Calibration of Lévy Term Structure Models

Ernst Eberlein[1] and Wolfgang Kluge[2]

[1] Department for Mathematical Stochastics
University of Freiburg
79104 Freiburg
Germany
eberlein@stochastik.uni-freiburg.de

[2] Wolfgang Kluge
BNP Paribas London Branch
London, NW1 6AA
United Kingdom
wolfgang.kluge@uk.bnpparibas.com

Summary. We review the Lévy driven interest rate theory which has been developed in recent years. The intimate relations between the various approaches as well as the differences are outlined. The main purpose of this article is to elaborate on calibration in the real-world as well as in the risk-neutral setting.

Key words: Lévy processes, market models, forward rate model, calibration

1 Introduction

Although the mathematical theory of Lévy processes in general originated in the first half of the last century, its use in finance started only in the last decade of that century. Since Brownian motion, itself a Lévy process, is so well understood and also since a broad community is familiar with diffusion techniques, it is not surprising that this technology became the basis for the classical models in finance. On the other side it is known for a long time, that the normal distribution which generates the Brownian motion and which is reproduced on any time horizon by this process, is only a poor approximation of the empirical return distributions observed in financial data. Of course, diffusion processes with random coefficients produce distributions different from the normal one, but the outcoming distribution on a given time horizon is not even known in general. It can only be determined approximately and visualized by Monte Carlo simulation. This remark holds for most of the

[2] The second author has been partially supported by a grant from the Deutsche Forschungsgemeinschaft (DFG).

extensions of the classical geometric Brownian motion model such as models with stochastic volatility or stochastic interest rates.

In [21] and [20] a genuine Lévy model for the pricing of derivatives was introduced by Madan and Seneta and Madan and Milne. Based on the three parameter variance gamma (V.G.) process as driving process they derived a pricing formula for standard call options. This approach was extended and refined in a series of papers by Madan and coauthors. We only mention the extension to the four parameter CGMY-model in [4], which added more flexibility to the initial V.G. model. Based on an extensive empirical study of stock price data, in an independent line of research, Eberlein and Keller introduced the hyperbolic Lévy model in [7]. Both processes, the V.G. as well as the hyperbolic Lévy process, are purely discontinuous and, therefore, in a sense opposite to the Brownian motion. Starting from empirical results the basic concern in [7] was to develop a model which produces distributions that fit the observed empirical return distributions as close as possible. This led to exponential Lévy models

$$S_t = S_0 \exp(L_t) \qquad (t \geq 0) \tag{1}$$

to describe stock prices or indices. The log returns, $\log S_t - \log S_{t-1}$, derived from model (1) are the increments of length 1 of the driving Lévy process L. Therefore, by feeding in the Lévy process L which is generated by an (infinitely divisible) empirical return distribution, at least on the time horizon 1 this model reproduces exactly the distribution which one sees in the data. This is not the case if one starts with a model for prices given by the stochastic differential equation

$$\mathrm{d}S_t = S_{t-}\,\mathrm{d}\widetilde{L}_t \tag{2}$$

or equivalently by the stochastic exponential

$$S_t = \mathcal{E}(\widetilde{L})_t \tag{3}$$

of a Lévy process $\widetilde{L} = (\widetilde{L}_t)_{t \geq 0}$.

A model based on normal inverse Gaussian (NIG) distributions was added by Barndorff-Nielsen in [2]. Normal inverse Gaussian Lévy processes have nice analytic properties. As the class of hyperbolic distributions, NIG distributions constitute a subclass of the class of generalized hyperbolic distributions. The stock price model based on this 5-parameter class was developed in [13] and [5]. V.G. distributions turned out to be another subclass. A further interesting class of Lévy models based on Meixner processes was introduced by Schoutens (see [23, 24]).

Calibration of the exponential Lévy model (1) at least with respect to the real-world (or historical) measure is conceptually straightforward since – as pointed out above – the return distribution is the one which generates the driving Lévy process. See [7] for calibration results in the case of the hyperbolic model. In this paper we study calibration of Lévy interest rate models. The corresponding theory has been developed in a series of papers starting with [14] and continuing with [11], [6], [12], [8], [18], [10]. During the extensions of the initial model it turned out that the natural driving processes for interest rate models are time-inhomogeneous Lévy processes. They are described in the next section. Section 3 is a brief review of the three basic approaches: the Lévy forward rate model (HJM-type model), the Lévy forward process model and the Lévy Libor model.

In each of these approaches a different quantity is modeled: the instantaneous forward rate $f(t,T)$, the forward process $F(t,T,U)$ corresponding to time points T and U, and the δ-(forward) Libor rate $L(t,T)$. The relation between the latter quantities is obvious, since $1+\delta L(t,T) = F(t,T,T+\delta)$. Although $L(t,T)$ and $F(t,T,T+\delta)$ differ only by an additive and multiplicative constant, the two specifications lead to models that behave quite differently. The reason is that the changes of the driving process have a different impact on the forward Libor rates. In the Lévy Libor model, forward Libor rates change by an amount that is relative to their current level while the change in the Lévy forward process model does not depend on the actual level. Let us note that by construction the forward process model is easier to handle and implement. On the other side, this model – as the classical HJM and therefore also the Lévy forward rate model – produces negative rates with some (small) probability. Negative rates are excluded in the Lévy Libor model.

It is shown in section 4 that there is also a close relation to the forward rate model. We prove that the forward process model can be seen as a special case of the forward rate model. In section 5 we describe how the Lévy forward rate model can be calibrated with respect to the real-world measure as well as with respect to the risk-neutral martingale measure. Some explicit calibration results for driving generalized hyperbolic Lévy processes are given.

In section 6 calibration of the Lévy forward process and the Lévy Libor model is discussed. Again generalized hyperbolic Lévy processes, in particular NIG processes, are considered in the explicit results. We would like to thank N. Koval for providing some of the figures in this section.

2 The driving process

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$, the filtration, satisfies the usual conditions, $T^* \in \mathbb{R}_+$ is a finite time horizon and $\mathcal{F} = \mathcal{F}_{T^*}$. The driving process $L = (L_t)_{t\in[0,T^*]}$ is a *time-inhomogeneous Lévy process*, i.e. an adapted process with *independent increments* and *absolutely continuous characteristics*, which is abbreviated by PIIAC in [16]. We can assume that the paths of the process are right continuous with left-hand limits. We also assume that the process starts in 0. The law of L_t is given by its characteristic function

$$\mathbb{E}[e^{i\langle u,L_t\rangle}] = \exp \int_0^t \Big[i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}} \Big) F_s(\mathrm{d}x) \Big] \,\mathrm{d}s. \tag{4}$$

Here, $b_s \in \mathbb{R}^d$, c_s is a symmetric nonnegative-definite $d\times d$-matrix, and F_s is a *Lévy measure*, i.e. a measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and satisfies $F_s(\{0\}) = 0$. By $\langle\cdot,\cdot\rangle$ we denote the Euclidian scalar product on $\mathbb{R}^d$, and $|\cdot|$ is the corresponding norm. We shall assume that

$$\int_0^{T^*} \Big(|b_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) \Big) \,\mathrm{d}s < \infty \tag{5}$$

where $\|\cdot\|$ denotes any norm on the $d\times d$-matrices. The triplet $(b, c, F) = (b_s, c_s, F_s)_{s\in[0,T^*]}$ represents the *local characteristics* of the process L. We shall impose a further moment assumption.

Assumption $\mathbb{EM}$: *There are constants M, $\varepsilon > 0$, such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$*

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\langle u, x\rangle F_s(\mathrm{d}x) \,\mathrm{d}s < \infty. \tag{6}$$

$\mathbb{EM}$ is a very natural assumption. It is equivalent to $\mathbb{E}[\exp\langle u, L_t\rangle] < \infty$ for all $t \in [0, T^*]$ and all u as above. In the interest rate models which we will consider, the underlying processes are always exponentials of stochastic integrals with respect to the driving processes L. In order to allow pricing of derivatives these underlying processes have to be *martingales* under the risk-neutral measure and, therefore, a priori have to have finite expectations, which is exactly assumption $\mathbb{EM}$.

In particular under $\mathbb{EM}$ the variable L_t itself has finite expectation and consequently we do not need a truncation function. The representation (4) simplifies to

$$\mathbb{E}[e^{i\langle u,L_t\rangle}] = \exp \int_0^t \Big[i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \Big) F_s(\mathrm{d}x) \Big] \,\mathrm{d}s. \tag{4$'$}$$

where the characteristic b_s is now different from the one in (4). We will always use the local characteristics (b, c, F) derived from (4$'$).

Another consequence of assumption $\mathbb{EM}$ is that L is a *special semimartingale* and thus its canonical representation has the simple form

$$L_t = \int_0^t b_s \,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x) \tag{7}$$

where W is a standard d-dimensional Brownian motion, $\sqrt{c_s}$ is a measurable version of the square root of c_s and μ^L is the random measure of jumps of L with compensator $\nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s$.

Assumption $\mathbb{EM}$, which will be assumed throughout the following chapters, holds for all processes we are interested in, in particular for processes generated by generalized hyperbolic distributions. It excludes processes generated by stable distributions in general, but these processes are a priori not appropriate for developing a martingale theory to price derivative products.

We denote by θ_s the *cumulant* associated with a process L as given in (7) with local characteristics (b_s, c_s, F_s), i.e,

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} (e^{\langle z,x\rangle} - 1 - \langle z, x\rangle) F_s(\mathrm{d}x). \tag{8}$$

3 Lévy term structure models

We give a short review of the three basic interest rate models which are driven by time-inhomogeneous Lévy processes. Although the focus is on different rates in these three approaches, the models are closely related. All three of them are appropriate to price the standard interest rate derivatives.

3.1 The Lévy forward rate model

Modeling the dynamics of instantaneous forward rates is the starting point in the Heath–Jarrow–Morton approach ([15]). The forward rate model driven by Lévy processes was introduced in [14] and developed further in [11], where in particular a risk-neutral version was identified. The model was extended to driving time-inhomogeneous Lévy processes in [6] and [8]. In the former reference a complete classification of all equivalent martingale measures was achieved. As an unexpected consequence of this analysis, it turned out that under the standard assumption of deterministic coefficients for 1-dimensional driving processes there is a single martingale measure and thus – as in the Black–Scholes option pricing theory – there is a unique way to price interest rate derivatives. Explicit pricing formulae for caps, floors, swaptions, and other derivatives as well as efficient algorithms to evaluate these formulae are given in [8] and [9].

Denote by $B(t, T)$ the price at time t of a zero coupon bond with maturity T. Obviously $B(T, T) = 1$ for any maturity date $T \in [0, T^*]$. Since zero coupon bond prices can be deduced from instantaneous forward rates $f(t, T)$ via $B(t, T) = \exp\big(-\int_t^T f(t, u)\,\mathrm{d}u\big)$ and vice versa, the term structure can be modeled by specifying either of them. Here we specify the forward rates. Its dynamics is given for any $T \in [0, T^*]$ by

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)\,\mathrm{d}s - \int_0^t \sigma(s, T)\,\mathrm{d}L_s \tag{9}$$

where $L = (L_t)_{t\in[0,T^*]}$ is a PIIAC with local characteristics (b, c, F). For details concerning assumptions on the coefficients $\alpha(t, T)$ and $\sigma(s, T)$ we refer to [6] and [8]. The simplest case and at the same time the most important one for the implementation of the model is the case where α and σ are deterministic functions. Defining

$$A(s,T) := \int_{s\wedge T}^{T} \alpha(s,u)\,\mathrm{d}u \quad \text{and} \quad \Sigma(s,T) = \int_{s\wedge T}^{T} \sigma(s,u)\,\mathrm{d}u \tag{10}$$

one can derive the corresponding zero coupon bond prices in the form

$$B(t,T) = B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s\right) \tag{11}$$

where $r(s) := f(s,s)$ denotes the short rate. Choosing $T = t$ in (11) the risk-free savings account $B_t = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right)$ can be written as

$$B_t = \frac{1}{B(0,t)}\exp\left(\int_0^t A(s,t)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s\right). \tag{12}$$

Now assume that $\Sigma(s,T)$ is deterministic and

$$0 \le \sigma^i(s,T) \le M \qquad (1 \in \{1,\dots,d\}) \tag{13}$$

where M is the constant from assumption $\mathbb{EM}$. From (11) one sees immediately that discounted bond prices $B(t,T)/B_t$ are martingales for all $T \in [0,T^*]$ if we choose

$$A(s,T) := \theta_s(\Sigma(s,T)), \tag{14}$$

since $\int_0^t \theta_s(\Sigma(s,T))\,\mathrm{d}s$ is the exponential compensator of $\int_0^t \Sigma(s,T)\,\mathrm{d}L_s$. Thus we are in an arbitrage-free market. Another useful representation of zero coupon bond prices which follows from (11) and (12) is

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\left(-\int_0^t A(s,t,T)\,\mathrm{d}s + \int_0^t \Sigma(s,t,T)\,\mathrm{d}L_s\right) \tag{15}$$

where we used the abbreviations

$$A(s,t,T) := A(s,T) - A(s,t) \quad \text{and} \quad \Sigma(s,t,T) = \Sigma(s,T) - \Sigma(s,t). \tag{16}$$

3.2 The Lévy forward process model

This model was introduced in [12]. The advantage of this approach is that the driving process remains a time-inhomogeneous Lévy process during the backward induction which is done to get the rates in uniform form. Thus one can avoid any approximation and the model is easy to implement.

Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T^*$ denote a discrete tenor structure and set $\delta_k = T_{k+1} - T_k$. For zero coupon bond prices $B(t,T_k)$ and $B(t,T_{k+1})$ the forward process is defined by

$$F(t,T_k,T_{k+1}) = \frac{B(t,T_k)}{B(t,T_{k+1})}. \tag{17}$$

Therefore, modeling forward processes means specifying the dynamics of ratios of successive bond prices.

Let L^{T^*} be a time-inhomogeneous Lévy process on a complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{F}, \mathbb{P}_{T^*})$. The probability measure $\mathbb{P}_{T^*}$ can be interpreted as the forward measure associated with the settlement date T^*. The moment condition $\mathbb{EM}$ is assumed as before. The local characteristics of L^{T^*} are denoted by (b^{T^*}, c, F^{T^*}). Two parameters, c and F^{T^*}, are free parameters whereas the drift characteristic b^{T^*} will be chosen to guarantee that the forward process is a martingale. Since we proceed by backward induction, let us use the notation $T_i^* := T_{N+1-i}$ and $\delta_i^* = \delta_{N+1-i}$ for $i \in \{0,\ldots,N+1\}$. The following ingredients are needed:

($\mathbb{FP}$.1) For any maturity T_i there is a bounded, continuous, deterministic function $\lambda(\cdot,T_i) : [0,T^*] \to \mathbb{R}^d$ which represents the volatility of the forward process $F(\cdot,T_i,T_{i+1})$. We require for all $k \in \{1,\ldots,N\}$

$$\Big|\sum_{i=1}^{k} \lambda^j(s,T_i)\Big| \le M \qquad (s \in [0,T^*], j \in \{1,\ldots,d\}) \tag{18}$$

where M ist the constant in assumption $\mathbb{EM}$ and $\lambda(s,T_i) = 0$ for $s > T_i$.

($\mathbb{FP}$.2) The initial term structure of zero coupon bond prices $B(0,T_i)$, $1 \le i \le N+1$, is strictly positive. Consequently the initial values of the forward processes are given by

$$F(0,T_i,T_{i+1}) = \frac{B(0,T_i)}{B(0,T_{i+1})}. \tag{19}$$

We begin to construct the forward process with the longest maturity and postulate

$$F(t,T_1^*,T^*) = F(0,T_1^*,T^*) \exp\left(\int_0^t \lambda(s,T_1^*)\,\mathrm{d}L_s^{T^*}\right). \tag{20}$$

Now we choose b^{T^*} such that $F(\cdot,T_1^*,T^*)$ becomes a $\mathbb{P}_{T^*}$-martingale. This is achieved via the following equation

$$\begin{aligned}\int_0^t \langle \lambda(s,T_1^*), b_s^{T^*}\rangle\,\mathrm{d}s \;&=\; -\frac{1}{2}\int_0^t \langle \lambda(s,T_1^*), c_s\lambda(s,T_1^*)\rangle\,\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s,T_1^*),x\rangle} - 1 - \langle \lambda(s,T_1^*),x\rangle\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x),\end{aligned} \tag{21}$$

where $\nu^{T^*}(\mathrm{d}s,\mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\,\mathrm{d}s$ is the $\mathbb{P}_{T^*}$-compensator of the random measure of jumps μ^L given by the process L^{T^*}. Using Lemma 2.6 in [17] one can express the ordinary exponential (20) as a stochastic exponential, namely

$$F(t,T_1^*,T^*) = F(0,T_1^*,T^*)\mathcal{E}_t(H(\cdot,T_1^*))$$

where

$$\begin{aligned} H(t,T_1^*) = &\int_0^t \sqrt{c_s}\lambda(s,T_1^*)\,\mathrm{d}W_s^{T^*} \\ &+ \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle\lambda(s,T_1^*),x\rangle} - 1\right)(\mu^L - \nu^{T^*})(\mathrm{d}s,\mathrm{d}x). \end{aligned} \tag{22}$$

Since $F(\cdot,T_1^*,T^*)$ is a martingale we can define the forward martingale measure associated with the date T_1^* by setting

$$\frac{\mathrm{d}\mathbb{P}_{T_1^*}}{\mathrm{d}\mathbb{P}_{T^*}} = \frac{F(T_1^*,T_1^*,T^*)}{F(0,T_1^*,T^*)} = \mathcal{E}_{T_1^*}(H(\cdot,T_1^*)). \tag{23}$$

Using Girsanov's Theorem for semimartingales (see [16, Theorem III.3.24]) we can identify from (22) the predictable processes β and Y which describe the measure change, namely

$$\beta(s) = \lambda(s,T_1^*) \quad \text{and} \quad Y(s,x) = \exp\langle\lambda(s,T_1^*),x\rangle$$

Consequently $W_t^{T_1^*} := W_t^{T^*} - \int_0^t \sqrt{c_s}\lambda(s,T_1^*)\,\mathrm{d}s$ is a standard Brownian motion under $\mathbb{P}_{T_1^*}$ and $\nu^{T_1^*}(\mathrm{d}t,\mathrm{d}x) := \exp\langle\lambda(s,T_1^*),x\rangle\nu^{T^*}(\mathrm{d}t,\mathrm{d}x)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L.

Now we construct the forward process $F(\cdot,T_2^*,T_1^*)$ by postulating

$$F(t,T_2^*,T_1^*) = F(0,T_2^*,T_1^*)\exp\left(\int_0^t \lambda(s,T_2^*)\,\mathrm{d}L_s^{T_1^*}\right),$$

where

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*}\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_1^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T_1^*})(\mathrm{d}s,\mathrm{d}x).$$

The drift characteristic $b^{T_1^*}$ can be chosen in an analoguous way as in (21) and we define the next measure change from the resulting equation. Proceeding this way we get all forward processes in the form

$$F(t,T_i^*,T_{i-1}^*) = F(0,T_i^*,T_{i-1}^*)\exp\left(\int_0^t \lambda(s,T_i^*)\,\mathrm{d}L_s^{T_{i-1}^*}\right) \tag{24}$$

with

$$L_t^{T_{i-1}^*} = \int_0^t b_s^{T_{i-1}^*}\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_{i-1}^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T_{i-1}^*})(\mathrm{d}s,\mathrm{d}x). \tag{25}$$

$W^{T^*_{i-1}}$ is here a $\mathbb{P}_{T^*_{i-1}}$-standard Brownian motion and $\nu^{T^*_{i-1}}$ is the $\mathbb{P}_{T^*_{i-1}}$-compensator of μ^L given by

$$\nu^{T^*_{i-1}}(\mathrm{d}t, \mathrm{d}x) = \exp\left(\sum_{j=1}^{i-1}\langle\lambda(t, T^*_j), x\rangle\right) F_t^{T^*}(\mathrm{d}x)\,\mathrm{d}t. \tag{26}$$

The drift characteristic $b^{T^*_{i-1}}$ satisfies

$$\begin{aligned}\int_0^t \langle\lambda(s, T^*_i), b_s^{T^*_{i-1}}\rangle\,\mathrm{d}s \;=\; & -\frac{1}{2}\int_0^t \langle\lambda(s, T^*_i), c_s\lambda(s, T^*_i)\rangle\,\mathrm{d}s \\ & - \int_0^t\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T^*_i),x\rangle} - 1 - \langle\lambda(s, T^*_i), x\rangle\right)\nu^{T^*_{i-1}}(\mathrm{d}s, \mathrm{d}x).\end{aligned} \tag{27}$$

All driving processes $L^{T^*_i}$ remain time-inhomogeneous Lévy processes under the corresponding forward measures, since they differ only by deterministic drift terms.

3.3 The Lévy Libor model

This approach has been described in full detail in [12], therefore, we just list some of the properties. As in section 3.2 the model is constructed by backward induction along the discrete tenor structure and is driven by a time-inhomogeneous Lévy process L^{T^*} which is given on a complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{F}, \mathbb{P}_{T^*})$. As in the Lévy forward process model, $\mathbb{P}_{T^*}$ should be regarded as the forward measure associated with the settlement day T^*. L^{T^*} is required to satisfy assumption $\mathbb{EM}$ and can be written in the form

$$L_t^{T^*} = \int_0^t b_s^{T^*}\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x) \tag{28}$$

where $\nu^{T^*}(\mathrm{d}t, \mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\,\mathrm{d}t$ is the compensator of μ^L. The ingredients needed for the model are:

($\mathbb{LR}$.1) For any maturity T_i there is a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$ which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$. In addition

$$\sum_{i=1}^N |\lambda^j(s, T_i)| \le M \qquad (s \in [0, T^*], j \in \{1, \ldots, d\})$$

where M is the constant from assumption $\mathbb{EM}$ and $\lambda(s, T_i) = 0$ for $s > T_i$.

($\mathbb{LR}$.2) The initial term structure $B(0, T_i)$, $1 \le i \le N+1$, is strictly positive and strictly decreasing (in i). Consequently the initial term structure $L(0, T_i)$ of forward Libor rates is given by

$$L(0, T_i) = \frac{1}{\delta_i}\left(\frac{B(0, T_i)}{B(0, T_{i+1})} - 1\right) > 0.$$

Now we can start the induction by postulating that

$$L(t,T_1^*) = L(0,T_1^*)\exp\left(\int_0^t \lambda(s,T_1^*)\,\mathrm{d}L_s^{T^*}\right). \tag{29}$$

The drift characteristic b^{T^*} is chosen as in (21) to make this process a martingale. Writing (29) as a stochastic exponential and exploiting the relation $F(t,T_1^*,T^*) = 1+\delta_1^* L(t,T_1^*)$ one gets the dynamics in terms of the forward process $F(\cdot,T_1^*,T^*)$. From this the measure change can be done as in section 3.2. As a result of the backward induction one gets for each tenor time point the forward Libor rates in the form

$$L(t,T_j^*) = L(0,T_j^*)\exp\left(\int_0^t \lambda(s,T_j^*)\,\mathrm{d}L_s^{T_{j-1}^*}\right) \tag{30}$$

under the corresponding forward martingale measure $\mathbb{P}_{T_{j-1}^*}$. The successive forward measures are related by the following equation

$$\frac{\mathrm{d}\mathbb{P}_{T_j^*}}{\mathrm{d}\mathbb{P}_{T_{j-1}^*}} = \frac{1+\delta_j L(T_j^*,T_j^*)}{1+\delta_j L(0,T_j^*)}. \tag{31}$$

The driving process $L^{T_{j-1}^*}$ in (30) has the canonical representation

$$L_t^{T_{j-1}^*} = \int_0^t b_s^{T_{j-1}^*}\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_{j-1}^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T_{j-1}^*})(\mathrm{d}s,\mathrm{d}x). \tag{32}$$

$W^{T_{j-1}^*}$ is a $\mathbb{P}_{T_{j-1}^*}$-Brownian motion via

$$W_t^{T_{j-1}^*} = W_t^{T_{j-2}^*} - \int_0^t \sqrt{c_s}\alpha(s,T_{j-1}^*,T_{j-2}^*)\,\mathrm{d}s$$

where

$$\alpha(t,T_k^*,T_{k-1}^*) = \frac{\delta_k^* L(t-,T_k^*)}{1+\delta_k^* L(t-,T_k^*)}\lambda(t,T_k^*). \tag{33}$$

Similarly $\nu^{T_{j-1}^*}$ is the $\mathbb{P}_{T_{j-1}^*}$-compensator of μ^L which is related to the $\mathbb{P}_{T_{j-2}^*}$-compensator via

$$\nu^{T_{j-1}^*}(\mathrm{d}s,\mathrm{d}x) = \beta(s,x,T_{j-1}^*,T_{j-2}^*)\nu^{T_{j-2}^*}(\mathrm{d}s,\mathrm{d}x)$$

where

$$\beta(t,x,T_k^*,T_{k-1}^*) = \frac{\delta_k^* L(t-,T_k^*)}{1+\delta_k^* L(t-,T_k^*)}\left(e^{\langle\lambda(t,T_k^*),x\rangle}-1\right)+1. \tag{34}$$

The backward induction guarantees that zero coupon bond prices $B(\cdot,T_j)$ discounted by $B(\cdot,T_k)$ i.e. ratios $B(\cdot,T_j)/B(\cdot,T_k)$ are $\mathbb{P}_{T_k}$-martingales for all

$j, k \in \{1, \ldots, N+1\}$, and thus we have an arbitrage-free market. This follows directly for successive tenor time points from the relation $1 + \delta L(t, T_j) = B(t, T_j)/B(t, T_{j+1})$ since $L(t, T_j)$ is by construction a $\mathbb{P}_{T_{j+1}}$-martingale. Expanding ratios with arbitrary tenor time points T_j and T_k into products of ratios with successive time points one gets the result from this special case. To see this, one has to use Proposition 3.8 in [16, p. 168], which is a fundamental result for the analysis of all interest rate models where forward martingale measures are used.

Note that the driving processes $L_t^{T^*_{j-1}}$ which are derived during the backward induction are no longer time-inhomogeneous Lévy processes. This is clear from (34) since due to the random term $\beta(s, x, T^*_{j-1}, T^*_{j-2})$, the compensator $\nu^{T^*_{j-1}}$ is no longer deterministic. One can force the process $\beta(\cdot, x, T^*_{j-1}, T^*_{j-2})$ to become deterministic by replacing $L(t-, T^*_k)/(1+\delta^*_k L(t-, T^*_k))$ by its starting value $L(0, T^*_k)/(1 + \delta^*_k L(0, T^*_k))$ in (34). This approximation is convenient for the implementation of the model, since then all driving processes are time-inhomogeneous Lévy processes. Since the process $Y(\cdot, x)$, which is used in the change from one compensator to the next in the forward process approach, is non-random, one can implement the model from section 3.2 without any approximation.

4 Embedding of the forward process model

In this section we show that the Lévy forward process model can be seen as a special case of the Lévy forward rate model. We will choose the parameters of the latter in such a way that we get the forward process specification as shown in (24)–(27). In the martingale case, which is defined by (14), according to (15) zero coupon bond prices can be represented in the form

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(\int_0^t \Big(\widetilde{\theta}_s(\varSigma(s,t)) - \widetilde{\theta}_s(\varSigma(s,T)) \Big) \, \mathrm{d}s + \int_0^t \varSigma(s,t,T) \, \mathrm{d}\widetilde{L}_s \right), \tag{35}$$

where $\widetilde{L}$ is a time-inhomogeneous Lévy process with characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ under the (spot martingale) measure $\mathbb{P}$. $\widetilde{L}$ satisfies assumption $\mathbb{EM}$ and $\widetilde{\theta}_s$ denotes the cumulant associated with the triplet $(\widetilde{b}_s, \widetilde{c}_s, \widetilde{F}_s)$. Recall that $\varSigma(s,T) = \int_{s \wedge T}^T \sigma(s,u) \, \mathrm{d}u$.

The forward martingale measure $\mathbb{P}_{T_i^*}$ associated with the settlement date T_i^* is related to the spot martingale measure $\mathbb{P}$ via the Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_{T_i^*}}{\mathrm{d}\mathbb{P}} = \frac{1}{B_{T_i^*} B(0, T_i^*)} \qquad \mathbb{P}\text{-a.s.}$$

Choosing $T = t = T_i^*$ in (11) one gets immediately the representation

$$\frac{\mathrm{d}\mathbb{P}_{T_i^*}}{\mathrm{d}\mathbb{P}} = \exp\left(- \int_0^{T_i^*} \widetilde{\theta}_s(\varSigma(s, T_i^*)) \, \mathrm{d}s + \int_0^{T_i^*} \varSigma(s, T_i^*) \, \mathrm{d}\widetilde{L}_s \right). \tag{36}$$

Since σ and, therefore, Σ are deterministic functions, $\widetilde{L}$ is also a time-inhomogeneous Lévy process with respect to $\mathbb{P}_{T_i^*}$ and its $\mathbb{P}_{T_i^*}$-characteristics $(\widetilde{b}^{T_i^*}, \widetilde{c}^{T_i^*}, \widetilde{F}^{T_i^*})$ are given by

$$\begin{aligned}
\widetilde{b}_s^{T_i^*} &= \widetilde{b}_s + \widetilde{c}_s \Sigma(s, T_i^*) + \int_{\mathbb{R}^d} \left(e^{\langle \Sigma(s,T_i^*),x\rangle} - 1 \right) x \, \widetilde{F}_s(\mathrm{d}x), \\
\widetilde{c}_s^{T_i^*} &= \widetilde{c}_s, \\
\widetilde{F}_s^{T_i^*}(\mathrm{d}x) &= e^{\langle \Sigma(s,T_i^*),x\rangle} \widetilde{F}_s(\mathrm{d}x).
\end{aligned} \tag{37}$$

Since $\widetilde{L}$ is also a $\mathbb{P}_{T_i^*}$-special semimartingale, it can be written in its $\mathbb{P}_{T_i^*}$-canonical representation as

$$\widetilde{L}_t = \int_0^t \widetilde{b}_s^{T_i^*} \, \mathrm{d}s + \int_0^t \sqrt{\widetilde{c}_s} \, \mathrm{d}W_s^{T_i^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*})(\mathrm{d}s, \mathrm{d}x), \tag{38}$$

where $W^{T_i^*}$ is a $\mathbb{P}_{T_i^*}$-standard Brownian motion and where $\widetilde{\nu}^{T_i^*}(\mathrm{d}s, \mathrm{d}x) := \widetilde{F}_s^{T_i^*}(\mathrm{d}x) \, \mathrm{d}s$ is the $\mathbb{P}_{T_i^*}$-compensator of $\mu^{\widetilde{L}}$, the random measure associated with the jumps of the process $\widetilde{L}$.

Using this representation in (35) we derive the forward process

$$\begin{aligned}
F(t, T_{i+1}^*, T_i^*) &= \frac{B(t, T_{i+1}^*)}{B(t, T_i^*)} \\
&= \frac{B(0, T_{i+1}^*)}{B(0, T_i^*)} \exp\bigg(\int_0^t \Big(\widetilde{\theta}_s(\Sigma(s, T_i^*)) - \widetilde{\theta}_s(\Sigma(s, T_{i+1}^*)) \Big) \, \mathrm{d}s \\
&\qquad + \int_0^t \Sigma(s, T_i^*, T_{i+1}^*) \, \mathrm{d}\widetilde{L}_s \bigg) \\
&= F(0, T_{i+1}^*, T_i^*) \exp\bigg(I_t^1 + I_t^2 + \int_0^t \sqrt{\widetilde{c}_s} \Sigma(s, T_i^*, T_{i+1}^*) \, \mathrm{d}W_s^{T_i^*} \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \langle \Sigma(s, T_i^*, T_{i+1}^*), x \rangle \big(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*} \big)(\mathrm{d}s, \mathrm{d}x) \bigg).
\end{aligned}$$

Here

$$\begin{aligned}
I_t^1 &:= \int_0^t \Big(\widetilde{\theta}_s(\Sigma(s, T_i^*)) - \widetilde{\theta}_s(\Sigma(s, T_{i+1}^*)) \Big) \, \mathrm{d}s \\
&= \int_0^t \bigg[- \langle \Sigma(s, T_i^*, T_{i+1}^*), \widetilde{b}_s \rangle \\
&\qquad + \frac{1}{2} \langle \Sigma(s, T_i^*), \widetilde{c}_s \Sigma(s, T_i^*) \rangle - \frac{1}{2} \langle \Sigma(s, T_{i+1}^*), \widetilde{c}_s \Sigma(s, T_{i+1}^*) \rangle \\
&\qquad + \int_{\mathbb{R}^d} \big(e^{\langle \Sigma(s,T_i^*),x\rangle} - e^{\langle \Sigma(s,T_{i+1}^*),x\rangle} + \langle \Sigma(s, T_i^*, T_{i+1}^*), x \rangle \big) \widetilde{F}_s(\mathrm{d}x) \bigg] \mathrm{d}s
\end{aligned}$$

and making use of the first equation in (37)

$$
\begin{aligned}
I_t^2 &:= \int_0^t \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{b}_s^{T_i^*} \rangle \,\mathrm{d}s \\
&= \int_0^t \Big[\langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{b}_s \rangle + \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{c}_s \Sigma(s,T_i^*) \rangle \\
&\qquad + \int_{\mathbb{R}^d} \langle \Sigma(s,T_i^*,T_{i+1}^*), x \rangle \big(e^{\langle \Sigma(s,T_i^*),x\rangle} - 1 \big) \widetilde{F}_s(\mathrm{d}x) \Big] \,\mathrm{d}s.
\end{aligned}
$$

Summing up I^1 and I^2 yields

$$
\begin{aligned}
I_t^1 + I_t^2 &= -\frac{1}{2} \int_0^t \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{c}_s \Sigma(s,T_i^*,T_{i+1}^*) \rangle \,\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \big(e^{\langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle} - 1 - \langle \Sigma(s,T_i^*,T_{i+1}^*), x \rangle \big) \widetilde{F}_s^{T_i^*}(\mathrm{d}x) \,\mathrm{d}s.
\end{aligned}
$$

Hence, the forward process is given by

$$
\begin{aligned}
&F(t,T_{i+1}^*,T_i^*) \\
&\quad = F(0,T_{i+1}^*,T_i^*) \exp\bigg(-\frac{1}{2} \int_0^t \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{c}_s \Sigma(s,T_i^*,T_{i+1}^*) \rangle \,\mathrm{d}s \\
&\qquad - \int_0^t \int_{\mathbb{R}^d} \big(e^{\langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle} - 1 - \langle \Sigma(s,T_i^*,T_{i+1}^*), x \rangle \big) \widetilde{F}_s^{T_i^*}(\mathrm{d}x) \,\mathrm{d}s \\
&\qquad + \int_0^t \sqrt{\widetilde{c}_s} \Sigma(s,T_i^*,T_{i+1}^*) \,\mathrm{d}W_s^{T_i^*} \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \langle \Sigma(s,T_i^*,T_{i+1}^*), x \rangle \big(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*} \big)(\mathrm{d}s,\mathrm{d}x) \bigg).
\end{aligned}
$$

Now we shall specify the model parameters, that is the volatility σ and the characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ of $\widetilde{L}$, in such a way that the forward process dynamics match the dynamics given in (24)–(27). First, we choose

$$
\Sigma(s,T_i^*,T_{i+1}^*) = \lambda(s,T_{i+1}^*).
$$

This can be reached by setting

$$
\sigma(s,u) := -\sum_{i=0}^{N} \frac{1}{\delta_{i+1}^*} \lambda(s,T_{i+1}^*) \mathbb{1}_{[T_{i+1}^*,T_i^*)}(u)
$$

since

$$
\Sigma(s,T_i^*,T_{i+1}^*) = -\int_{T_{i+1}^*}^{T_i^*} \sigma(s,u) \,\mathrm{d}u = \lambda(s,T_{i+1}^*).
$$

Of course there are many other possibilities to specify σ. It could also be chosen to be continuous or smooth in the second variable.

Next, we specify the triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$. $\widetilde{b}_s$ can be chosen arbitrary. We set $\widetilde{c}_s = c_s$ and

$$\widetilde{F}_s(\mathrm{d}x) = \exp\langle -\Sigma(s, T^*), x\rangle F_s^{T^*}(\mathrm{d}x), \tag{39}$$

where F^{T^*} is the third characteristic of the driving process L^{T^*} in the Lévy forward process model. Then using the third equation in (37)

$$\begin{aligned}
\widetilde{F}_s^{T_i^*}(\mathrm{d}x) &= \exp\langle \Sigma(s, T_i^*) - \Sigma(s, T^*), x\rangle F_s^{T^*}(\mathrm{d}x) \\
&= \exp\Big(\sum_{j=1}^{i} \langle \Sigma(s, T_j^*) - \Sigma(s, T_{j-1}^*), x\rangle\Big) F_s^{T^*}(\mathrm{d}x) \\
&= \exp\Big(\sum_{j=1}^{i} \langle \lambda(s, T_j^*), x\rangle\Big) F_s^{T^*}(\mathrm{d}x)
\end{aligned}$$

and we arrive at the forward process

$$F(t, T_{i+1}^*, T_i^*) = F(0, T_{i+1}^*, T_i^*) \exp\Big(\int_0^t \lambda(s, T_{i+1}^*)\, \mathrm{d}\widetilde{L}_s^{T_i^*}\Big),$$

where

$$\widetilde{L}_t^{T_i^*} = \int_0^t b_s^{T_i^*}\, \mathrm{d}s + \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_i^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*})(\mathrm{d}s, \mathrm{d}x).$$

The $\mathbb{P}_{T_i^*}$-compensator $\widetilde{\nu}^{T_i^*}$ of $\mu^{\widetilde{L}}$ is given by

$$\widetilde{\nu}^{T_i^*}(\mathrm{d}t, \mathrm{d}x) = \exp\Big(\sum_{j=1}^{i} \langle \lambda(t, T_j^*), x\rangle\Big) F_t^{T^*}(\mathrm{d}x)\, \mathrm{d}t$$

and finally $(b_s^{T_i^*})$ satisfies

$$\begin{aligned}
&\int_0^t \langle \lambda(s, T_{i+1}^*), b_s^{T_i^*}\rangle\, \mathrm{d}s \\
&\qquad = -\frac{1}{2} \int_0^t \langle \lambda(s, T_{i+1}^*), \widetilde{c}_s \lambda(s, T_{i+1}^*)\rangle\, \mathrm{d}s \\
&\qquad\quad - \int_0^t \int_{\mathbb{R}^d} \Big(e^{\langle \lambda(s, T_{i+1}^*), x\rangle} - 1 - \langle \lambda(s, T_{i+1}^*), x\rangle\Big) \widetilde{\nu}^{T_i^*}(\mathrm{d}s, \mathrm{d}x).
\end{aligned}$$

Remark 1. This embedding works only for driving processes that are time-inhomogeneous Lévy processes. If both models are driven by a process with stationary increments, that is $F_s^{T^*}$ and $\widetilde{F}_s$ do not depend on s, in general we cannot embed the forward process model in the forward rate model.

5 Calibration of the Lévy forward rate model

5.1 The real-world measure

In this section we consider the Lévy forward rate model with a time-homogeneous driving process L, i.e. L has stationary increments. The goal is to estimate the parameters of the driving process under the real-world measure. For this purpose we use market data of discount factors (zero coupon bond prices) for one year up to ten years, quoted between September 17, 1999, and September 17, 2001, i.e. for 522 trading days.

The parameter estimation in the forward rate model is substantially more difficult than in a stock price model. The reason is that we have a number of different assets, namely ten bonds in the case of our data set (in theory of course an infinite number), but only one driving process. Therefore, we have to find a way to extract the parameters of the driving process from the *log returns* of all ten bond prices.

Let us start by considering the logarithm of the ratio between the bond price and its forward price on the day before, i.e.

$$\mathrm{LR}(t,T) := \log \frac{B(t+1,t+T)}{B(t,t+1,t+T)}.$$

Here, $B(t,t+1,t+T)$ is the forward price of $B(t+1,t+T)$ at time t, i.e.

$$B(t,t+1,t+T) := \frac{B(t,t+T)}{B(t,t+1)}.$$

We call LR the *daily log return.* Using (15) we get

$$\begin{aligned} \mathrm{LR}(t,T) &= \log B(t+1,t+T) - \log B(t,t+T) + \log B(t,t+1) \\ &= -\int_t^{t+1} A(s,t+1,t+T)\,\mathrm{d}s + \int_t^{t+1} \Sigma(s,t+1,t+T)\,\mathrm{d}L_s. \end{aligned} \tag{40}$$

In what follows, we consider for simplicity the Ho–Lee volatility structure, i.e. $\Sigma(s,T) = \widehat{\sigma}(T-s)$ for a constant $\widehat{\sigma}$, which we set equal to one without loss of generality. Similar arguments can be carried out for other stationary volatility structures, as e.g. the Vasiček volatility function. By *stationary* we mean that $\Sigma(s,T)$ depends only on $(T-s)$. We assume that the drift term also satisfies some stationarity condition, namely

$$A(s,T) = A(0,T-s) \qquad \text{for } s \le t.$$

In the risk-neutral case given by (14) this stationarity follows from the stationarity of the volatility function $\Sigma(s,T)$. We get

$$-\int_t^{t+1} A(s,t+1,t+T)\,\mathrm{d}s = -\int_0^1 A(s,1,T)\,\mathrm{d}s =: d(T) \tag{41}$$

independent of t and

$$\int_t^{t+1} \Sigma(s, t+1, t+T)\, \mathrm{d}L_s = (T-1)(L_{t+1} - L_t).$$

Consequently,

$$\mathrm{LR}(t,T) = d(T) + c(T)Y_{t+1} \tag{42}$$

where $c(T) := (T-1)$ is deterministic and

$$Y_{t+1} := L_{t+1} - L_t \;\sim\; L_1$$

is $\mathcal{F}_{t+1}$ measurable and does not depend on T.

To estimate the parameters of the driving process we first determine the daily log returns, i.e. for $k \in \{0, 1, \dots, 520\}$, $n \in \{1, \dots, 10\}$

$$\mathrm{LR}(k, k+(n \text{ years})) = \log B(k+1, k+(n \text{ years})) \;+\; \log \frac{B(k, k+1)}{B(k, k+(n \text{ years}))}.$$

Unfortunately, we can only get $B(k, k+(n \text{ years}))$ directly from our data set. To determine $B(k+1, k+(n \text{ years}))$ and $B(k, k+1)$ we use an idea developed in [22, Section 5.3] and interpolate the negative of the logarithm of the bond prices with a cubic spline. We do this procedure separately for each day of the data set. Figure 1 shows the interpolation for the first day (even for maturities up to 30 years).

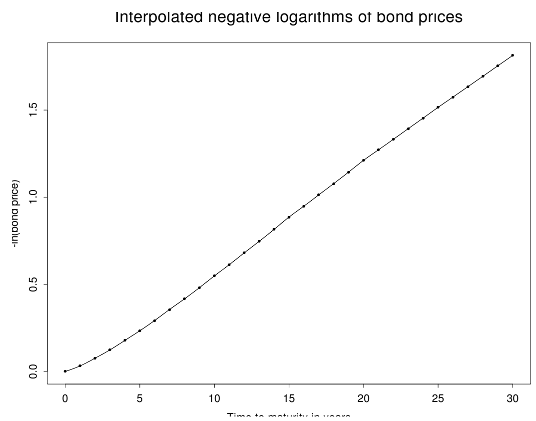

Fig. 1. Negative logarithms of bond prices on September 17, 1999 and interpolating cubic spline.

Since $\mathbb{E}[L_1] = 0$, we know that

$$\mathrm{LR}(t,T) - \mathbb{E}[\mathrm{LR}(t,T)] = (T-1)Y_{t+1}, \tag{43}$$

i.e. the centered log returns are affine linear in T. Moreover, $Y_1, Y_2, \ldots Y_{521}$ are independent and equal to L_1 in distribution. The corresponding samples $y_1, y_2, \ldots, y_{521}$ could be calculated for a fixed $n \in \{1, 2, \ldots, 10\}$ via

$$y_{k+1} := \frac{\mathrm{LR}(k, k + (n \text{ years})) - \bar{x}_n}{(n \text{ years}) - 1} \qquad \text{with} \tag{44}$$

$$\bar{x}_n := \frac{1}{521} \sum_{k=0}^{520} \mathrm{LR}(k, k + (n \text{ years})). \tag{45}$$

However, since the centered empirical log returns $\mathrm{LR}(k, k + (n \text{ years})) - \bar{x}_n$ are not exactly affine linear in n (compare Figure 2), the y_{k+1} in (44) would then depend on n. Remember that the distribution of L_1 in the Lévy forward rate model does not depend on the time to maturity of the bonds. Therefore, we take a different approach and use the points

$$((1 \text{ year})-1, \mathrm{LR}(k, k+(1 \text{ year}))-\bar{x}_1), \ldots, ((10 \text{ years})-1, \mathrm{LR}(k, k+(10 \text{ years}))-\bar{x}_{10})$$

for a linear regression through the origin. The gradient of the straight line yields the value for y_{k+1}. Figure 2 shows the centered empirical log returns and the regression line for the first day of the data set. Repeating this procedure for each day provides us with the samples $y_1, y_2, \ldots, y_{521}$ which can now be used to estimate the parameters of L_1 by using maximum likelihood estimation.

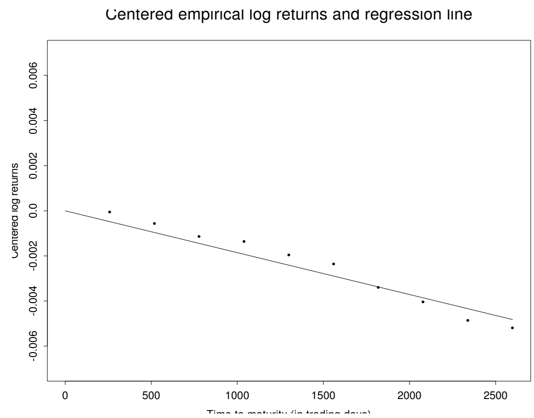

Fig. 2. Centered empirical log returns and regression line for the first day of the data set.

The parametric class of distributions we use here are generalized hyperbolic distributions (see e.g. [5]) or subclasses such as hyperbolic [7] or normal inverse Gaussian (NIG) distributions [2]. The resulting densities for our data set are shown in Figure 3. Figure 4 shows the same densities on a log-scale,

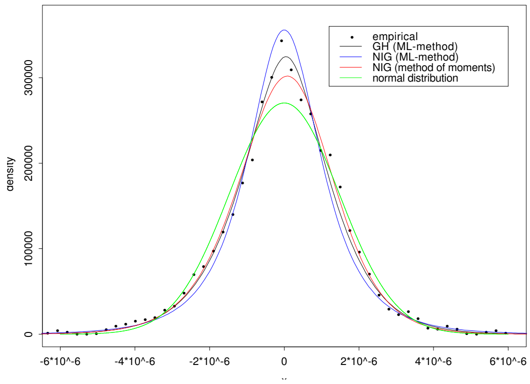

Fig. 3. Densities of empirical and estimated distributions.

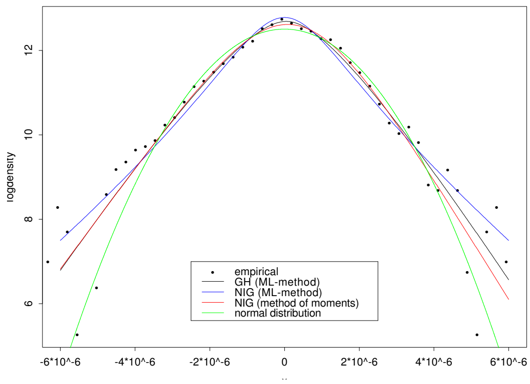

Fig. 4. Log-densities of empirical and estimated distributions.

Table 1. Estimated parameters of the distribution L_1 under the measure $\mathbb{P}$.

α	β	δ	μ	λ	Method
1474224	-34659	1.1892e-12	7.5642e-08	2.37028	Max-Likelihood (GH)
590033	-14	1.3783e-06	3.2426e-11	-0.5	Max-Likelihood (NIG)
1195475	-114855	2.5614e-06	2.4723e-07	-0.5	Moments (NIG)

which allows to see the fit in the tails. The estimated distribution parameters corresponding to the densities in Figure 3 are given in Table 1.

One of the densities in Figures 3 and 4 was estimated using the method of moments. This is a somewhat simpler approach where one exploits the relation between moments of order i and the i-th cumulant (see [1, 26.1.13 and 26.1.14]). Since for generalized hyperbolic distributions the cumulants are explicitly known, one can express the moments (up to order 4) as functions of the distribution parameters λ, α, β, δ, and μ. Applying the usual estimators for moments one gets the parameters.

5.2 The risk-neutral measure

There are very liquid markets for the basic interest rate derivatives such as caps, floors, and swaptions. Therefore, market prices for these instruments – typically quoted in terms of their (implied) volatilities with respect to the standard Gaussian model – contain a maximum of information. A cap is a series of call options on subsequent variable interest rates, namely Libor rates. Each option is called a caplet. It is easy to see that the payoff of a caplet can be expressed as the payoff of a put option on a zero coupon bond. In the same way a floor is a series of floorlets and each floorlet is equivalent to a call option on a bond. Thus to price a floorlet one has to price a call on a bond. According to the general no-arbitrage valuation theory the time-0-value of a call with strike K and maturity t on a bond with maturity T is

$$C_0(t,T,K) := \mathbb{E}\left[\frac{1}{B_t}(B(t,T)-K)^+\right], \tag{46}$$

where the expectation is taken with respect to the risk-neutral measure (*spot martingale measure*). Since one would need the joint distribution of B_t and $B(t,T)$ to evaluate this expectation, it is more efficient to express this expectation with respect to the forward measure associated with time t. One then gets

$$C_0(t,T,K) = B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T)-K)^+]. \tag{47}$$

Once one has numerically efficient algorithms to compute these expectations, one can calibrate the model by minimizing the differences between model prices and market quotes simultaneously across all available option maturities and strikes. This has been described in detail in [8].

Let us mention that in the stationary case one can derive risk-neutral parameters from the real-world parameters which we estimated in 5.1. Because of the martingale drift condition (14) we get from (41)

$$d(T) = \int_0^1 (A(s,1)-A(s,T))\,\mathrm{d}s = \int_0^1 (\theta(\Sigma(\sigma,1)) - \theta(\Sigma(s,T)))\,\mathrm{d}s \tag{48}$$

where θ is the logarithm of the moment generating function of $\mathcal{L}(L_1)$ under the risk-neutral measure. By stationarity $\mathcal{L}(Y_{n+1}) = \mathcal{L}(L_1)$ and $\mathbb{E}[L_1] = 0$, therefore, (42) implies

$$\mathbb{E}[\mathrm{LR}(t,T)] = d(T). \tag{49}$$

The arithmetic mean of the empirical log returns (45) is an estimator for the expectation on the left side. By a minimization procedure one can now extract the distribution parameters of $\mathcal{L}(L_1)$, which are implicit on the right hand side of (48) from this equation.

In the case of generalized hyperbolic Lévy processes as driving processes it is known from [22] that the parameters μ and δ do not change when one switches from the real-world to the risk-neutral distribution. Therefore, one has only to extract the remaining parameters λ, α, β via minimization of the distance between average log returns and the integral on the right side in (48).

It is clear that due to the highly liquid market in interest rate derivatives the direct fit to market quotes as described at the beginning of this section provides more reliable calibration results than the derivation from real-world parameters. It will, therefore, be preferred by practitioners.

6 Calibration of forward process and Libor models

Recall that a cap is a sequence of call options on subsequent Libor rates. Each single option is called a caplet. Let a discrete tenor structure $0 = T_0 < T_1 < \cdots < T_{n+1} = T^*$ be given as before. The time-T_j payoff of a caplet which is settled in arrears is

$$N\delta_{j-1}(L(T_{j-1}, T_{j-1}) - K)^+ \tag{50}$$

where K is the strike and N the notional amount which we assume to be 1. The corresponding payoff of a floorlet settled in arrears is

$$N\delta_{j-1}(K - L(T_{j-1}, T_{j-1}))^+. \tag{51}$$

The time-t price of the cap is then

$$C_t(K) = \sum_{j=1}^{N+1} B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\left[\delta_{j-1}(L(T_{j-1}, T_{j-1}) - K)^+|\mathcal{F}_t\right] \tag{52}$$

Given this formula, the Libor model is the natural approach to price caps since then the Libor rates $L(T_{j-1}, T_{j-1})$ are given in the simple form (30) with respect to the forward martingale measure $\mathbb{P}_{T_j}$.

Numerically efficient ways based on bilateral Laplace transforms to evaluate the expectations in (52) are described in [12]. For numerical purposes the non-deterministic compensators which arise during the backward induction in the Lévy Libor model can be approximated by deterministic ones. Concretely, the stochastic ratios $\delta_j L(s-, T_j)/(1 + \delta_j L(s-, T_j)$ are replaced by their deterministic initial values $\delta_j L(0, T_j)/(1 + \delta_j L(0, T_j))$. An alternative approximation method which is numerically much faster, is described in [18, Section 3.2.1].

Instead of basing the pricing on the Libor model one can use the forward process approach outlined in Section 3.2. It is then more natural to write the caplet payoff (50) in the form

$$\big(1 + \delta_{j-1} L(T_{j-1}, T_{j-1}) - \widetilde{K}_{j-1}\big)^+ \tag{53}$$

where $\widetilde{K}_{j-1} = 1 + \delta_{j-1}K$. Since $1 + \delta_{j-1}L(T_{j-1}, T_{j-1}) = F(T_{j-1}, T_{j-1}, T_j)$, the pricing formula is then instead of (52)

$$C_t(K) = \sum_{j=1}^{N+1} B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\big[(F(T_{j-1}, T_{j-1}, T_j) - \widetilde{K}_{j-1})^+ | \mathcal{F}_t\big]. \tag{54}$$

The implementation of this approach leads to a much faster algorithm since the backward induction is more direct in the case of the forward process model. Also, any approximation can be avoided here.

In the implementations we use mildly time-inhomogeneous Lévy processes, namely those which are piecewise homogeneous Lévy processes. In order to catch the terms structure of smiles, which one sees in implied volatility surfaces (see [8, Figure 1]), with sufficient accuracy, typically three Lévy parameter sets are needed: one Lévy process corresponding to short maturities up to 1 year roughly, a second one for maturities between 1 and 5 years, and a third Lévy process corresponding to long maturities from 5 to 10 years. Instead of predetermining the breakpoints where the Lévy parameters change, one can actually include the choice of the breakpoints in the estimation procedure. These random breakpoints improve the calibration results further.

According to ($\mathbb{FP}$.1) and ($\mathbb{LR}$.1) a volatility structure has to be chosen in both models. In [3] a broad spectrum of suitable volatility structures is discussed, which can be used in the forward process or the Libor model. A sufficiently flexible structure is given by

$$\lambda(t, T_j) = a(T_j - t)\exp(-b(T_j - t)) + c \tag{55}$$

with three parameters a, b, c. Note that we consider 1-dimensional processes and consequently also scalar volatility functions in all calibrations. Figure 5 shows a variety of shapes produced by formula (55).

Without loss of generality one can set $a = 1$, since this parameter can be included in the specification of the Lévy process. To see this take for example a Lévy process L generated by a normal inverse Gaussian distribution, i.e. $\mathcal{L}(L_1) = \mathrm{NIG}(\alpha, \beta, \delta, \mu)$ and $a \neq 1$. Define $\widetilde{L} = aL$ and $\widetilde{\lambda}(t, T) = \lambda(t, T)/a$ then (compare e.g. [5]) $\mathcal{L}(\widetilde{L}_1) = \mathrm{NIG}(\frac{\alpha}{|a|}, \frac{\beta}{|a|}, |a|\delta, a\mu)$ and $\int_0^t \lambda(s, T)\,\mathrm{d}L_s = \int_0^t \widetilde{\lambda}(s, T)\,\mathrm{d}\widetilde{L}_s$. Thus the model $(\widetilde{\lambda}, \widetilde{L})$ is exactly of the same type with parameter $a = 1$.

As far as the driving process L is concerned we consider one dimensional processes generated by generalized hyperbolic distributions $\mathrm{GH}(\lambda, \alpha, \beta, \delta, \mu)$ or subclasses like hyperbolic, where $\lambda = 1$, or normal inverse Gaussian distributions, where $\lambda = -\frac{1}{2}$. The generalized hyperbolic distribution is given by its characteristic function

$$\varPhi_{\mathrm{GH}}(u) = e^{\mathrm{i}\mu u}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + \mathrm{i}u)^2}\right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \tag{56}$$

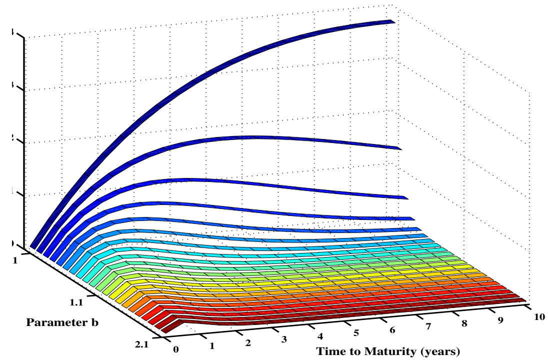

Fig. 5. A variety of shapes for the instantaneous volatility curve produced by (55) with $a = 1$, $b \in [0.1, 2.1]$, $c = 0.1$. Source: [19].

where K_λ denotes the modified Bessel function of the third kind with index λ. For normal inverse Gaussian distributions this simplifies to

$$\Phi_{\mathrm{NIG}}(\mu) = e^{\mathrm{i}\mu u} \frac{\exp(\delta\sqrt{\alpha^2 - \beta^2})}{\exp(\delta\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2})}. \tag{57}$$

The class of generalized hyperbolic distributions is so flexible that one does not have to consider higher-dimensional driving processes. Note that it would not be appropriate to classify a model driven by a one-dimensional Lévy process as a one-factor model, since the driving Lévy process itself is already a high-dimensional object. The notion of an x-factor model ($x = 1, 2, \ldots, n$) should be reserved for the world of classical Gaussian models.

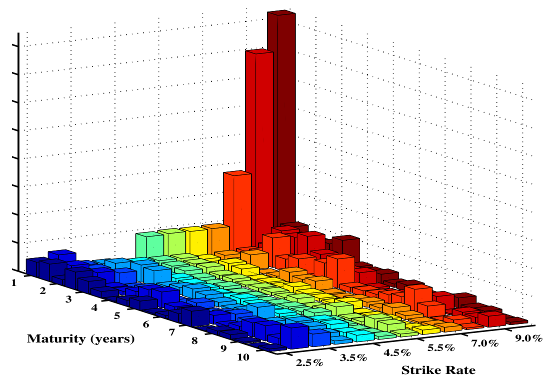

Fig. 6. Data set I. Absolute errors of EUR caplet calibration: Forward process model. Source: [19].

To extract the model parameters from market quotes, one considers for each caplet the (possibly squared) difference between the market and model price. The objective function which has to be minimized is then the weighted sum over all strikes and over all caplets along the tenor structure. A possible choice of weights would be the at the money prices for the respective maturity. Figure 6 shows how close one gets to the empirical volatility surface on the basis of the forward process model. The figure shows the absolute difference between model and market prices expressed in volatilities. For the relevant part of the moneyness–maturity plane the differences are below 1 %. The large deviations at the short end are of no importance since the one year caplet prices are of the order of magnitude 10^{-9} for the strike rates 8, 9 and 10 %. The underlying data set consists of cap prices in the Euro market on February 19, 2002. The differences one gets for the same data set on the basis of the Lévy Libor model are shown in Figure 7.

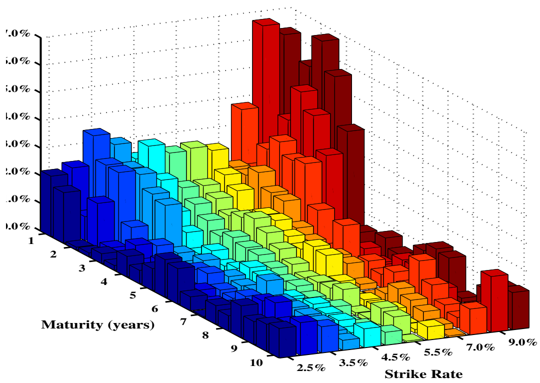

Fig. 7. Data set I. Absolute errors of EUR caplet calibration: Libor rate model. Source: [19].

Comparing the two calibration results, one sees that the forward process approach yields a more accurate fit than the Libor approach. In [10] both, the Lévy Libor as well as the Lévy forward process approach, have been extended to a multicurrency setting which takes the interplay between interest rates and foreign exchange rates into account. This model is also driven by a single time-inhomogeneous Lévy process, namely the process which drives the most distant forward Libor rate (or forward process) in the domestic market. Implementation of this sophisticated model was tested for up to three currencies (EUR, USD, and GBP).

References

1. M. Abramowitz and I. A. Stegun (1968). *Handbook of Mathematical Functions.* Dover Publications.
2. O. E. Barndorff-Nielsen (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics 2*, 41–68.
3. D. Brigo and F. Mercurio (2001). *Interest Rate Models: Theory and Practice.* Springer.
4. P. Carr, H. Geman, D. Madan, and M. Yor (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business 75*, 305–332.
5. E. Eberlein (2001). Application of generalized hyperbolic Lévy motions to finance. In O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy Processes: Theory and Applications*, pp. 319–337. Birkhäuser Verlag.
6. E. Eberlein, J. Jacod, and S. Raible (2005). Lévy term structure models: no-arbitrage and completeness. *Finance and Stochastics 9*, 67–88.
7. E. Eberlein and U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli 1*, 281–299.
8. E. Eberlein and W. Kluge (2006a). Exact pricing formulae for caps and swaptions in a Lévy term structure model. *The Journal of Computational Finance 9*(2), 99–125.
9. E. Eberlein and W. Kluge (2006b). Valuation of floating range notes in Lévy term structure models. *Mathematical Finance 16*, 237–254.
10. E. Eberlein and N. Koval (2006). A cross-currency Lévy market model. *Quantitative Finance 6.*
11. E. Eberlein and F. Özkan (2003). The defaultable Lévy term structure: ratings and restructuring. *Mathematical Finance 13*, 277–300.
12. E. Eberlein and F. Özkan (2005). The Lévy Libor model. *Finance and Stochastics 9*, 327–348.
13. E. Eberlein and K. Prause (2002). The generalized hyperbolic model: financial derivatives and risk measures. In H. Geman, D. Madan, S. Pliska, and T. Vorst (Eds.), *Mathematical Finance – Bachelier Congress 2000*. Springer.
14. E. Eberlein and S. Raible (1999). Term structure models driven by general Lévy processes. *Mathematical Finance 9*, 31–53.
15. E. Heath, R. A. Jarrow, and A. Morton (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica 60*, 77–105.
16. J. Jacod and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer.
17. J. Kallsen and A. N. Shiryaev (2002). The cumulant process and Esscher's change of measure. *Finance and Stochastics 6*, 397–428.
18. W. Kluge (2005). *Time-inhomogeneous Lévy processes in interest rate and credit risk models.* Ph.D. thesis, University of Freiburg, Germany.
19. N. Koval (2005). *Time-inhomogeneous Lévy processes in cross-currency market models.* Ph.D. thesis, University of Freiburg, Germany.
20. D. Madan and F. Milne (1987). Option pricing with V.G. martingale components. *Mathematical Finance 1*(4), 39–55.
21. D. Madan and E. Seneta (1990). The variance gamma (V.G.) model for share market returns. *Journal of Business 63*, 511–524.
22. S. Raible (2000). *Lévy processes in finance: Theory, numerics, and empirical facts.* Ph.D. thesis, University of Freiburg, Germany.

23. W. Schoutens (2001). Meixner processes in finance. Report 2001-002, EURANDOM, Eindhoven.
24. W. Schoutens (2003). *Lévy Processes in Finance: Pricing Financial Derivatives.* Wiley.