $\int_{\{|x|\leq 1\}} |x|\lambda(\mathrm{d}x) = \infty$;
> **(b):** $c = 0$, $\lambda(\mathbb{R}) = \infty$ and $\int_{\{|x|\leq 1\}} |x|\lambda(\mathrm{d}x) < \infty$.

Statement 4.2. Let L be a Lévy process and assume that

> **(a):** L has *infinite variation*, or
> **(b):** L has *infinite activity* and is *regular upwards*. Regular upwards means that $P(\tau_0 = 0) = 1$ where $\tau_0 := \inf\{t > 0 : L_t(\omega) > 0\}$.

Then, Lemma 49.3 in Sato (1999) yields that $\overline{L}_t$ has a *continuous* distribution for every $t \in [0,T]$. The statement for the infimum of a Lévy process is analogous.

4.2. **Examples.** Next, we describe the most popular Lévy processes for applications in mathematical finance, namely the generalized hyperbolic (GH) process, the CGMY process and the Meixner process. We present their characteristic functions, which are essential for the application of Fourier transform methods for option pricing, and its domain of definition. We also discuss their path properties which are relevant for option pricing. For an interesting

survey on the path properties of Lévy processes we refer to Kyprianou and Loeffen (2005).

Example 4.3 (GH model). Let $H = (H_t)_{0 \leq t \leq T}$ be a generalized hyperbolic process with $\mathcal{L}(H_1) = \mathrm{GH}(\lambda, \alpha, \beta, \delta, \mu)$, cf. Eberlein (2001, p. 321) or Eberlein and Prause (2002). The characteristic function of H_1 is

$$\varphi_{H_1}(u) = e^{iu\mu}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\big(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\big)}{K_\lambda\big(\delta\sqrt{\alpha^2 - \beta^2}\big)}, \qquad (4.1)$$

where K_λ denotes the Bessel function of the third kind with index λ (cf. Abramowitz and Stegun 1968); the moment generating function exists for $u \in (-\alpha - \beta, \alpha - \beta)$. The sample paths of a generalized hyperbolic Lévy process have infinite variation. Thus, by Statements 4.1 and 4.2, we can deduce that the laws of both a GH Lévy process and its supremum do not have atoms.

The class of generalized hyperbolic distributions is not closed under convolution, hence the distribution of H_t is no longer a generalized hyperbolic one. Nevertheless, the characteristic function of $\mathcal{L}(H_t)$ is given explicitly by

$$\varphi_{H_t}(u) = (\varphi_{H_1}(u))^t.$$

A class closed under certain convolutions is the class of normal inverse Gaussian distributions, where $\lambda = -\frac{1}{2}$; cf. Barndorff-Nielsen (1998). In that case, $\mathcal{L}(H_t) = \mathrm{NIG}(\alpha, \beta, \delta t, \mu t)$ and the characteristic function resumes the form

$$\varphi_{H_t}(u) = e^{iu\mu t}\frac{\exp(\delta t\sqrt{\alpha^2 - \beta^2})}{\exp(\delta t\sqrt{\alpha^2 - (\beta + iu)^2})}. \qquad (4.2)$$

Another interesting subclass is given by the hyperbolic distributions which arise for $\lambda = 1$; the hyperbolic model has been introduced to finance by Eberlein and Keller (1995).

Example 4.4 (CGMY model). Let $H = (H_t)_{0 \leq t \leq T}$ be a CGMY Lévy process, cf. Carr, Geman, Madan, and Yor (2002); another name for this process is (generalized) tempered stable process (see e.g. Cont and Tankov (2004)). The Lévy measure of this process has the form

$$\lambda^{CGMY}(\mathrm{d}x) = C\frac{e^{-Mx}}{x^{1+Y}}1_{\{x>0\}}\mathrm{d}x + C\frac{e^{Gx}}{|x|^{1+Y}}1_{\{x<0\}}\mathrm{d}x,$$

where the parameter space is $C, G, M > 0$ and $Y \in (-\infty, 2)$. Moreover, the characteristic function of H_t, $t \in [0, T]$, is

$$\varphi_{H_t}(u) = \exp\Big(t\,C\,\Gamma(-Y)\big[(M - iu)^Y + (G + iu)^Y - M^Y - G^Y\big]\Big), \quad (4.3)$$

for $Y \neq 0$, and the moment generating function exists for $u \in [-G, M]$.

The sample paths of the CGMY process have unbounded variation if $Y \in [1, 2)$, bounded variation if $Y \in (0, 1)$, and are of compound Poisson type if $Y < 0$. Moreover, the CGMY process is regular upwards if $Y > 0$; cf. Kyprianou and Loeffen (2005). Hence, by Statements 4.1 and 4.2, the laws of a CGMY Lévy process, and its supremum, do not have atoms if $Y \in (0, 2)$.

The CGMY process contains the Variance Gamma process (cf. Madan and Seneta 1990) as a subclass, for $Y = 0$. The characteristic function of H_t, $t \in [0,T]$, is

$$\varphi_{H_t}(u) = \exp\left(t\, C \left[-\log\left(1 - \frac{iu}{M}\right) - \log\left(1 + \frac{iu}{G}\right) \right] \right), \qquad (4.4)$$

and the moment generating function exists for $u \in [-G, M]$. The paths of the variance gamma process have bounded variation, infinite activity and are regular upwards. Thus, the laws of a VG Lévy process and its supremum do not have atoms.

Example 4.5 (Meixner model). Let $H = (H_t)_{0 \le t \le T}$ be a Meixner process with $\mathcal{L}(H_1) = \mathrm{Meixner}(\alpha, \beta, \delta)$, $\alpha > 0$, $-\pi < \beta < \pi$, $\delta > 0$, cf. Schoutens and Teugels (1998) and Schoutens (2002). The characteristic function of H_t, $t \in [0,T]$, is

$$\varphi_{H_t}(u) = \left(\frac{\cos \frac{\beta}{2}}{\cosh \frac{\alpha u - i\beta}{2}} \right)^{2\delta t}, \qquad (4.5)$$

and the moment generating function exists for $u \in \left(\frac{\beta - \pi}{\alpha}, \frac{\beta + \pi}{\alpha} \right)$. The paths of a Meixner process have infinite variation. Hence the laws of a Meixner Lévy process and its supremum do not have atoms.

5. Applications in finance

In this section, we derive valuation formulas for lookback options, one-touch options and equity default swaps, in models driven by Lévy processes. We combine the results on the Wiener–Hopf factorization and the characteristic function of the supremum of a Lévy process from this paper, with the results on Fourier transform valuation formulas derived in EGP. Note that the results presented in the sequel are valid for all the examples discussed in section 4.

We model the price process of a financial asset $S = (S_t)_{0 \le t \le T}$ as an exponential Lévy process, i.e. a stochastic process with representation

$$S_t = S_0 e^{L_t}, \qquad 0 \le t \le T \qquad (5.1)$$

(shortly: $S = S_0\, e^L$). Every Lévy process L, subject to Assumption ($\mathbb{EM}$), has the canonical decomposition

$$L_t = bt + \sqrt{c}\, W_t + \int_0^t \int_{\mathbb{R}} x(\mu - \nu)(\mathrm{d}s, \mathrm{d}x), \qquad (5.2)$$

where $W = (W_t)_{0 \le t \le T}$ denotes a P-standard Brownian motion and μ denotes the random measure associated with the jumps of L; cf. Jacod and Shiryaev (2003, Chapter II).

Let $\mathcal{M}(P)$ denote the class of martingales on the stochastic basis $\mathcal{B}$. The martingale condition for an asset S is

$$S = S_0\, e^L \in \mathcal{M}(P) \Leftrightarrow b + \frac{c}{2} + \int_{\mathbb{R}} (e^x - 1 - x)\lambda(\mathrm{d}x) = 0; \qquad (5.3)$$

cf. Eberlein et al. (2008) for the details. That is, throughout the rest of this paper, we will assume that P is a *martingale measure* for S.

5.1. **Lookback options.** The results on the characteristic function of the supremum of a Lévy process, cf. section 3, allow us to price lookback options in models driven by Lévy processes using Fourier methods. Excluded are only compound Poisson processes. Assuming that the asset price evolves as an exponential Lévy process, a fixed strike lookback call option with payoff

$$(\overline{S}_T - K)^+ = (S_0 e^{\overline{L}_T} - K)^+ \tag{5.4}$$

can be viewed as a call option where the driving process is the *supremum* of the underlying Lévy processes L. Therefore, the price of a lookback call option is provided by the following result.

Theorem 5.1. *Let L be a Lévy process that satisfies Assumption* $(\mathbb{EM})$. *The price of a fixed strike lookback call option with payoff* (5.4) *is given by*

$$\mathbb{C}_T(\overline{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi_{\overline{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} du, \tag{5.5}$$

where

$$\varphi_{\overline{L}_T}(-u - iR) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{T(Y+iv)}}{Y + iv} \frac{\overline{\kappa}(Y + iv, 0)}{\overline{\kappa}(Y + iv, iu - R)} dv, \tag{5.6}$$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

Proof. We aim at applying Theorem 2.2 in EGP, hence we must check if conditions (C1)–(C3) (of EGP) are satisfied. Assumption $(\mathbb{EM})$, coupled with Corollary 2.5, yields that $M_{\overline{L}_T}(R)$ exists for $R \in (-\infty, M)$, hence condition (C2) is satisfied. Now, the Fourier transform of the payoff function $f(x) = (e^x - K)^+$ is

$$\widehat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)},$$

and conditions (C1) and (C3) are satisfied for $R \in (1, \infty)$; cf. Example 5.1 in EGP. Further, the extended characteristic function $\varphi_{\overline{L}_T}$ of $\overline{L}_T$ is provided by Theorem 3.6 and equals (5.6) for $R \in (-\infty, M)$ and $Y > \alpha^*(M)$. Finally, Theorem 2.2 in EGP delivers the asserted valuation formula (5.5). $\qquad \square$

Remark 5.2. Completely analogous formulas can be derived for the fixed strike lookback put option with payoff $(K - \underline{S}_T)^+$ using the results for the infimum of a Lévy process. Moreover, floating strike lookback options can be treated by the same formulas making use of the duality relationships proved in Eberlein and Papapantoleon (2005) and Eberlein, Papapantoleon, and Shiryaev (2008).

5.2. **One-touch options.** Analogously, we can derive valuation formulas for one-touch options in assets driven by Lévy processes using Fourier transform methods; here, the exceptions are compound Poisson processes and non-regular upwards, finite variation, Lévy processes. Assuming that the asset price evolves as an exponential Lévy process, a one-touch call option with payoff

$$1_{\left\{\overline{S}_T>B\right\}} = 1_{\left\{\overline{L}_T>\log\left(\frac{B}{S_0}\right)\right\}} \tag{5.7}$$

can be valued as a digital call option where the driving process is the supremum of the underlying Lévy process.

Theorem 5.3. *Let L be a Lévy process with infinite variation, or a regular upwards process with infinite activity, that satisfies Assumption* (𝔼𝕄). *The price of a one-touch option with payoff* (5.7) *is given by*

$$\mathbb{DC}_T(\overline{S}; B) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} S_0^{R+iu} \varphi_{\overline{L}_T}(u-iR) \frac{B^{-R-iu}}{R+iu} \mathrm{d}u \tag{5.8}$$

$$= P\big(\overline{L}_T > \log(B/S_0)\big),$$

for $R \in (0, M)$ and $Y > \alpha^*(M)$, where $\varphi_{\overline{L}_T}$ is given by (5.6).

Proof. We will apply Theorem 2.7 in EGP, hence we must check conditions (D1)–(D2). As in the proof of Theorem 5.1, Assumption (𝔼𝕄) shows that condition (D2) is satisfied for $R \in (-\infty, M)$, while Theorem 3.6 provides the characteristic function of $\overline{L}_T$, given by (5.6). Example 5.2 in EGP yields that the Fourier transform of the payoff function $f(x) = 1_{\{x>\log B\}}$ equals

$$\widehat{f}(iR-u) = \frac{B^{-R-iu}}{R+iu}, \tag{5.9}$$

and condition (D1) is satisfied for $R \in (0, \infty)$. In addition, if the measure $P_{\overline{L}_T}$ is atomless, then the valuation function is continuous and has bounded variation. Now, by Statement 4.2, we know that the measure $P_{\overline{L}_T}$ is atomless exactly when L has infinite variation, or has infinite activity and is regular upwards. Therefore, Theorem 2.7 in EGP applies, and results in the valuation formula (5.8) for the one-touch call option. □

Remark 5.4. Completely analogous valuation formulas can be derived for the digital put option with payoff $1_{\{\underline{S}_T<B\}}$.

Remark 5.5. Summarizing the results of this paper and of EGP, when dealing with *continuous* payoff functions the valuation formulas can be applied to *all* Lévy processes. When dealing with *discontinuous* payoff functions, then the valuation formulas apply to most Lévy processes *apart* from *compound Poisson* type processes without diffusion component, and finite variation Lévy processes which are not *regular upwards*. This is true for both non-path-dependent as well as for *path-dependent exotic* options.

Remark 5.6. Arguing analogously to Theorems 5.1 and 5.3, we can derive the price of options with a "general" payoff function $f(\overline{L}_T)$. For example, one could consider payoffs of the form $[(\overline{S}_T - K)^+]^2$ or $\overline{S}_T 1_{\{\overline{S}_T>B\}}$; cf. Raible

(2000, Table 3.1) and Example 5.3 in EGP for the corresponding Fourier transforms.

5.3. **Equity default swaps.** Equity default swaps were recently introduced in financial markets, and offer a link between equity and credit risk. The structure of an equity default swap imitates that of a credit default swap: the protection buyer pays a fixed premium in exchange for an insurance payment in case of 'default'. In this case 'default', also called the 'equity event', is defined as the first time the asset price process drops below a fixed barrier, typically 30% or 50% of the initial value S_0.

Let us denote by τ_B the first passage time below the barrier level B, i.e.

$$\tau_B = \inf\{t \geq 0; S_t \leq B\}.$$

The protection buyer pays a fixed premium denoted by $\mathcal{K}$ at the dates $T_1, T_2, \ldots, T_N = T$, provided that default has not occurred, i.e. $T_i < \tau_B$. In case of default, the protection seller makes the insurance payment $\mathcal{C}$, which is typically 50% of the initial value. The premium $\mathcal{K}$ is fixed such that the value of the equity default swap at inception is zero, hence we get

$$\mathcal{K} = \frac{\mathcal{C}E\left[e^{-r\tau_B}1_{\{\tau_B \leq T\}}\right]}{\sum_{i=1}^{N} E\left[e^{-rT_i}1_{\{\tau_B > T_i\}}\right]}, \tag{5.10}$$

where r denotes the risk-free interest rate.

Now, using that $1_{\{\tau_B \leq t\}} = 1_{\{\underline{S}_t \leq B\}}$ which immediately translates into

$$P(\tau_B \leq t) = E\left[1_{\{\tau_B \leq t\}}\right] = E\left[1_{\{\underline{S}_t \leq B\}}\right], \tag{5.11}$$

and that

$$E\left[e^{-r\tau_B}1_{\{\tau_B \leq T\}}\right] = \int_0^T e^{-rt} P_{\tau_B}(\mathrm{d}t),$$

the quantities in (5.10) can be calculated using the valuation formulas for one-touch options.

References

Abramowitz, M. and I. Stegun (Eds.) (1968). *Handbook of Mathematical Functions* (5th ed.). Dover.

Alili, L. and A. E. Kyprianou (2005). Some remarks on first passage of Lévy process, the American put and pasting principles. *Ann. Appl. Probab. 15*, 2062–2080.

Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus*. Cambridge University Press.

Asmussen, S., F. Avram, and M. R. Pistorius (2004). Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl. 109*, 79–111.

Asmussen, S., D. Madan, and M. Pistorius (2007). Pricing equity default swaps under an approximation to the CGMY Lévy model. *J. Comput. Finance 11*, 79–93.

Avram, F., A. Kyprianou, and M. R. Pistorius (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab. 14*, 215–238.

Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance Stoch. 2*, 41–68.

Barndorff-Nielsen, O. E., T. Mikosch, and S. Resnick (Eds.) (2001). *Lévy Processes: Theory and Applications*. Birkhäuser.

Bertoin, J. (1996). *Lévy processes*. Cambridge University Press.

Bingham, N. H. (1975). Fluctuation theory in continuous time. *Adv. Appl. Probab. 7*, 705–766.

Boyarchenko, S. I. and S. Z. Levendorskiĭ (2002a). Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Ann. Appl. Probab. 12*, 1261–1298.

Boyarchenko, S. I. and S. Z. Levendorskiĭ (2002b). *Non-Gaussian Merton-Black-Scholes Theory*. World Scientific.

Carr, P., H. Geman, D. B. Madan, and M. Yor (2002). The fine structure of asset returns: An empirical investigation. *J. Business 75*, 305–332.

Cont, R. and P. Tankov (2004). *Financial Modelling with Jump Processes*. Chapman and Hall/CRC Press.

Doetsch, G. (1950). *Handbuch der Laplace-Transformation*. Birkhäuser.

Eberlein, E. (2001). Application of generalized hyperbolic Lévy motions to finance. In O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick (Eds.), *Lévy Processes: Theory and Applications*, pp. 319–336. Birkhäuser.

Eberlein, E., K. Glau, and A. Papapantoleon (2010). Analysis of Fourier transform valuation formulas and applications. *Appl. Math. Finance 17*, 211–240.

Eberlein, E. and U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli 1*, 281–299.

Eberlein, E. and A. Papapantoleon (2005). Equivalence of floating and fixed strike Asian and lookback options. *Stochastic Process. Appl. 115*, 31–40.

Eberlein, E., A. Papapantoleon, and A. N. Shiryaev (2008). On the duality principle in option pricing: semimartingale setting. *Finance Stoch. 12*, 265–292.

Eberlein, E. and K. Prause (2002). The generalized hyperbolic model: financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, and T. Vorst (Eds.), *Mathematical Finance – Bachelier Congress 2000*, pp. 245–267. Springer.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications* (2nd ed.), Volume II. Wiley.

Greenwood, P. and J. Pitman (1980a). Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. Appl. Probab. 12*, 893–902.

Greenwood, P. and J. Pitman (1980b). Fluctuation identities for random walk by path decomposition at the maximum. *Adv. Appl. Probab. 12*, 291–293.

Hilber, N., N. Reich, C. Schwab, and C. Winter (2009). Numerical methods for Lévy processes. *Finance Stoch. 13*, 471–500.

Hubalek, F. and A. E. Kyprianou (2010). Old and new examples of scale functions for spectrally negative Lévy processes. In R. Dalang,

M. Dozzi, and F. Russo (Eds.), *Seminar on Stochastic Analysis, Random Fields and Applications VI*, Progress in Probability. Birkhäuser. (forthcoming).

Jacod, J. and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer.

Jeannin, M. and M. Pistorius (2009). A transform approach to compute prices and greeks of barrier options driven by a class of Lévy processes. *Quant. Finance*. (forthcoming).

Kudryavtsev, O. and S. Levendorskiĭ (2006). Pricing of first touch digitals under normal inverse Gaussian processes. *Int. J. Theor. Appl. Finance 9*, 915–949.

Kudryavtsev, O. and S. Levendorskiĭ (2009). Fast and accurate pricing of barrier options under Lévy processes. *Finance Stoch. 13*, 531–562.

Kuznetsov, A. (2009). Wiener–Hopf factorization for a family of Lévy processes related to theta functions. Preprint, York University.

Kuznetsov, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.*. (forthcoming).

Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.

Kyprianou, A. E. and R. Loeffen (2005). Lévy processes in finance distinguished by their coarse and fine path properties. In A. Kyprianou, W. Schoutens, and P. Wilmott (Eds.), *Exotic Option Pricing and Advanced Lévy Models*, pp. 1–28. Wiley.

Kyprianou, A. E. and M. R. Pistorius (2003). Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Probab. 13*, 1077–1098.

Kyprianou, A. E. and V. Rivero (2008). Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab. 13*, 1672–1701.

Kyprianou, A. E., V. Rivero, and R. Song (2009). Convexity and smoothness of scale functions and de Finetti's control problem. *J. Theoret. Probab.*. (forthcoming).

Kyprianou, A. E., W. Schoutens, and P. Wilmott (Eds.) (2005). *Exotic Option Pricing and Advanced Lévy Models*. Wiley.

Kyprianou, A. E. and B. A. Surya (2005). On the Novikov–Shiryaev optimal stopping problem in continuous time. *Elect. Comm. Probab. 10*, 146–154.

Levendorskiĭ, S., O. Kudryavtsev, and V. Zherder (2005). The relative efficiency of numerical methods for pricing American options under Lévy processes. *J. Comput. Finance 9*, 69–98.

Madan, D. B. and E. Seneta (1990). The variance gamma (VG) model for share market returns. *J. Business 63*, 511–524.

Raible, S. (2000). *Lévy processes in finance: Theory, numerics, and empirical facts*. Ph. D. thesis, Univ. Freiburg.

Rudin, W. (1987). *Real and Complex Analysis* (3rd ed.). McGraw-Hill.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

Schoutens, W. (2002). The Meixner process: Theory and applications in
    finance. In O. E. Barndorff-Nielsen (Ed.), *Mini-proceedings of the 2nd
    MaPhySto Conference on Lévy Processes*, pp. 237–241.

Schoutens, W. (2003). *Lévy Processes in Finance: Pricing Financial
    Derivatives*. Wiley.

Schoutens, W. and J. L. Teugels (1998). Lévy processes, polynomials and
    martingales. *Comm. Statist. Stochastic Models 14*, 335–349.

Spitzer, F. (1964). *Principles of Random Walk*. Van Nostrand.

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