A Cross-Currency Lévy Market Model

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Abstract

The Lévy Libor or market model which was introduced in Eberlein and Özkan (2005) is extended to a multi-currency setting. As an application we derive closed form pricing formulas for cross-currency derivatives. Foreign caps and floors and cross-currency swaps are studied in detail. Numerically efficient pricing algorithms based on bilateral Laplace transforms are derived. A calibration example is given for a two-currency setting (EUR, USD).

Key words: multi-currency model, cross-currency derivatives, foreign forward caps and floors, cross-currency swaps, time-inhomogeneous Lévy processes, forward martingale measure

1 Introduction

For corporations which operate in a globalised economy, the fluctuations in foreign exchange and interest rates represent two significant sources of risk. It has been understood for a long time that these risk factors are linked by fundamental economic relationships. For a brief review of attempts to model this complex combination of risks, see Musiela and Rutkowski (1997, chap. 17). The starting point for the present paper was essentially Schlögl (2002), where various possibilities to extend the classical lognormal market model (Brace, Gatarek and Musiela (1997), Miltersen, Sandmann and Sondermann (1997), Jamshidian (1997)) to a multi-currency setting are discussed.

The standard models which are implemented in the financial industry to describe the evolution of interest rates are based on Brownian motion as the driving process. Due to the limited flexibility of the normal distribution and the process generated from it, already for a single market it is difficult to calibrate such a model to quotes of derivatives like caps, floors and swaptions. The market practice is to communicate prices of derivatives in terms of their Black volatilities, i.e. volatilities in a log-normal model. The corresponding surfaces across all strikes and maturities expose a sophisticated term structure of smiles (see e.g. Eberlein and Kluge (2004, Figure 1)). In order to catch these empirical surfaces as accurate as possible we use a wider class of distributions and the processes which they generate. Thus we are entering the realm of Lévy processes or one could say we use semimartingale techniques.

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It is actually the combination of modern stochastic analysis in terms of semimartingales with classical results on Lévy processes which makes this approach attractive. It is not our goal to push the models to utmost generality from the mathematical point of view – although as one sees from the canonical representation for semimartingales, the latter are only slightly more general than Lévy processes. In order to successfully implement theses models one has to look at the right balance between flexibility of the model, i.e. increasing the number of parameters, and tractability. Estimation of the parameters is a crucial issue here. The class of generalized hyperbolic distributions and the purely discontinuous processes which they generate, turn out to be a reasonable choice. Sometimes subclasses such as hyperbolic, normal inverse Gaussian or variance gamma do as well. The only large class which is not contained in this family are stable distributions, but stable distributions are not a first choice in finance anyhow. Valuation of derivatives is based on martingale theory and the latter needs finite moments as an intrinsic ingredient. It is this what stable distributions cannot provide. Let us mention that Cauchy as well as Student-t distributions appear as limiting cases of generalized hyperbolic distributions (see Eberlein and von Hammerstein (2004)) . Thus from the point of view of tail behavior one can approximate the class of heavy tailed distributions as close as one wants.

The initial Lévy term structure model (Eberlein and Raible (1999)) which shows already a better performance than the log-normal one, was extended to a model which is driven by a time-inhomogeneous Lévy process in Eberlein, Jacod and Raible (2005) and Eberlein and Kluge (2006). Time-inhomogeneity adds the necessary flexibility towards the time dependence of smiles. The Lévy Libor or market model (Eberlein and Özkan (2005)) which is designed to price Libor-dependent securities was developed from the very beginning for time-inhomogeneous driving processes, since time-inhomogeneity comes in naturally when one changes from one forward measure to the next during the backward induction. To consider expectations with respect to the appropriate forward martingale measures is the key point on which this approach relies. Forward measures provide the elegance for this theory.

It is evident that the calibration problem becomes even more demanding as soon as one considers derivatives which are related to several currencies. A cross-currency derivative is by definition a security which depends on at least two economics. Usually this will be the domestic and a foreign market. Note that there are basket options that are written on a weighted average of a number of foreign interest rates. Examples are basket caps and basket floors. Cross-currency derivatives allow to manage the combination of interest rate and foreign exchange risk. In this paper we extend the Lévy market model to a multi-currency setting. The closely related forward process approach is discussed as well. As an application we derive closed form pricing formulas for foreign caps and floors, cross-currency swaps as well as quanto caplets. Further cross-currency instruments can be priced along the same lines.

The outline of the paper is as follows. In section 2 we introduce the driving time-inhomogeneous d-dimensional Lévy process and define the dynamics of the forward exchange rates for a set of m foreign currencies. The single currency Lévy Libor approach is extended to a multi-currency setting in section 3. By construction a link between the different fixed-income markets exists only on the level of the processes which drive the most distant Libor rates.

The links which are induced on the level of arbitrary tenor time points by a no arbitrage consideration are studied in detail in section 4. On the basis of this model closed form pricing formulas can be derived for a number of cross-currency derivatives, which are studied in section 5. That the model is not too complex to be implemented is shown in the last section. Time-inhomogeneity is used in a mild sense in the implementations, namely in the sense of driving processes which are piecewise (time-homogeneous) Lévy processes. Typically we used three or four of them, which are related to short, middle range and long maturities. This way we end up with a reasonable number of parameters. Generalized hyperbolic Lévy processes turned out to be a good choice here as well. The calibration results are presented for derivatives depending on two currencies (EUR, USD). The largest deviations between market and model prices expressed in volatilities occur for short maturities and thus in an area where absolute values of the prices are extremely small. Thus these deviations can be neglected. Similar empirical results were obtained in a three-currency market where derivatives in British Pounds were included. Numerically the so-called forward process approach is always more efficient than the Libor approach. The latter one needs approximations which can be avoided in the more direct forward process approach. On the other side negative interest rates are not excluded if one starts with a Lévy model for the forward process.

2 The foreign forward exchange rate model

Let T^* be a fixed time horizon for all market activities. We consider an international economy consisting of $m+1$ markets (currencies) indexed by $i \in \{0, \dots, m\}$, where 0 stands for the domestic market. The choice of the domestic market is arbitrary and depends on the particular pricing problem under consideration. We introduce a discrete-tenor structure $\mathbb{T} := \{T_0 < T_1 < \cdots < T_N < T_{N+1} = T^*\}$, which is the same for the domestic market and for all foreign markets.

All processes considered in what follows are defined on a common probability space $\big(\Omega, \mathcal{F}, \mathbb{P}^{0,T^*}\big)$, where the measure $\mathbb{P}^{0,T^*}$ is interpreted as the domestic forward measure associated with the date T^*. The probability space is endowed with a canonical filtration $(\mathcal{F}_t)_{0 \le t \le T^*}$, associated with a d-dimensional time-inhomogeneous Lévy process $(L_t^{0,T^*})_{0 \le t \le T^*}$. More specifically, $L^{0,T^*} = (L_1^{0,T^*}, \dots, L_d^{0,T^*})^\top$ is a process with independent increments and absolutely continuous characteristics (shortly PIIAC), which is defined by

$$L_t^{0,T^*} = \int_0^t b_s^{0,T^*} \, \mathrm{d}s + \int_0^t c_s \, \mathrm{d}W_s^{0,T^*} + \int_0^t \int_{\mathbb{R}^d} x \, (\mu - \nu_{0,T^*}) \, (\mathrm{d}s, \mathrm{d}x). \tag{2.1}$$

The law of L_t^{0,T^*} is determined by the following characteristic function:

$$\begin{aligned} \mathbb{E}\left[\exp\left(iu^\top L_t^{0,T^*}\right)\right] = \exp\Bigg(\int_0^t \Bigg(& iu^\top b_s^{0,T^*} - \frac{1}{2} u^\top \mathcal{C}_s u \\ & + \int_{\mathbb{R}^d} \left(\exp\left(iu^\top x\right) - 1 - iu^\top x\right) \lambda_s^{0,T^*}(\mathrm{d}x)\Bigg) \mathrm{d}s\Bigg). \end{aligned} \tag{2.2}$$

Here $b_t^{0,T^*}, u \in \mathbb{R}^d$, and $\mathcal{C}_t$ is a symmetric positive semidefinite $d \times d$ matrix. In equation (2.1) W_t^{0,T^*} denotes a $\mathbb{P}^{0,T^*}$-standard Brownian motion with values in $\mathbb{R}^d$ and c_t is a measurable version of the square root of $\mathcal{C}_t$. The measure μ is the random measure of jumps of the process L^{0,T^*}, and $\nu_{0,T^*}(\mathrm{d}s, \mathrm{d}x) = \lambda_s^{0,T^*}(\mathrm{d}x)\,\mathrm{d}s$ is the $\mathbb{P}^{0,T^*}$-compensator of μ, where $\lambda_t^{0,T^*}(\mathrm{d}x)$ is a measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and $\lambda_t^{0,T^*}(\{0\}) = 0$ for all $t \in [0, T^*]$. We assume that the three characteristics b_t^{0,T^*} and $\mathcal{C}_t$ and ν_{0,T^*} satisfy the following integrability condition:

$$\int_0^{T^*} \left(|b_s^{0,T^*}| + \|\mathcal{C}_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, \lambda_s^{0,T^*}(\mathrm{d}x) \right) \mathrm{d}s < \infty.$$

Furthermore, we make the following assumption:

($\mathbb{EM}$) There exists a positive constant M such that

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\left(u^\top x\right) \lambda_s^{0,T^*}(\mathrm{d}x)\,\mathrm{d}s < \infty$$

for all $u \in [-M, M]^d$.

Note that under assumption ($\mathbb{EM}$) all moments of L_t^{0,T^*} are finite (see e.g. Lemma 6 in Eberlein and Kluge (2006)). In particular the first moment is finite. As a consequence we do not need a truncation function in the representations (2.1) or (2.2).

To introduce a model for the foreign forward exchange rate associated with the final horizon date T^*, we use the following inputs

($\mathbb{FXR}$.1) We assume for every market $i \in \{0, \ldots, m\}$ a strictly decreasing and strictly positive family $B^i(0, T_j)$ $(j = 0, \ldots, N+1)$ of domestic and foreign zero-coupon bond prices and for $i \in \{1, \ldots, m\}$ a family $X^i(0)$ of positive spot exchange rates (expressed in units of domestic currency per unit of foreign currency) to be given. Consequently, the initial value of the foreign forward exchange rate corresponding to the most distant tenor time point T^* is given by

$$F_{X^i}(0, T^*) = \frac{B^i(0, T^*) X^i(0)}{B^0(0, T^*)}.$$

($\mathbb{FXR}$.2) For every foreign market $i \in \{1, \ldots, m\}$ there is a continuous deterministic function $\xi^i(\cdot, T^*) : [0, T^*] \to \mathbb{R}_+^d$. We assume for every coordinate $k,\ 1 \le k \le d$ of the vector $\xi^i(s, T^*)$ that $(\xi^i(s, T^*))_k \le \overline{M}$ for all $s \in [0, T^*]$ and $i \in \{1, \ldots, m\}$, where $\overline{M}$ is chosen in such a way that

$$\overline{M} < \frac{M}{N+2},$$

and M is the constant from assumption ($\mathbb{EM}$).

(𝔽𝕏ℝ.3) For every $i \in \{1, \dots, m\}$ the dynamics of the foreign forward exchange rate for the date T^* is given by

$$F_{X^i}(t, T^*) = F_{X^i}(0, T^*) \exp\left(\int_0^t \gamma^i(s, T^*)\, \mathrm{d}s + \int_0^t \xi^i(s, T^*)^\top \mathrm{d}L_s^{0,T^*} \right) \tag{2.3}$$

where

$$\begin{aligned} \gamma^i(s, T^*) &= -\xi^i(s, T^*)^\top b_s^{0,T^*} - \frac{1}{2}|\xi^i(s, T^*)^\top c_s|^2 \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\xi^i(s,T^*)^\top x} - 1 - \xi^i(s, T^*)^\top x \right) \lambda_s^{0,T^*}(\mathrm{d}x). \end{aligned}$$

The drift coefficients $\gamma^i(\cdot, T^*)$ are chosen such that equation (2.3) can be written equivalently in the form

$$\begin{aligned} F_{X^i}(t, T^*) &= F_{X^i}(0, T^*)\mathcal{E}_t \left(\int_0^\cdot \xi^i(s, T^*)^\top c_s \, \mathrm{d}W_s^{0,T^*} \right. \\ &\quad \left. + \int_0^\cdot \int_{\mathbb{R}^d} \left(\exp\left(\xi^i(s, T^*)^\top x \right) - 1 \right) (\mu - \nu_{0,T^*}) (\mathrm{d}s, \mathrm{d}x) \right), \end{aligned} \tag{2.4}$$

where $\mathcal{E}_t(Z)$ denotes as usual the stochastic exponential of a process Z at time t. Evidently, by construction $F_{X^i}(\cdot, T^*)$ is a local $\mathbb{P}^{0,T^*}$-martingale. It is actually a martingale since it is the stochastic exponential of a process which is a PIIAC and a local martingale (see Proposition 4.4 in Eberlein, Jacod and Raible (2005)). As a consequence

$$\mathbb{E}_{\mathbb{P}^{0,T^*}} \left[\frac{F_{X^i}(t, T^*)}{F_{X^i}(0, T^*)} \right] = 1.$$

Observe that we can define the foreign forward martingale measure associated with the date T^* and market $i \in \{1, \dots, m\}$ by setting its Radon–Nikodým derivative to be equal to the stochastic exponential in equation (2.4). More specifically,

$$\left. \frac{\mathrm{d}\mathbb{P}^{i,T^*}}{\mathrm{d}\mathbb{P}^{0,T^*}} \right|_{\mathcal{F}_t} = \frac{F_{X^i}(t, T^*)}{F_{X^i}(0, T^*)}, \qquad \text{for } i \in \{1, \dots, m\}. \tag{2.5}$$

Hence, by Girsanov's theorem for semimartingales a $\mathbb{P}^{i,T^*}$-standard Brownian motion and the $\mathbb{P}^{i,T^*}$-compensator of the random measure of jumps μ are given by

$$\begin{aligned} W_t^{i,T^*} &= W_t^{0,T^*} - \int_0^t c_s \xi^i(s, T^*)\, \mathrm{d}s, \\ \nu_{i,T^*}(\mathrm{d}t, \mathrm{d}x) &= \exp\left(\xi^i(t, T^*)^\top x \right) \nu_{0,T^*}(\mathrm{d}t, \mathrm{d}x), \end{aligned} \tag{2.6}$$

where $i \in \{1, \dots, m\}$.

The *inverse forward exchange rate* $\widehat{F}_{X^i}$ at time t for the date T^* is defined by

$$\widehat{F}_{X^i}(t, T^*) := \frac{1}{F_{X^i}(t, T^*)}.$$

Applying Itô's formula to equation (2.4), we obtain the following dynamics of the inverse forward exchange rate under the domestic forward measure $\mathbb{P}^{0,T^*}$

$$\frac{\mathrm{d}\widehat{F}_{X^i}(t,T^*)}{\widehat{F}_{X^i}(t-,T^*)} = -\,\xi^i(t,T^*)^\top c_t\,\mathrm{d}W_t^{0,T^*} - \int_{\mathbb{R}^d}\left(\exp\left(\xi^i(t,T^*)^\top x\right)-1\right)(\mu-\nu_{0,T^*})(\mathrm{d}t,\mathrm{d}x)$$
$$+\,|\xi^i(t,T^*)^\top c_t|^2\,\mathrm{d}t + \int_{\mathbb{R}^d}\frac{\left(\exp\left(\xi^i(t,T^*)^\top x\right)-1\right)^2}{\exp\left(\xi^i(t,T^*)^\top x\right)}\mu(\mathrm{d}t,\mathrm{d}x).$$

Using (2.6) we can write this in the form

$$\begin{aligned}\widehat{F}_{X^i}(t,T^*) = \widehat{F}_{X^i}(0,T^*)\mathcal{E}_t\Bigg(&-\int_0^\cdot \xi^i(s,T^*)^\top c_s\,\mathrm{d}W_s^{i,T^*}\\ &+\int_0^\cdot\int_{\mathbb{R}^d}\left(\frac{1}{\exp\left(\xi^i(s,T^*)^\top x\right)}-1\right)(\mu-\nu_{i,T^*})(\mathrm{d}s,\mathrm{d}x)\Bigg).\end{aligned} \tag{2.7}$$

Thus, being the stochastic exponential of a local martingale, the inverse forward exchange rate is itself a local martingale under the forward martingale measure $\mathbb{P}^{i,T^*}$. Using the same arguments as for $F_{X^i}(t,T^*)$ we can show that it is actually a martingale and, thus,

$$\mathbb{E}_{\mathbb{P}^{i,T^*}}\left[\frac{\widehat{F}_{X^i}(t,T^*)}{\widehat{F}_{X^i}(0,T^*)}\right] = 1.$$

Hence, the stochastic exponential on the right-hand side of equation (2.7) can also be considered as a density process. More specifically, for $i=1,\ldots,m$ we have

$$\left.\frac{\mathrm{d}\mathbb{P}^{0,T^*}}{\mathrm{d}\mathbb{P}^{i,T^*}}\right|_{\mathcal{F}_t} = \frac{\widehat{F}_{X^i}(t,T^*)}{\widehat{F}_{X^i}(0,T^*)}. \tag{2.8}$$

3 The cross-currency Lévy Libor rate model

In this section we present the cross-currency Lévy Libor model, which is an extension of the Lévy Libor model introduced in Eberlein and Özkan (2005).

Let us consider the discrete-tenor structure $\mathbb{T}$, which was defined at the beginning of the previous section. To construct the model in the multi-currency setting we make the following additional assumptions:

($\mathbb{CLM}$.1) For every market $i\in\{0,\ldots,m\}$ and every maturity T_j with $j\in\{0,\ldots,N\}$ there is a deterministic function $\lambda^i(\cdot,T_j):[0,T^*]\to\mathbb{R}_+^d$, which is continuous and which represents the volatility of the forward Libor rate $L^i(\cdot,T_j)$ in market i. We assume for every coordinate $k,\ 1\le k\le d$, of the vector $\lambda^i(s,T_j)$, that $(\lambda^i(s,T_j))_k\le\overline{M}$ for all $s\in[0,T^*]$, where $\overline{M}$ is the constant from assumption ($\mathbb{FXR}$.2) and $\lambda^i(s,T_j)=0$ for $s>T_j$.

($\mathbb{CLM}$.2) The initial term structure $L^i(0,T_j)$ of forward Libor rates in the i-th market is given by

$$L^i(0,T_j) = \frac{1}{\delta}\left(\frac{B^i(0,T_j)}{B^i(0,T_{j+1})}-1\right), \tag{3.1}$$

where $\delta = T_{j+1}-T_j,\ j=0,\ldots,N$.

The forward Libor rates are constructed for every market $i \in \{0,\dots,m\}$ by a backward induction. We start with the most distant rate $L^i(t,T_N)$ and postulate that under $\mathbb{P}^{i,T^*}$ it has the form

$$L^i(t,T_N) = L^i(0,T_N)\exp\left(\int_0^t \lambda^i(s,T_N)^\top \,\mathrm{d}L_s^{i,T^*}\right),$$

where

$$L_t^{i,T^*} = \int_0^t b_s^{i,T^*}\,\mathrm{d}s + \int_0^t c_s\,\mathrm{d}W_s^{i,T^*} + \int_0^t\int_{\mathbb{R}^d} x\,(\mu - \nu_{i,T^*})\,(\mathrm{d}s,\mathrm{d}x). \tag{3.2}$$

with W^{i,T^*} and ν_{i,T^*} given by equations (2.6) in the case of $i \in \{1,\dots,m\}$ and by (2.1) for $i=0$.

Observe that under Assumption ($\mathbb{EM}$), the process L_t^{i,T^*} is a special semimartingale. Obviously, $\int_0^t \lambda^i(s,T_N)\,\mathrm{d}L_s^{i,T^*}$ exists for every $t \in (0,T_N]$ and the process $\left(\int_0^t \lambda^i(s,T_N)\,\mathrm{d}L_s^{i,T^*}\right)_{0\le t\le T_N}$ is a special semimartingale too. By Proposition 2.2.5 in Koval (2005) it is also exponentially special.

We want the Libor rate process to be a martingale under $\mathbb{P}^{i,T^*}$. To assure the martingality we choose the drift term b_t^{i,T^*} such that $\int_0^t \lambda^i(s,T_N)^\top b_s^{i,T^*}\mathrm{d}s$ is equal to the exponential compensator of $\int_0^t \lambda^i(s,T_N)^\top \left(\mathrm{d}L_s^{i,T^*} - b_s^{i,T^*}\mathrm{d}s\right)$. More specifically,

$$\begin{aligned}\int_0^t \lambda^i(s,T_N)^\top b_s^{i,T^*}\,\mathrm{d}s =\ & -\frac{1}{2}\int_0^t |\lambda^i(s,T_N)^\top c_s|^2\,\mathrm{d}s \\ & -\int_0^t\int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(s,T_N)^\top x\right) - 1 - \lambda^i(s,T_N)^\top x\right)\nu_{i,T^*}(\mathrm{d}s,\mathrm{d}x),\end{aligned}$$

Hence, we obtain the following representation of the Libor rate $L^i(t,T_N)$

$$\begin{aligned}\frac{L^i(t,T_N)}{L^i(0,T_N)} = \mathcal{E}_t\Bigg(&\int_0^\cdot \lambda^i(s,T_N)^\top c_s\,\mathrm{d}W_s^{i,T^*} \\ &+ \int_0^\cdot\int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(s,T_N)^\top x\right) - 1\right)(\mu - \nu_{i,T^*})\,(\mathrm{d}s,\mathrm{d}x)\Bigg).\end{aligned}$$

Recall that the forward process corresponding to the time points T_N and T^* is related to the Libor rate by

$$F_{B^i}(t,T_N,T^*) = 1 + \delta L^i(t,T_N), \quad i = 0,\dots,m. \tag{3.3}$$

Thus, from the dynamics of $L^i(t,T_N)$ we can immediately derive the dynamics of $F_{B^i}(t,T_N,T^*)$

$$\begin{aligned}\frac{\mathrm{d}F_{B^i}(t,T_N,T^*)}{F_{B^i}(t-,T_N,T^*)} = \frac{\delta L^i(t-,T_N)}{1+\delta L^i(t-,T_N)}\Bigg(&\lambda^i(t,T_N)^\top c_t\,\mathrm{d}W_t^{i,T^*} \\ &+ \int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(t,T_N)^\top x\right) - 1\right)(\mu - \nu_{i,T^*})\,(\mathrm{d}t,\mathrm{d}x)\Bigg).\end{aligned} \tag{3.4}$$

Observe that the coefficients of the stochastic differential equation

$$\begin{cases} \dfrac{\delta L^i(t-,T_N)}{1+\delta L^i(t-,T_N)}\lambda^i(t,T_N)^\top c_t, \quad \text{and} \\ \dfrac{\delta L^i(t-,T_N)}{1+\delta L^i(t-,T_N)}\left(\exp\left(\lambda^i(t,T_N)^\top x\right) - 1\right) \end{cases}$$

are no longer deterministic. For numerical purposes it is desirable to stay within the class of PIIAC processes. This can be achieved by replacing the random term

$$l^i(t-,T_N) := \frac{\delta L^i(t-,T_N)}{(1+\delta L^i(t-,T_N))} \tag{3.5}$$

in the two coefficients above with a deterministic one, namely $l^i(0,T_N) = \delta L^i(0,T_N)/(1+\delta L^i(0,T_N))$. A similar approximation has already been used in Brace and Womersley (2000) for the derivation of the approximate swaption formula in the lognormal forward Libor model, as well as in Schlögl (2002). In implementations of the Lévy Libor model we make use of this approximation. For the development of the model itself, the approximation is not relevant.

Now we write for all $t \in [0,T_N]$

$$\alpha^i(t,T_N,T^*)^\top := l^i(t-,T_N)\lambda^i(t,T_N)^\top c_t$$

and

$$\beta^i(t,x,T_N,T^*) := l^i(t-,T_N)\left[\exp\left(\lambda^i(t,T_N)^\top x\right)-1\right]+1.$$

Then the forward process, given by equation (3.4), admits the following representation:

$$\begin{aligned} F_{B^i}(t,T_N,T^*) = F_{B^i}(0,T_N,T^*)\,\mathcal{E}_t\bigg(&\int_0^\cdot \alpha^i(s,T_N,T^*)^\top\,\mathrm{d}W_s^{i,T^*} \\ &+\int_0^\cdot\int_{\mathbb{R}^d}\left(\beta^i(s,x,T_N,T^*)-1\right)(\mu-\nu_{i,T^*})(\mathrm{d}s,\mathrm{d}x)\bigg). \end{aligned} \tag{3.6}$$

Using this dynamics, we can define the forward measure $\mathbb{P}^{i,T_N}$, associated with the date T_N and the market i, by setting

$$\begin{aligned} \frac{\mathrm{d}\mathbb{P}^{i,T_N}}{\mathrm{d}\mathbb{P}^{i,T^*}} = \mathcal{E}_{T_N}\bigg(&\int_0^\cdot \alpha^i(t,T_N,T^*)^\top\,\mathrm{d}W_t^{i,T^*}+ \\ &+\int_0^\cdot\int_{\mathbb{R}^d}\left(\beta^i(t,x,T_N,T^*)-1\right)(\mu-\nu_{i,T^*})(\mathrm{d}t,\mathrm{d}x)\bigg). \end{aligned}$$

Again using Girsanov's theorem we obtain a $\mathbb{P}^{i,T_N}$-standard Brownian motion W^{i,T_N} and the $\mathbb{P}^{i,T_N}$-compensator of the random measure of jumps μ

$$\begin{aligned} W_t^{i,T_N} &= W_t^{i,T^*} - \int_0^t \alpha^i(s,T_N,T^*)\,\mathrm{d}s, \\ \nu_{i,T_N}(\mathrm{d}t,\mathrm{d}x) &= \beta^i(t,x,T_N,T^*)\nu_{i,T^*}(\mathrm{d}t,\mathrm{d}x). \end{aligned} \tag{3.7}$$

Now we define the forward Libor rate $L^i(t,T_{N-1})$ by postulating that under the forward measure $\mathbb{P}^{i,T_N}$

$$L^i(t,T_{N-1}) = L^i(0,T_{N-1})\exp\left(\int_0^t \lambda^i(s,T_{N-1})^\top\,\mathrm{d}L_s^{i,T_N}\right),$$

where

$$L_t^{i,T_N} = \int_0^t b_s^{i,T_N}\,\mathrm{d}s + \int_0^t c_s\,\mathrm{d}W_s^{i,T_N} + \int_0^t\int_{\mathbb{R}^d} x\,(\mu-\nu_{i,T_N})(\mathrm{d}s,\mathrm{d}x).$$

To ensure the $\mathbb{P}^{i,T_N}$-martingality of $L^i(t,T_{N-1})$, we choose again the drift characteristic b^{i,T_N} such that $\int_0^t \lambda^i(s,T_{N-1})^\top b_s^{i,T_N}\mathrm{d}s$ is equal to the exponential compensator of $\int_0^t \lambda^i(s,T_{N-1})^\top \cdot \left(\mathrm{d}L_s^{i,T_N} - b_s^{i,T_N}\mathrm{d}s\right)$ and proceed as for $L^i(t,T_N)$.

Carrying forward this procedure, we get for each time point T_{j-1} with $j=1,\dots,N+1$ in the tenor structure $\mathbb{T}$ a Libor rate process, which has the following form under the forward martingale measure $\mathbb{P}^{i,T_j}$ in the i-th market

$$L^i(t,T_{j-1}) = L^i(0,T_{j-1})\exp\left(\int_0^t \lambda^i(s,T_{j-1})^\top \,\mathrm{d}L_s^{i,T_j}\right), \tag{3.8}$$

where

$$L_t^{i,T_j} = \int_0^t b_s^{i,T_j}\,\mathrm{d}s + \int_0^t c_s\,\mathrm{d}W_s^{i,T_j} + \int_0^t\int_{\mathbb{R}^d} x\left(\mu - \nu_{i,T_j}\right)(\mathrm{d}s,\mathrm{d}x), \tag{3.9}$$

and the drift term b_t^{i,T_j} is chosen such that $L^i(t,T_{j-1})$ is a martingale. More specifically,

$$\begin{aligned}\int_0^t \lambda^i(s,T_{j-1})^\top b_s^{i,T_j}\,\mathrm{d}s \;=\; & -\frac{1}{2}\int_0^t |\lambda^i(s,T_{j-1})^\top c_s|^2\,\mathrm{d}s \\ & -\int_0^t\int_{\mathbb{R}^d}\left(e^{\lambda^i(s,T_{j-1})^\top x} - 1 - \lambda^i(s,T_{j-1})^\top x\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x).\end{aligned} \tag{3.10}$$

The forward martingale measures $\mathbb{P}^{i,T_{j-1}}$ corresponding to the i-th market are defined successively by the densities

$$\frac{\mathrm{d}\mathbb{P}^{i,T_{j-1}}}{\mathrm{d}\mathbb{P}^{i,T_j}} = \frac{F_{B^i}(T_{j-1},T_{j-1},T_j)}{F_{B^i}(0,T_{j-1},T_j)} = \frac{1+\delta L^i(T_{j-1},T_{j-1})}{1+\delta L^i(0,T_{j-1})},$$

where the forward processes $F_{B^i}(\cdot,T_{j-1},T_j)$ are given in the form

$$\frac{\mathrm{d}F_{B^i}(t,T_{j-1},T_j)}{F_{B^i}(t-,T_{j-1},T_j)} = \alpha^i(t,T_{j-1},T_j)^\top\,\mathrm{d}W_t^{i,T_j} + \int_{\mathbb{R}^d}\left(\beta^i(t,x,T_{j-1},T_j) - 1\right)\left(\mu - \nu_{i,T_j}\right)(\mathrm{d}t,\mathrm{d}x). \tag{3.11}$$

The forward Brownian motion associated with the date T_{j-1} is given by

$$W_t^{i,T_{j-1}} = W_t^{i,T^*} - \int_0^t \alpha^i(s,T_{j-1},T^*)\,\mathrm{d}s$$

and

$$\nu_{i,T_{j-1}}(\mathrm{d}t,\mathrm{d}x) = \beta^i(t,x,T_{j-1},T^*)\nu_{i,T^*}(\mathrm{d}t,\mathrm{d}x) \tag{3.12}$$

is the $\mathbb{P}^{i,T_{j-1}}$-compensator of μ, where for $j=1,\dots,N+1$ we have

$$\begin{aligned}\alpha^i(t,T_{j-1},T^*)^\top &:= \sum\nolimits_{k=j-1}^N \alpha^i(t,T_k,T_{k+1})^\top = \sum\nolimits_{k=j-1}^N l^i(t-,T_k)\lambda^i(t,T_k)^\top c_t \\ \beta^i(t,x,T_{j-1},T^*) &:= \prod\nolimits_{k=j-1}^N \beta^i(t,x,T_k,T_{k+1}) \\ &= \prod\nolimits_{k=j-1}^N \left(l^i(t-,T_k)\left[\exp\left(\lambda^i(t,T_k)^\top x\right) - 1\right] + 1\right)\end{aligned} \tag{3.13}$$

As explained in Eberlein and Özkan (2005), one can alternatively base the backward induction on the forward process. This approach allows a direct implementation without any approximations and thus, simplifies the calibration to the market data. On the other side with this approach one has to accept negative Libor rates with some probability. Let us briefly outline the forward process approach. We postulate

$$F_{B^i}\left(t, T_{j-1}, T_j\right) = 1 + \delta L^i(t, T_{j-1}) = \left(1 + \delta L^i(0, T_{j-1})\right) \exp\left(\int_0^t \lambda^i(s, T_{j-1})^\top \, \mathrm{d}L_s^{i,T_j}\right) \tag{3.14}$$

for $i = 0, \ldots, m$. Having chosen the drift characteristic b_t^{i,T_j} as in equation (3.10), we can write the last equation as a stochastic differential equation

$$\begin{aligned}\frac{\mathrm{d}F_{B^i}(t, T_{j-1}, T_j)}{F_{B^i}(t-, T_{j-1}, T_j)} &= \lambda^i(t, T_{j-1})^\top c_t \, \mathrm{d}W_t^{i,T_j} \\ &\quad + \int_{\mathbb{R}^d} \left(\exp\left(\lambda^i(t, T_{j-1})^\top x\right) - 1\right) \left(\mu - \nu_{i,T_j}\right)(\mathrm{d}t, \mathrm{d}x)\end{aligned}$$

with the initial condition

$$F_{B^i}(0, T_{j-1}, T_j) = \frac{B^i(0, T_{j-1})}{B^i(0, T_j)}, \qquad j = 1, \ldots, N+1.$$

Then we define

$$\frac{\mathrm{d}\mathbb{P}^{i,T_{j-1}}}{\mathrm{d}\mathbb{P}^{i,T_j}} = \frac{F_{B^i}\left(T_{j-1}, T_{j-1}, T_j\right)}{F_{B^i}\left(0, T_{j-1}, T_j\right)}.$$

In this alternative setting the forward Brownian motion and the $\mathbb{P}^{i,T_{j-1}}$-compensator of μ are given by

$$\begin{aligned}W_t^{i,T_{j-1}} &= W_t^{i,T_j} - \int_0^t c_s \lambda^i(s, T_{j-1}) \, \mathrm{d}s \\ &= W_t^{i,T^*} - \int_0^t c_s \Big(\sum\nolimits_{k=j-1}^{N} \lambda^i(s, T_k)\Big) \, \mathrm{d}s, \\ \nu_{i,T_{j-1}}(\mathrm{d}t, \mathrm{d}x) &= \exp\left(\lambda^i(t, T_{j-1})^\top x\right) \nu_{i,T_j}(\mathrm{d}t, \mathrm{d}x) \\ &= \exp\Big(\sum\nolimits_{k=j-1}^{N} \lambda^i(t, T_k)^\top x\Big) \nu_{i,T^*}(\mathrm{d}t, \mathrm{d}x).\end{aligned} \tag{3.15}$$

4 Relationship between the domestic and the foreign markets

In the previous section we have shown, how the arbitrage free markets can be modeled on the basis of a cross-currency (time-inhomogeneous) Lévy Libor model. In particular, given a discrete-tenor structure $\mathbb{T}$, for each market $i \in \{0, \ldots, m\}$ the forward measures associated with dates T_j, $j = 1, \ldots, N+1$ were constructed, using a backward induction procedure starting from the horizon date $T^* = T_{N+1}$. The relationships (Radon–Nikodým densities) between the successive forward measures were established separately in each market. Naturally, the next step is to link the domestic and the foreign markets (see Figure 4.1). For diffusion models Schlögl (2002) succeeded to do this by a backward induction applied to the forward exchange rate processes. We will use the following standard relationship between two successive foreign forward exchange rates on the i-th foreign market

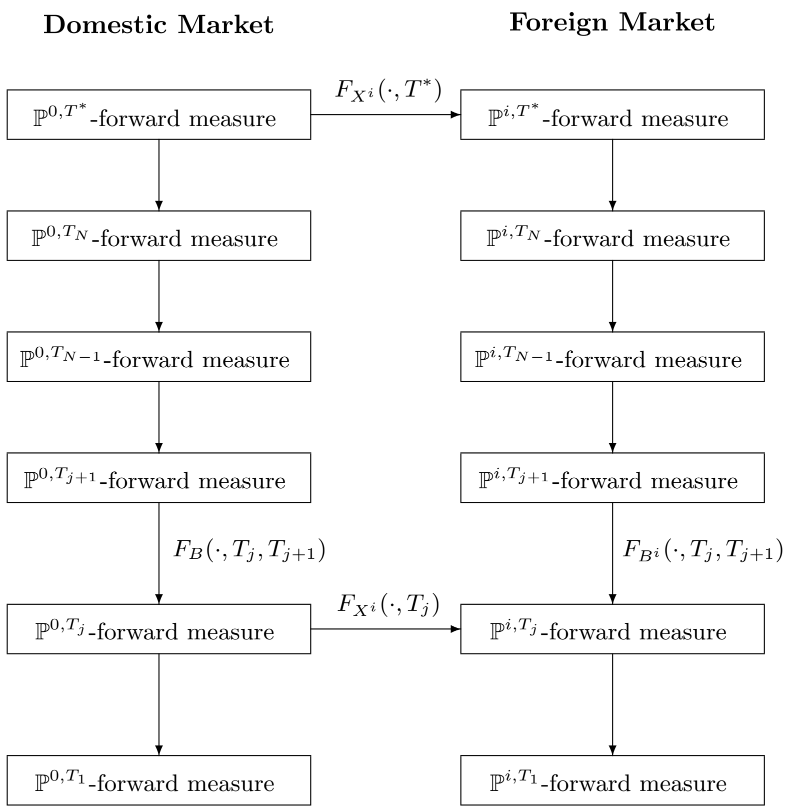

Figure 4.1 Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.

($\mathbb{FXR}$.4) The forward exchange rates in the i-th foreign market ($i \in \{1, \ldots, m\}$) are related by

$$F_{X^i}(t, T_j) = F_{X^i}(t, T_{j+1}) \frac{F_{B^i}(t, T_j, T_{j+1})}{F_{B^0}(t, T_j, T_{j+1})}, \qquad j = 0, \ldots, N, \text{ and } i = 1, \ldots, m.$$

Based on this product representation we can now derive the dynamics of $F_{X^i}(t, T_j)$.

Theorem 4.1 *For every tenor time point T_j $j = 1, \ldots, N$ and every market $i = 1, \ldots, m$ under the domestic forward measure $\mathbb{P}^{0,T_j}$ the dynamics of the forward exchange rate for date T_j is given by*

$$\begin{aligned} \frac{\mathrm{d}F_{X^i}(t, T_j)}{F_{X^i}(t-, T_j)} &= \zeta^i(t, T_j, T_{j+1})^\top \, \mathrm{d}W_t^{0,T_j} \\ &\quad + \int_{\mathbb{R}^d} \left(\overline{\zeta}^i(t, x, T_j, T_{j+1}) - 1\right) \left(\mu - \nu_{0,T_j}\right)(\mathrm{d}t, \mathrm{d}x), \end{aligned} \tag{4.1}$$

where the coefficients $\zeta^i(t, T_j, T_{j+1})$ and $\overline{\zeta}^i(t, x, T_j, T_{j+1})$ are given recursively by the following formula

$$\zeta^i(t, T_j, T_{j+1}) = \alpha^i(t, T_j, T_{j+1}) - \alpha^0(t, T_j, T_{j+1}) + \zeta^i(t, T_{j+1}, T_{j+2})$$

with $c_t \xi^i(t,T^*)$ *as starting point and*

$$\overline{\zeta}^i(t,x,T_j,T_{j+1}) = \frac{\beta^i(t,x,T_j,T_{j+1})}{\beta^0(t,x,T_j,T_{j+1})}\,\overline{\zeta}^i(t,x,T_{j+1},T_{j+2})$$

with $\exp\left(\xi^i(t,T^*)^\top x\right)$ *as starting point.*

Proof. In the Appendix.

Looking back at the definition of $\alpha^i(t,T_j,T_{j+1})$ (see (3.13)) we can reexpress $\zeta^i(t,T_j,T_{j+1})$ and $\overline{\zeta}^i(t,x,T_j,T_{j+1})$ in terms of the domestic and forward Libor rate coefficients

$$\begin{aligned}\zeta^i(t,T_j,T_{j+1})^\top &= l^i(t-,T_j)\,\lambda^i(t,T_j)^\top c_t \;-\; l^0(t-,T_j)\,\lambda^0(t,T_j)^\top c_t \;+\; \zeta^i(t,T_{j+1},T_{j+2})^\top\,,\\ \overline{\zeta}^i(t,x,T_j,T_{j+1}) &= \frac{l^i(t-,T_j)\left[\exp\left(\lambda^i(t,T_j)^\top x\right)-1\right]+1}{l^0(t-,T_j)\left[\exp\left(\lambda^0(t,T_j)^\top x\right)-1\right]+1}\,\overline{\zeta}^i(t,x,T_{j+1},T_{j+2}),\end{aligned} \tag{4.2}$$

with $l^i(t-,T_j)$ defined in the same way as $l^i(t-,T_N)$ is defined in (3.5). Evidently the coefficients $\zeta^i(t,T_j,T_{j+1})$ and $\overline{\zeta}^i(t,x,T_j,T_{j+1})$ are not deterministic, since $l^i(t-,T_j)$ and $l^0(t-,T_j)$ are not. Hence, we cannot model the foreign forward exchange rate as an exponential of a time-inhomogeneous Lévy process under all domestic forward measures simultaneously. This problem has been encountered already in the case of the Libor market model, where Libor rates are assumed to be driven by a Brownian motion.

In the Gaussian setting various possible combinations of lognormality assumptions for the forward exchange rates and the Libor rates in different markets have been analyzed systematically in Schlögl (2002). In particular, if lognormal dynamics are assumed for forward Libor rates in two currencies, the forward exchange rate, linking these two currencies, can be chosen to be lognormal for one maturity only. For all other maturities the dynamics can be obtained using no-arbitrage relationships similar to (4.2) above. Alternatively, one could choose forward Libor rates in only one currency, say the domestic one, to be lognormal, and postulate lognormal dynamics for all forward exchange rates, while the dynamics of foreign Libor rates is determined by the no-arbitrage relationships. The first option for the lognormal Libor market model is described in more detail in Mikkelsen (2002).

The stochastic differential equation derived in the theorem shows that $F_{X^i}(\cdot,T_j)$ is a local $\mathbb{P}^{0,T_j}$-martingale. To see that it is actually a martingale with respect to $\mathbb{P}^{0,T_j}$, we refer to the product representation in ($\mathbb{FXR}$.4). Start with time point T_N, then

$$F_{X^i}(t,T_N) = F_{B^i}(t,T_N,T^*)F_{X^i}(t,T^*)F_{B^0}(t,T_N,T^*)^{-1}.$$

Now $F_{B^i}(t,T_N,T^*)$ is a $\mathbb{P}^{i,T^*}$-martingale. $F_{X^i}(t,T^*)$ is according to (2.5) the density $\left.\frac{d\mathbb{P}^{i,T^*}}{d\mathbb{P}^{0,T^*}}\right|_{\mathcal{F}_t}$ up to a constant factor and $F_{B^0}(t,T^*,T_N)$ is according to chapter 3 the density $\left.\frac{d\mathbb{P}^{0,T^*}}{d\mathbb{P}^{0,T_N}}\right|_{\mathcal{F}_t}$ again up to a constant factor. Applying now Proposition III.3.8 from Jacod and Shiryaev (1987) twice, the martingality of $F_{X^i}(t,T_N)$ follows.

We carry this argument forward by induction and get the result for every T_j. Using this martingale propertey one sees that the T_j-forward measure in the i-th foreign market is directly related to the T_j-forward measure in the domestic market by

$$\frac{d\mathbb{P}^{i,T_j}}{d\mathbb{P}^{0,T_j}} = \frac{F_{X^i}(T_j,T_j)}{F_{X^i}(0,T_j)}$$

and thus the functions $\zeta^i(\cdot,T_j,T_{j+1})$ and $\overline{\zeta}^i(\cdot,x,T_j,T_{j+1})$ determine a change of measure such that under $\mathbb{P}^{i,T_j}$

$$W_t^{i,T_j} = W_t^{0,T_j} - \int_0^t \zeta^i(s,T_j,T_{j+1})\,\mathrm{d}s \tag{4.3}$$

is a standard Brownian motion and

$$\nu_{i,T_j}(\mathrm{d}t,\mathrm{d}x) = \overline{\zeta}^i(t,x,T_j,T_{j+1})\nu_{0,T_j}(\mathrm{d}t,\mathrm{d}x) \tag{4.4}$$

is the compensator of the random measure of jumps of μ.

It is necessary to note that there is another possibility to connect domestic and foreign fixed income markets by setting up the backward induction procedure (see assumption ($\mathbb{FXR}$.4)) on the basis of the domestic and foreign forward processes directly. Starting with equation (3.14) the expressions for $\zeta^i(t,T_j,T_{j+1})$ and $\overline{\zeta}^i(t,x,T_j,T_{j+1})$ are given in this case by

$$\begin{aligned} \zeta^i(t,T_j,T_{j+1})^\top &= \lambda^i(t,T_j)^\top c_t - \lambda^0(t,T_j)^\top c_t + \zeta^i(t,T_{j+1},T_{j+2})^\top, \\ \overline{\zeta}^i(t,x,T_j,T_{j+1}) &= \exp\left(\left(\lambda^i(t,T_j) - \lambda^0(t,T_j)\right)^\top x\right)\overline{\zeta}^i(t,x,T_{j+1},T_{j+2}), \end{aligned} \tag{4.5}$$

where $i = 1,\ldots,m$ and $j = 1,\ldots,N$.

With this forward process approach one can avoid any approximation in the implementations. In a sense, this is analogous to the Gaussian Heath, Jarrow and Morton (1992) approach, where forward bond prices are lognormal and the multicurrency extension is very tractable (see Frey and Sommer (1996)), versus the case of the slightly more complicated lognormal Libor market model.

5 Pricing cross-currency derivatives

In this section we examine the risk-neutral valuation of foreign market interest rate derivatives in the framework of the cross-currency Lévy Libor model, which we developed in the previous sections. We derive explicit pricing formulas for foreign caps and floors, cross-currency swaps and quanto-options. All these instruments are widely used by market participants to manage the combinations of interest rate and currency risks.

5.1 Foreign forward caps and floors

An interest rate cap is a contract, where the seller has an obligation to pay cash to the holder if a Libor rate exceeds a mutually agreed level at some future date or dates. Hence, the cap is an insurance against increasing interest rates, when the holder of the cap has a liability with a variable interest rate. An interest rate floor, on the contrary, is an insurance against declining interest rates. In this case, the seller has an obligation to pay cash to the holder if the Libor rate goes below a preassigned level. A foreign forward cap (or a foreign forward floor) represents a series of caplets (floorlets), each of which is a call (put) option on a foreign forward Libor rate, respectively. A caplet for the period $[T_{j-1}, T_j]$ with strike K^i and a nominal amount Z ensures the holder an amount of

$$\delta Z \left[L^i(T_{j-1}, T_{j-1}) - K^i\right]^+$$

at time T_j, where $L^i(T_{j-1}, T_{j-1})$ is a foreign Libor rate at time T_{j-1} for the period $[T_{j-1}, T_j]$. Note that the caplet expires at time T_{j-1}, but the payoff is received at the end of the accrual period, i.e at time $T_{j-1} + \delta = T_j$. For simplicity we will always assume that $Z = 1$.

Thus, the value of a foreign T_{N+1}-maturity cap, which we denote by FC^i, at time $0 < T_0$ can be obtained by considering the risk-neutral expectation of its discounted payoff, which is given in terms of the corresponding forward measures by

$$\mathrm{FC}^i(0, T_{N+1}) = \delta \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{\mathbb{P}^{i,T_j}} \left[\left(L^i(T_{j-1}, T_{j-1}) - K^i \right)^+ \right], \tag{5.1}$$

whereas the price $\mathrm{FF}^i(0, T_{N+1})$ at time $0 < T_0$ of the T_{N+1}-maturity foreign interest rate floor is given by

$$\mathrm{FF}^i(0, T_{N+1}) = \delta \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{\mathbb{P}^{i,T_j}} \left[\left(K^i - L^i\left(T_{j-1}, T_{j-1}\right) \right)^+ \right].$$

In the following exposition we shall refer to this as the *Libor rate approach.*

Alternatively, we can express the prices of the foreign cap and floor FC^i and FF^i, by using the foreign forward process $F_{B^i}(t, T_{j-1}, T_j) = 1 + \delta L^i(t, T_{j-1})$ as an underlying. In this case the formulas are rewritten in the form

$$\begin{aligned} \mathrm{FC}^i(0, T_{N+1}) &= \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{\mathbb{P}^{i,T_j}} \left[\left(1 + \delta L^i(T_{j-1}, T_{j-1}) - \widetilde{K}^i \right)^+ \right], \\ \mathrm{FF}^i(0, T_{N+1}) &= \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{\mathbb{P}^{i,T_j}} \left[\left(\widetilde{K}^i - 1 - \delta L^i(T_{j-1}, T_{j-1}) \right)^+ \right], \end{aligned} \tag{5.2}$$

where $\widetilde{K}^i = 1 + \delta K^i$. We shall call this the *forward process approach.* Due to the lack of space we shall focus on the results for this method of valuation, because it was extensively used in the numerical implementation of the model. As already mentioned in the previous sections, taking the foreign forward processes in the form (3.14) as the basic quantities allows us to avoid any approximations. Similar results for the *Libor rate approach* can be found in Koval (2005). To keep formulas simple we restrict the discussion in section 5.1 to the case of purely discontinuous processes, since all processes which we use in the implementations, are of this type.

Recall that in the purely discontinuous case the dynamics of the foreign forward process under the foreign forward measure $\mathbb{P}^{i,T_j}$ admits the following representation:

$$\begin{aligned}\left(1+\delta L^i(T_{j-1},T_{j-1})\right) &= \left(1+\delta L^i(0,T_{j-1})\right)\exp\left(\int_0^{T_{j-1}} \lambda^i(s,T_{j-1})^\top b_s^{i,T_j}\,\mathrm{d}s\right.\\ &\quad\left.+\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\lambda^i(s,T_{j-1})^\top x\left(\mu-\nu_{i,T_j}\right)(\mathrm{d}s,\mathrm{d}x)\right),\end{aligned}$$

where the drift term $b_t^{i,\,T_j}$ satisfies condition (3.10) with $c_s\equiv 0$. We put for $0\le t\le T_{j-1}$

$$X^i_{T_{j-1}}(t) := \int_0^t \lambda^i(s,T_{j-1})^\top\,\mathrm{d}L_s^{i,T_j} = \ln\frac{1+\delta L^i(t,T_{j-1})}{1+\delta L^i(0,T_{j-1})}.$$

Observe that the predictable characteristics under $\mathbb{P}^{i,T_j}$ for this process are

$$\begin{aligned}B^{X^i_{T_{j-1}}}(t) &= -\int_0^t\int_{\mathbb{R}^d}\left(e^{\lambda^i(s,T_{j-1})^\top x}-1-\lambda^i(s,T_{j-1})^\top x\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x),\\ C^{X^i_{T_{j-1}}} &\equiv 0,\\ \nu^{X^i_{T_{j-1}}}([0,t]\times G) &= \int_0^{T^*}\int_{\mathbb{R}^d}\mathbb{1}_{[0,t]\times G}\left(s,\lambda^i(s,T_{j-1})^\top x\right)\,\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x),\end{aligned}$$

where $G\in\mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right)$. Since $X^i_{T_{j-1}}$ is a PIIAC, its distribution at each time point t and in particular at T_{j-1} is infinitely divisible. By Theorem 9.1 in Sato (1999), we obtain with respect to $\mathbb{P}^{i,T_j}$ the following representation of its characteristic function

$$\begin{aligned}\chi^{i,T_{j-1}}(z) &= \exp\left(iz\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\lambda^i(s,T_{j-1})^\top x+1-\exp\left(\lambda^i(s,T_{j-1})^\top x\right)\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x)\right.\\ &\quad\left.+\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\exp\left(iz\lambda^i(s,T_{j-1})^\top x\right)-1-iz\lambda^i(s,T_{j-1})^\top x\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x)\right)\\ &= \exp\left(\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(e^{iz\lambda^i(s,T_{j-1})^\top x}-ize^{\lambda^i(s,T_{j-1})^\top x}-(1-iz)\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x)\right)\end{aligned}\tag{5.3}$$

for $z\in\mathbb{R}$, where by equations (3.15) and (2.6) the compensator ν_{i,T_j} satisfies

$$\nu_{i,T_j}(\mathrm{d}t,\mathrm{d}x)=\exp\left(\left(\sum\nolimits_{k=j}^N\lambda^i(t,T_k)+\xi^i(t,T^*)\right)^\top x\right)\nu_{0,T^*}(\mathrm{d}t,\mathrm{d}x).\tag{5.4}$$

We will assume that $\int_{-\infty}^{\infty}|\chi^{i,T_{j-1}}(u)|\,\mathrm{d}u<\infty$, so that the distribution of $X^i_{T_{j-1}}$ has a Lebesgue density. Then the price of the foreign caplet is given by the following

Theorem 5.1 *Set* $\widetilde{\xi}^i_j=\ln\left(\widetilde{K}^i\right)-\ln\left(1+\delta L^i(0,T_{j-1})\right)$ *and let* $R<-1$ *be chosen such that* $\chi^{i,T_{j-1}}(iR)<\infty$. *Then the price at time* $t=0$ *of the* j*-th foreign caplet is given by*

$$C^i(0,T_j,\widetilde{K}^i)=B^i(0,T_j)\widetilde{K}^i\frac{\exp\left(\widetilde{\xi}^i_jR\right)}{2\pi}\int_{-\infty}^{\infty}\exp\left(iu\widetilde{\xi}^i_j\right)\frac{\chi^{i,T_{j-1}}(iR-u)}{(R+iu)(1+R+iu)}\,\mathrm{d}u.$$

Proof. The proof is completely analogous to the proof of Theorem 5.1 in Eberlein and Özkan (2005), therefore we omit it. □

The integral which has to be evaluated to get the caplet price, contains the characteristic function $\chi^{i,T_{j-1}}(z)$ at a complex argument z. We want to show that this characteristic function exists along the integration path and we will derive an explicit form. For this let us introduce a time-inhomogeneous Lévy process $\widetilde{L}^{0,T^*}$ by setting

$$\widetilde{L}_t^{0,T^*} := L_t^{0,T^*} - \int_0^t b_s^{0,T^*} \mathrm{d}s. \tag{5.5}$$

Evidently, the characteristics of $\widetilde{L}_t^{0,T^*}$ satisfy $B_t^{\widetilde{L}} = 0,\ \ C_t^{\widetilde{L}} = 0,\ \ \nu^{\widetilde{L}}(\mathrm{d}t, \mathrm{d}x) = \nu_{0,T^*}(\mathrm{d}t, \mathrm{d}x)$.

To simplify notation, let us further denote by θ_s the cumulant associated with the Lévy–Khintchine triplet $(b_s^{0,T^*}, \mathcal{C}_s, \lambda_s^{0,T^*})$ given in the representation (2.2), i.e.

$$\theta_s(z) = z^\top b_s^{0,T^*} + \frac{1}{2} z^\top \mathcal{C}_s z + \int_{\mathbb{R}^d} (e^{z^\top x} - 1 - z^\top x) \lambda_s^{0,T^*}(\mathrm{d}x).$$

Proposition 5.2 *For all $z \in \mathbb{C}$ such that $\mathrm{Im}\,(z) = R$ and $R \in \left[-1 - \frac{M-(N-j+2)\overline{M}-\overline{M}}{\overline{M}}, -1\right)$ the characteristic function of $X^i_{T_{j-1}}(T_{j-1})$ satisfies $\chi^{i,T_{j-1}}(z) < \infty$. Furthermore, it admits the following representation:*

$$\begin{aligned}\chi^{i,T_{j-1}}(z) = \exp\Bigg(\int_0^{T_{j-1}} \big[&\theta_s(iz\lambda^i(s,T_{j-1}) + \psi_j^i(s,T^*)) \\ &-iz\theta_s(\lambda^i(s,T_{j-1}) + \psi_j^i(s,T^*)) + (iz-1)\theta_s(\psi_j^i(s,T^*))\big]\, \mathrm{d}s\Bigg)\end{aligned} \tag{5.6}$$

where $\psi_j^i(t,T^*) := \xi^i(t,T^*) + \sum_{k=j}^N \lambda^i(t,T_k)$.

Proof. According to Proposition 8 in Eberlein and Kluge (2006) we derive

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_{0,T^*}}\left[\exp\left(\int_0^{T_{j-1}} \left(iz\lambda^i(s,T_{j-1}) + \psi_j^i(s,T^*)\right)^\top \mathrm{d}\widetilde{L}_s^{0,T^*}\right)\right] \\ &= \exp\left(\int_0^{T_{j-1}} \theta_s\left(iz\lambda^i(s,T_{j-1}) + \psi_j^i(s,T^*)\right) \mathrm{d}s\right) \\ &= \exp\left(\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(e^{\left(iz\lambda^i(s,T_{j-1})+\psi_j^i(s,T^*)\right)^\top x} - 1 - \left(iz\lambda^i(s,T_{j-1}) + \psi_j^i(s,T^*)\right)^\top x\right)\nu_{0,T^*}(\mathrm{d}s,\mathrm{d}x)\right).\end{aligned} \tag{5.7}$$

where the assumption on the boundedness of the integrand in Proposition 8 is satisfied since choosing the value of the parameter R from $\left[-1 - \frac{M-(N-j+2)\overline{M}-\overline{M}}{\overline{M}}, -1\right)$ we obtain for each coordinate $l \in \{1, \dots, d\}$

$$\begin{aligned}&\left|\mathrm{Re}\left(iz\left(\lambda^i(s,T_{j-1})\right)_l + \left(\psi_j^i(s,T^*)\right)_l\right)\right| \\ &\le |-R|\left|\left(\lambda^i(s,T_{j-1})\right)_l\right| + \textstyle\sum_{k=j}^N \left|\left(\lambda^i(s,T_k)\right)_l\right| + \left|\left(\xi^i(s,T^*)\right)_l\right| \\ &< M - (N-j+2)\overline{M} + (N-j+1)\overline{M} + \overline{M} = M.\end{aligned} \tag{5.8}$$

With the same argument we get

$$\begin{aligned}
&\exp\left(-iz\ln\left(\mathbb{E}_{\mathbb{P}^{0,T^*}}\left[\exp\left(\int_0^{T_{j-1}}\left(\lambda^i(s,T_{j-1})+\psi_j^i(s,T^*)\right)^\top \mathrm{d}\widetilde{L}_s^{0,T^*}\right)\right]\right)\right)\\
&\quad=\exp\left(-iz\int_0^{T_{j-1}}\theta_s(\lambda^i(s,T_{j-1})+\psi_j^i(s,T^*))\,\mathrm{d}s\right) \qquad (5.9)\\
&\quad=\exp\left(-iz\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(e^{\left(\lambda^i(s,T_{j-1})+\psi_j^i(s,T^*)\right)^\top x}-1\right.\right.\\
&\qquad\qquad\left.\left.-\left(\lambda^i(s,T_{j-1})+\psi_j^i(s,T^*)\right)^\top x\right)\nu_{0,T^*}(\mathrm{d}s,\mathrm{d}x)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left((iz-1)\ln\left(\mathbb{E}_{\mathbb{P}^{0,T^*}}\left[\exp\left(\int_0^{T_{j-1}}\psi_j^i(s,T^*)^\top \mathrm{d}\widetilde{L}_s^{0,T^*}\right)\right]\right)\right)\\
&\quad=\exp\left((iz-1)\int_0^{T_{j-1}}\theta_s(\psi_j^i(s,T^*))\,\mathrm{d}s\right) \qquad (5.10)\\
&\quad=\exp\left((iz-1)\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\exp\left(\psi_j^i(s,T^*)^\top x\right)-1-\psi_j^i(s,T^*)^\top x\right)\nu_{0,T^*}(\mathrm{d}s,\mathrm{d}x)\right).
\end{aligned}$$

Multiplying the third lines in (5.7), (5.9), and (5.10) one easily gets the representation of the characteristic function as given in (5.3) in connection with (5.4). Multiplying the second lines of the same equations one gets the right-hand side of (5.6). □

5.2 Cross-currency swaps

A cross-currency (or differential) swap is an interest rate swap agreement, in which at least one of the Libor rates involved is related to a foreign market. More precisely, a floating-for-floating cross-currency $(i;l;0)$ swap is a financial instrument which allows swapping two foreign Libor rates whose payments are made in units of the domestic currency. At each of the payment dates T_j, $j=1,\ldots,N+1$, the Libor rate $L^i(T_{j-1},T_{j-1})$ of currency i is received and the corresponding Libor rate $L^l(T_{j-1},T_{j-1})$ of currency l is paid. As usual we assume the notional amount to be 1. More generally one could consider a floating-for-floating cross-currency swap of type $(i;l;m)$ where payments are made in units of currency m. We shall derive the risk-neutral value based on the *forward process approach*. The *Libor rate approach* would force us to make approximations in this context as well.

Theorem 5.3 *The time 0 value of a floating-for-floating $(i;l;0)$ cross-currency forward swap in units of the domestic currency is*

$$\begin{aligned}
CCFS_{[i;l;0]}(0)=\sum\nolimits_{j=1}^{N+1}B^0(0,T_j)&\left(\frac{B^i(0,T_{j-1})}{B^i(0,T_j)}\exp\left(\mathcal{D}^i(0,T_{j-1},T_j)\right)\right. \qquad (5.11)\\
&\left.-\frac{B^l(0,T_{j-1})}{B^l(0,T_j)}\exp\left(\mathcal{D}^l(0,T_{j-1},T_j)\right)\right)
\end{aligned}$$

where

$$\mathcal{D}^i(0,T_{j-1},T_j) = -\int_0^{T_{j-1}} \lambda^i(s,T_{j-1})^\top c_s \zeta^i(s,T_j,T_{j+1})\,\mathrm{d}s$$
$$- \int_0^{T_{j-1}}\int_{\mathbb{R}^d} \left(\exp\left(\lambda^i(s,T_{j-1})^\top x\right)-1\right)\left(\overline{\zeta}_i(s,x,T_j,T_{j+1})-1\right)\nu_{0,T_j}(\mathrm{d}s,\mathrm{d}x)$$

and $\zeta^i(s,T_j,T_{j+1}) = c_s\left(\xi^i(s,T^*)+\sum_{k=j}^N\left(\lambda^i(s,T_k)-\lambda^0(s,T_k)\right)\right)$, *whereas* $\overline{\zeta}_i(s,x,T_j,T_{j+1}) = \exp\left(\xi^i(s,T^*)^\top x+\sum_{k=j}^N(\lambda^i(s,T_k)-\lambda^0(s,T_k))^\top x\right)$ *and analogously for* $\mathcal{D}^l$.

Proof. At tenor time point T_j the cashflow of this swap is $\delta(L^i(T_{j-1},T_{j-1})-L^l(T_{j-1},T_{j-1}))$. We write this as $1+\delta L^i(T_{j-1},T_{j-1})-(1+\delta L^l(T_{j-1},T_{j-1}))$ since we shall price the claim based on the forward process approach. Using domestic forward measures we get

$$CCFS_{[i;l;0]}(0) = \sum_{j=1}^{N+1} B^0(0,T_j)\left(\mathbb{E}_{\mathbb{P}^{0,T_j}}\left[1+\delta L^i(T_{j-1},T_{j-1})-\left(1+\delta L^l(T_{j-1},T_{j-1})\right)\right]\right).$$

Consequently we have to compute the expectation $\mathbb{E}_{\mathbb{P}^{0,T_j}}[1+\delta L^i(T_{j-1},T_{j-1})]$ and analogously for i replaced by l. Under $\mathbb{P}^{i,T_j}$ we have by construction

$$1+\delta L^i(t,T_{j-1}) = (1+\delta L^i(0,T_{j-1}))\exp\left(\int_0^t \lambda^i(s,T_{j-1})^\top\,\mathrm{d}L_s^{i,T_j}\right),$$

where L_s^{i,T_j} is given by (3.9) with b_s^{i,T_j} chosen appropriately. Therefore, we get

$$\begin{aligned}&1+\delta L^i(T_{j-1},T_{j-1})\\ &= (1+\delta L^i(0,T_{j-1}))\exp\left(\int_0^{T_{j-1}}\lambda^i(s,T_{j-1})^\top c_s\,\mathrm{d}W_s^{i,T_j}\right.\\ &\quad -\frac{1}{2}\int_0^{T_{j-1}}|\lambda^i(s,T_{j-1})^\top c_s|^2\,\mathrm{d}s+\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\lambda^i(s,T_{j-1})^\top x(\mu-\nu_{i,T_j})(\mathrm{d}s,\mathrm{d}x)\\ &\quad \left.-\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(s,T_{j-1})^\top x\right)-1-\lambda^i(s,T_{j-1})^\top x\right)\nu_{i,T_j}(\mathrm{d}s,\mathrm{d}x)\right).\end{aligned}$$

Defining

$$\begin{aligned}\mathcal{M}^i(0,T_{j-1},T_j) &= \int_0^{T_{j-1}}\lambda^i(s,T_{j-1})^\top c_s\,\mathrm{d}W_s^{0,T_j}-\frac{1}{2}\int_0^{T_{j-1}}|\lambda^i(s,T_{j-1})^\top c_s|^2\,\mathrm{d}s\\ &\quad+\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\lambda^i(s,T_{j-1})^\top x\,(\mu-\nu_{0,T_j})(\mathrm{d}s,\mathrm{d}x)\\ &\quad-\int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(s,T_{j-1})^\top x\right)-1-\lambda^i(s,T_{j-1})^\top x\right)\nu_{0,T_j}(\mathrm{d}s,\mathrm{d}x)\end{aligned}$$

and using (4.3), (4.4) we get the last exponent in the form $\mathcal{M}^i(0,T_{j-1},T_j)+\mathcal{D}^i(0,T_{j-1},T_j)$. Now $\mathcal{D}^i(0,T_{j-1},T_j)$ is non-random and $\mathbb{E}_{\mathbb{P}^{0,T_j}}[\exp\left(\mathcal{M}^i(0,T_{j-1},T_j)\right)]=1$, therefore,

$$\mathbb{E}_{\mathbb{P}^{0,T_j}}[1+\delta L^i(T_{j-1},T_{j-1})] = (1+\delta L^i(0,T_{j-1}))\exp\left(\mathcal{D}^i(0,T_{j-1},T_j)\right).$$

From this the result follows. The explicit expressions for $\zeta^i(s,T_j,T_{j+1})$ and $\overline{\zeta}_i(s,x,T_j,T_{j+1})$ are immediate consequences of (4.5). □

Remark 5.4 *For a* $(i;0;0)$ *cross-currency swap, i.e. if the second Libor rate is the domestic one,* (5.11) *simplifies to*

$$CCFS_{[i;0;0]}(0) = \sum\nolimits_{j=1}^{N+1} B^0(0,T_j)\frac{B^i(0,T_{j-1})}{B^i(0,T_j)}\exp\left(\mathcal{D}^i(0,T_{j-1},T_j)\right) - \sum\nolimits_{j=1}^{N+1} B^0(0,T_{j-1}).$$

The last formula is a consequence of the $\mathbb{P}^{0,T_j}$-martingality of the forward process $(1+\delta L^0(t,T_{j-1}))_{0\leq t\leq T_{j-1}}$.

5.3 Quanto caplets

A quanto caplet with strike $\mathcal{K}^i$, which expires at time T_{j-1}, pays to its holder at time T_j the amount of

$$QCpl^i(T_j,T_j,\mathcal{K}^i) = \delta\bar{X}^i\left(L^i\left(T_{j-1},T_{j-1}\right)-\mathcal{K}^i\right)^+,$$

where $\bar{X}^i$ for $i=1,\ldots,m$ is the preassigned foreign exchange rate and $j=1,\ldots,N+1$. The value of a quanto caplet at time $t=0$ can be obtained by considering the risk-neutral expectation of its discounted payoff, which is given again in terms of the corresponding forward measures by

$$\begin{aligned} QCpl^i(0,T_j,\mathcal{K}^i) &= B^0(0,T_j)\,\mathbb{E}_{\mathbb{P}^{0,T_j}}\Big[\delta\bar{X}^i\left(L^i\left(T_{j-1},T_{j-1}\right)-\mathcal{K}^i\right)^+\Big] \\ &= B^0(0,T_j)\bar{X}^i\,\mathbb{E}_{\mathbb{P}^{0,T_j}}\Big[\Big(1+\delta L^i\left(T_{j-1},T_{j-1}\right)-\left(1+\delta\mathcal{K}^i\right)\Big)^+\Big]. \end{aligned}$$

Using the dynamics of the forward process $(1+\delta L^i(t,T_{j-1}))_{0\leq t\leq T_{j-1}}$ and the quantities $\mathcal{D}^i(0,T_{j-1},T_j)$ and $\mathcal{M}^i(0,T_{j-1},T_j)$ as in the proof of Theorem 5.3 we write this as

$$\begin{aligned} & B^0(0,T_j)\bar{X}^i\,\mathbb{E}_{\mathbb{P}^{0,T_j}}\Big[\Big(\left(1+\delta L^i(0,T_{j-1})\right)e^{\mathcal{M}^i(0,T_{j-1},T_j)+\mathcal{D}^i(0,T_{j-1},T_j)}-\left(1+\delta\mathcal{K}^i\right)\Big)^+\Big] \\ &\quad = B^0(0,T_j)\bar{X}^i\left(1+\delta\mathcal{K}^i\right)\mathbb{E}_{\mathbb{P}^{0,T_j}}\Big[\left(\exp\left(\mathcal{M}^i(0,T_{j-1},T_j)-\xi_j\right)-1\right)^+\Big], \end{aligned}$$

where $\xi_j = \ln\left(1+\delta\mathcal{K}^i\right)-\ln\left(1+\delta L^i(0,T_{j-1})\right)-\mathcal{D}^i(0,T_{j-1},T_j)$. Defining $v(x)=(e^{-x}-1)^+$ and assuming that the distribution of $\mathcal{M}^i(0,T_{j-1},T_j)$ has a density $\rho(x)$ we get

$$\begin{aligned} QCpl^i(0,T_j,\mathcal{K}^i) &= B^0(0,T_j)\bar{X}^i\left(1+\delta\mathcal{K}^i\right)\int_{-\infty}^{\infty} v(\xi_j-x)\rho(x)\,\mathrm{d}x \\ &= B^0(0,T_j)\bar{X}^i\left(1+\delta\mathcal{K}^i\right)(v*\rho)(\xi_j). \end{aligned}$$

Now proceeding in the same way as in Raible (2000) p.64 or in the proof of Theorem 5.1 in Eberlein and Özkan (2005) we get the following result

$$QCpl^i(0,T_j,\mathcal{K}^i) = B^0(0,T_j)\bar{X}^i\left(1+\delta\mathcal{K}^i\right)\frac{\exp(\xi_j R)}{2\pi}\int_{-\infty}^{\infty}\exp(iu\xi_j)\,\frac{\chi^{\mathcal{M}^i,T_{j-1}}(iR-u)}{(R+iu)(R+1+iu)}\,\mathrm{d}u,$$

where $R < -1$ is chosen such that the characteristic function $\chi^{\mathcal{M}^i,T_{j-1}}(u)$ of the random variable $\mathcal{M}^i(0,T_{j-1},T_j)$ satisfies $\chi^{\mathcal{M}^i,T_{j-1}}(iR) < \infty$. The existence of such an R is shown as in Proposition 5.2. The explicit form of this characteristic function is

$$\chi^{\mathcal{M}^i,T_{j-1}}(u) = \exp\Bigg(-\frac{u(u+i)}{2}\int_0^{T_{j-1}} \left|\lambda^i(s,T_{j-1})^\top c_s\right|^2 \mathrm{d}s \\ + \int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(e^{iu\lambda^i(s,T_{j-1})^\top x} - iue^{\lambda^i(s,T_{j-1})^\top x} - (1-iu)\right)\nu_{0,T_j}(\mathrm{d}s,\mathrm{d}x)\Bigg).$$

6 Model calibration

While the pricing problem is mainly concerned with computing values of options, given the model parameters, here we are interested in backing out the parameters, describing the risk-neutral dynamics of domestic and foreign Libor rates from the market prices of interest rate options. The data set we consider here consists of prices for interest rate caps in two different currencies: Euro (EUR) and US Dollar (USD) with maturities ranging from 1 to 10 years, and five different strike rates in each currency. We have chosen the EUR market to be the domestic one.

The market quotes prices of caps and floors mainly by their implied volatilities (annualized and in percentage). This entails the existence of a "standard" market model, because otherwise it would make no sense to quote prices in the language of a parameter in a model. The following formula for the price of the T_{N+1}-maturity cap (settled in arrears at dates T_j, $j = 2,\dots,N+1$), is in fact the market standard, and the lognormal Libor model behind it is called the *market model*.

$$\mathrm{FC}^N_{\mathrm{mkt}}(t,T_N) = \delta\sum_{j=2}^{N+1} B(t,T_j)\left[L(0,T_{j-1})\Phi(D_1) - K\Phi(D_2)\right],$$

where $\Phi(\cdot)$ denotes the standard Gaussian distribution function and

$$D_1 = \frac{\ln\left(\frac{L(0,T_{j-1})}{K}\right) + \frac{1}{2}\sigma_N^2 T_{j-1}}{\sigma_N\sqrt{T_{j-1}}}, \qquad D_2 = D_1 - \sigma_N\sqrt{T_{j-1}}.$$

To describe the calibration procedure, we first have to specify the main components of the model. These are the driving process L and the volatility structures λ^i for $i = 0,\dots,m$ and ξ^i for $i = 1,\dots,m$. We have chosen the first one to be a "piecewise Lévy process". More specifically, we divide the time interval $[0,T^*]$, where $T^* = 10$ years into $\mathcal{N}$ subintervals, which might have different lengths. With each subinterval we associate a homogeneous Lévy process with NIG-distributed increments of length 1. This particular choice of the driving process can be justified by the behaviour of the traders in fixed income markets. They distinguish between certain maturity ranges such as short, middle, long, and we relate these ranges to piecewise stationary parameters. Let us mention in this context that the traditional classification in a one-, two-, etc. factor model, which is used for models driven by Brownian motions, is not

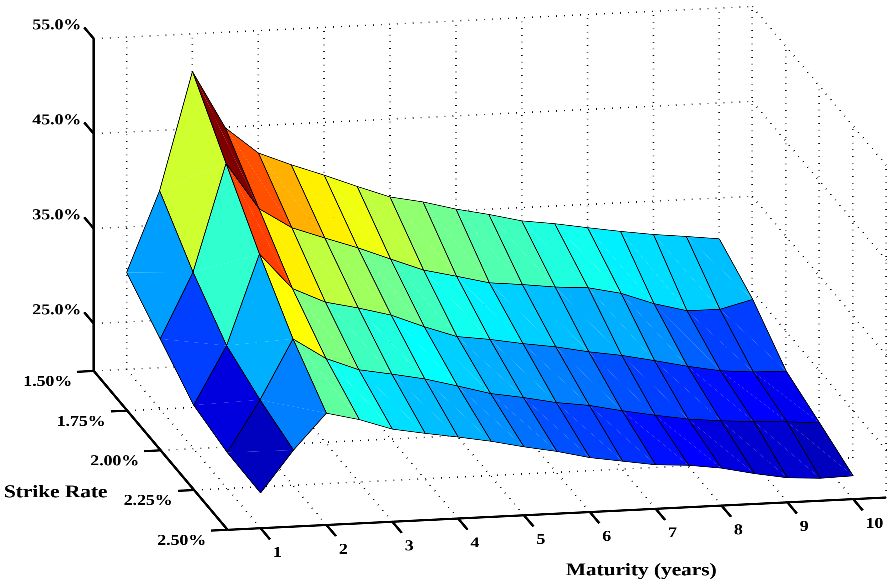

Figure 6.1 Euro caplet implied volatility surface on July 20, 2003.

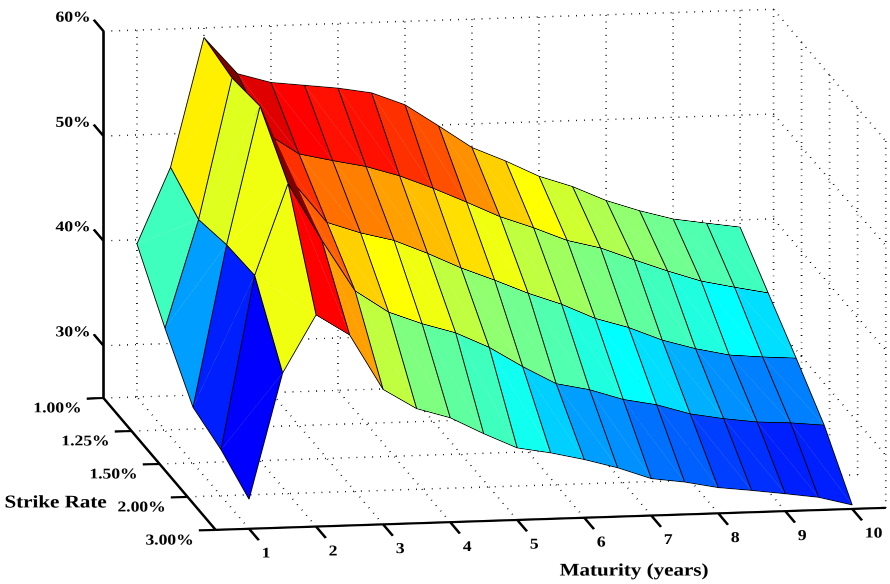

Figure 6.2 USD caplet implied volatility surface on July 20, 2003.

appropriate here. The driving process, even if one-dimensional, is itself a highdimensional object.

The second ingredient to our model – the volatility structure λ^i – has the following parametric form: for $a^i \in \mathbb{R}$ and $b^i, c^i > 0$

$$\begin{aligned} \lambda^i(t, T_j; a^i, b^i, c^i) &= a^i(T_j - t)\exp\left(-b^i(T_j - t)\right) + c^i \\ &= a^i \tau_j \exp\left(-b^i \tau_j\right) + c^i, \end{aligned} \tag{6.1}$$

where $\tau_j = T_j - t$ and $i = 0, \ldots, m$. The volatility structure of the foreign forward exchange rate ξ^i has the same representation, i.e. for $\widetilde{a}^i \in \mathbb{R}$ and $\widetilde{b}^i, \widetilde{c}^i > 0$ we have $\xi^i(t, T^*; \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}^i) = \widetilde{a}^i(T^* - t)\exp\left(-\widetilde{b}^i(T^* - t)\right) + \widetilde{c}^i$ for $i = 1, \ldots, m$. As for the functional form chosen for λ^i and ξ^i, the presence of a linear term together with a decaying exponential allows for the existence of a hump in the curve. We will always set the value of parameters a^i and $\widetilde{a}^i$ to be equal to one, since these constants can be included into the driving process L. A broad overview of different volatility structures, which can be applied in Libor rate models, is given in Brigo and Mercurio (2001).

We calibrate the forward process model, introduced in the previous section, to the market prices of caplets. More specifically, we choose the weighted sum of squared errors between the model and market prices of caplets E, given by

$$E(m, N, \mathcal{K}_1, \ldots, \mathcal{K}_m, \Delta) = \sum_{i=0}^{m} \sum_{j=1}^{N} \sum_{l=1}^{\mathcal{K}_i} w_{jl}^i \left[\mathrm{Cpl}^i_{\mathrm{mdl}}(0, T_j, K_l^i, \Delta) - \mathrm{Cpl}^i_{\mathrm{mkt}}(0, T_j, K_l^i)\right]^2 \tag{6.2}$$

to be the objective function. In the equation above m stands for the number of foreign currencies, N for the number of different caplets, $\mathcal{K}_i$ for the number of strike rates for each currency i with $i = 0, \ldots, m$, and T_j for the maturity of the options. The model price $\mathrm{Cpl}^i_{\mathrm{mdl}}$ depends on the set $\Delta = (\mathcal{T}, \Gamma, \Lambda, \Xi)$, where

- $\mathcal{T}$ is the partition of the time interval $[0, T^*]$,
- Γ is the set of distribution parameters,
- Λ is the set of parameters describing the Libor rate volatility structure,
- Ξ is the set of parameters describing the forward exchange rate volatility structure.

More specifically, $\mathcal{T} = \left\{(\mathcal{T}_0, \ldots, \mathcal{T}_{\mathcal{N}}) \in \mathbb{R}^{\mathcal{N}+1},\ \mathcal{T}_0 = 0,\ \mathcal{T}_{\mathcal{N}} = T^*\right\}$,

$$\begin{aligned} \Gamma &= \left\{ \begin{pmatrix} \alpha_1, & \beta_1, & \delta_1, \\ \vdots & \vdots & \vdots \\ \alpha_{\mathcal{N}}, & \beta_{\mathcal{N}}, & \delta_{\mathcal{N}} \end{pmatrix} \in \mathbb{R}^{3\mathcal{N}}, \quad 0 \le |\beta_l| < \alpha_l,\ \delta_l > 0 \quad \text{for } l = 1, \ldots, \mathcal{N} \right\}, \\ \Lambda &= \left\{ \begin{pmatrix} b_0, & c_0, \\ \vdots & \vdots \\ b_m, & c_m \end{pmatrix} \in \mathbb{R}^{2m+2},\ b_l, c_l \ge 0,\ \text{for } l = 0, \ldots, m \right\}, \\ \Xi &= \left\{ \begin{pmatrix} \widetilde{b}_1, & \widetilde{c}_1, \\ \vdots & \vdots \\ \widetilde{b}_m, & \widetilde{c}_m \end{pmatrix} \in \mathbb{R}^{2m},\ \widetilde{b}_l, \widetilde{c}_l \ge 0,\ \text{for } l = 1, \ldots, m \right\}. \end{aligned} \tag{6.3}$$

Note that in the set Γ each triple $(\alpha_l,\ \beta_l,\ \delta_l)$ corresponds to the time interval $(\mathcal{T}_{l-1}, \mathcal{T}_l]$ for $l = 1, \dots, \mathcal{N}$. The weights w^i_{jl} in (6.2) can be chosen in different ways. For example, in Eberlein and Kluge (2006) they are given by the reciprocal ATM prices for the respective maturity.

Calibration problem: Given the set of observed market prices of caplets

$$\mathrm{Cpl}_{\mathrm{mkt}} = \left\{ \mathrm{Cpl}^i_{\mathrm{mkt}}(0, T_j, K^i_l) \;\middle|\; i = 0, \dots, m;\ j = 1, \dots, N;\ K^i_l,\ l = 1, \dots, \mathcal{K}^i \right\},$$

we have to find such a set of parameters Δ that minimizes the value of the objective function E given in (6.2).

Notice that the final time horizon, after which any market activity is no longer considered, is equal to 10 years. It means that the last payoff occurs at time $T = 10$ years. However, the option embedded in the last caplet matures at time $T = 9.5$ years.

Thus, the number of parameters Q in our model, which are subject to calibration, is equal to $Q = (\mathcal{N} - 1) + 3\mathcal{N} + 2(m + 1) + 2m = 4(\mathcal{N} + m) + 1$, where $(\mathcal{N} - 1)$ is the number of partition parameters, $3\mathcal{N}$ is the number of distribution parameters, $2(m + 1)$ is the number of parameters describing the Libor volatility structures, and $2m$ is the number of parameters describing the volatilities of foreign forward exchange rates. It follows from equations (6.2) and (6.3) that the calibration problem is a constrained nonlinear least squares problem.

In case of our data set the objective function E satisfies

$$E(1, 19, 5, 5) = \sum_{i=0}^{1} \sum_{j=1}^{19} \sum_{l=1}^{5} \left[\mathrm{Cpl}^i_{\mathrm{mdl}}(0, T_j, K^i_l, \Delta) - \mathrm{Cpl}^i_{\mathrm{mkt}}(0, T_j, K^i_l) \right]^2,$$

where $i \in \{\mathrm{EUR}, \mathrm{USD}\}$. The minimal value of the objective function E is given by $E = 2.21136e - 07$, which corresponds to the partition parameters set $\mathcal{T}$ and the distribution parameters set Γ, presented in Table 6.1 below. The optimal values of the parameters for the

	$\mathcal{T}_j$ **(years)**	**Interval**	α	β	δ
$\mathcal{T}_1$	0.89	$(0, \mathcal{T}_1]$	356.43	53.77	0.0530
$\mathcal{T}_2$	1.40	$(\mathcal{T}_1, \mathcal{T}_2]$	138.92	99.53	0.1580
$\mathcal{T}_3$	9.15	$(\mathcal{T}_2, \mathcal{T}_3]$	60.34	2.53	0.0101
$\mathcal{T}_4$	9.50	$(\mathcal{T}_3, \mathcal{T}_4]$	150.25	137.05	0.0235

Table 6.1 Values of partition and distribution parameters

Libor rate volatility structures and the parameters describing the volatility of the EUR/USD forward exchange rate are given in Table 6.2. The absolute value of the differences between volatilities implied by the market prices and those from the forward process model for EUR caplets is shown in Figure 6.3. In Figure 6.4 one can see the absolute difference between implied volatilities of the market and the forward process model prices for USD caplets.

The results of the model calibration to EUR and USD market prices show that both volatility structures, illustrated in Figures 6.1 and 6.2, are fitted well. The largest deviations from the market implied volatilities are given by 2.75 % and 2.57 % for EUR and USD model

Volatility structure:	b	c
EUR Libor rate	3.67	0.11
USD Libor rate	3.32	0.13
EUR/USD forward FX rate	0.46	0.06

Table 6.2 Parameter values: forward process model.

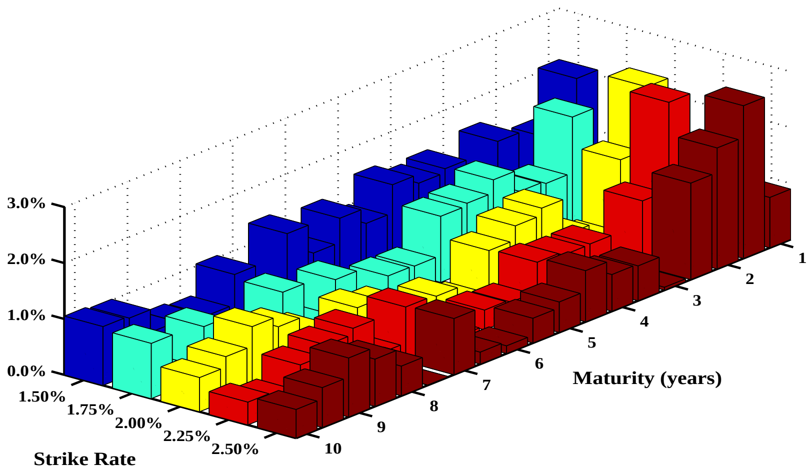

Figure 6.3 Absolute errors of EUR caplet calibration.

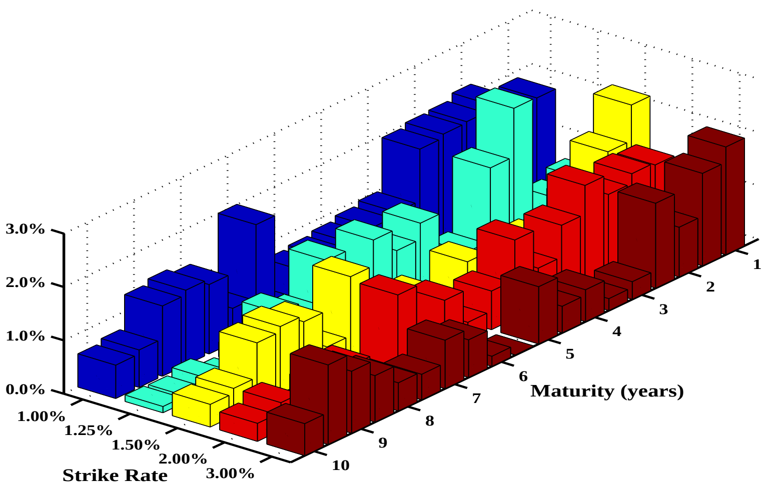

Figure 6.4 Absolute errors of USD caplet calibration.

implied volatilities respectively. These correspond to absolute differences of 7.56e–05 EUR and 4.24e–05 USD in the prices and are therefore negligible.

A Appendix

Proof of Theorem 4.1. Starting with the maturity T_N, we denote by

$$\begin{aligned} F_{\frac{B^i}{B^0}}(t,T_N,T^*) &:= \frac{F_{B^i}(t,T_N,T^*)}{F_{B^0}(t,T_N,T^*)} \\ &= F_{B^i}(t,T_N,T^*)\,F_{B^0}(t,T^*,T_N) \qquad \text{for } t\in[0,T_N] \end{aligned}$$

the mixed forward price ratio for the dates T_N and T^*, where $i=1,\ldots,m$. The dynamics of $F_{\frac{B^i}{B^0}}(\cdot,T_N,T^*)$ under the domestic forward measure $\mathbb{P}^{0,T^*}$ can be determined using the integration by parts formula (see Corollary II.6.2 in Protter (1990)). More precisely,

$$\begin{aligned} \mathrm{d}F_{\frac{B^i}{B^0}}(t,T_N,T^*) &= F_{B^i}(t-,T_N,T^*)\,\mathrm{d}F_{B^0}(t,T^*,T_N) + F_{B^0}(t-,T^*,T_N)\,\mathrm{d}F_{B^i}(t,T_N,T^*) \\ &\quad + \mathrm{d}[F_{B^i}(\cdot,T_N,T^*),F_{B^0}(\cdot,T^*,T_N)]_t\,. \end{aligned}$$

Let us first consider the domestic inverse forward rate $F_{B^0}(t,T^*,T_N) := F_{B^0}(t,T_N,T^*)^{-1}$. Applying Itô's formula to equation (3.11), we obtain its dynamics under the domestic forward measure $\mathbb{P}^{0,T^*}$

$$\begin{aligned} \frac{\mathrm{d}F_{B^0}(t,T^*,T_N)}{F_{B^0}(t-,T^*,T_N)} &= -\alpha^0(t,T_N,T^*)^\top\,\mathrm{d}W_t^{0,T^*} - \int_{\mathbb{R}^d}\left(\beta^0(t,x,T_N,T^*)-1\right)(\mu-\nu_{0,T^*})(\mathrm{d}t,\mathrm{d}x) \\ &\quad + \left|\alpha^0(t,T_N,T^*)\right|^2\mathrm{d}t + \int_{\mathbb{R}^d}\frac{\left(\beta^0(t,x,T_N,T^*)-1\right)^2}{\beta^0(t,x,T_N,T^*)}\mu(\mathrm{d}t,\mathrm{d}x) \\ &= -\alpha^0(t,T_N,T^*)^\top\,\mathrm{d}W_t^{0,T^*} + \left|\alpha^0(t,T_N,T^*)\right|^2\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^d}\left(\frac{1}{\beta^0(t,x,T_N,T^*)}-1\right)(\mu-\nu_{0,T^*})(\mathrm{d}t,\mathrm{d}x) \\ &\quad + \int_{\mathbb{R}^d}\left(\frac{1}{\beta^0(t,x,T_N,T^*)}-1+\beta^0(t,x,T_N,T^*)-1\right)\nu_{0,T^*}(\mathrm{d}t,\mathrm{d}x). \end{aligned} \tag{A.1}$$

Combining equation (3.6) with equations (2.6), we obtain the following dynamics of the foreign forward process $F_{B^i}(\cdot,T_N,T^*)$ with $i=1,\ldots,m$ under the domestic forward measure $\mathbb{P}^{0,T^*}$

$$\begin{aligned} \frac{\mathrm{d}F_{B^i}(t,T_N,T^*)}{F_{B^i}(t-,T_N,T^*)} &= \alpha^i(t,T_N,T^*)^\top\left(\mathrm{d}W_t^{0,T^*} - c_t\,\xi^i(t,T^*)\,\mathrm{d}t\right) \\ &\quad + \int_{\mathbb{R}^d}\left(\beta^i(t,x,T_N,T^*)-1\right)(\mu-\nu_{0,T^*})(\mathrm{d}t,\mathrm{d}x) \\ &\quad - \int_{\mathbb{R}^d}\left(\beta^i(t,x,T_N,T^*)-1\right)\left(\exp\left(\xi^i(t,T^*)^\top x\right)-1\right)\nu_{0,T^*}(\mathrm{d}t,\mathrm{d}x). \end{aligned} \tag{A.2}$$

Equations (A.1) and (A.2) yield

$$\begin{aligned} &F_{B^i}(t-,T_N,T^*)\,\mathrm{d}F_{B^0}(t,T^*,T_N) \\ &= F_{B^i}(t-,T_N,T^*)\,F_{B^0}(t-,T^*,T_N)\Big[-\alpha^0(t,T_N,T^*)^\top\,\mathrm{d}W_t^{0,T^*} + \left|\alpha^0(t,T_N,T^*)\right|^2\mathrm{d}t \end{aligned}$$

$$+ \int_{\mathbb{R}^d} \left(\frac{1}{\beta^0(t,x,T_N,T^*)} - 1 \right) (\mu - \nu_{0,T^*})(\mathrm{d}t, \mathrm{d}x)$$
$$+ \int_{\mathbb{R}^d} \left(\frac{1}{\beta^0(t,x,T_N,T^*)} - 1 + \beta^0(t,x,T_N,T^*) - 1 \right) \nu_{0,T^*}(\mathrm{d}t, \mathrm{d}x) \Bigg],$$

and

$$\begin{aligned}
&F_{B^0}(t-,T^*,T_N)\,\mathrm{d}F_{B^i}(t,T_N,T^*) = \\
&= F_{B^0}(t-,T^*,T_N)\,F_{B^i}(t-,T_N,T^*) \Big[\alpha^i(t,T_N,T^*)^\top \left(\mathrm{d}W_t^{0,T^*} - c_t\,\xi^i(t,T^*)\,\mathrm{d}t \right) \\
&\quad + \int_{\mathbb{R}^d} \left(\beta^i(t,x,T_N,T^*) - 1 \right) (\mu - \nu_{0,T^*})(\mathrm{d}t, \mathrm{d}x) \\
&\quad - \int_{\mathbb{R}^d} \left(\beta^i(t,x,T_N,T^*) - 1 \right) \left(\exp\left(\xi^i(t,T^*)^\top x \right) - 1 \right) \nu_{0,T^*}(\mathrm{d}t, \mathrm{d}x) \Big].
\end{aligned}$$

Applying Theorem I.4.52 in Jacod and Shiryaev (1987), we get

$$\begin{aligned}
[F_{B^i}(\cdot,T_N,T^*), F_{B^0}(\cdot,T^*,T_N)]_t &= \left\langle F^c_{B^i}(t,T_N,T^*), F^c_{B^0}(t,T^*,T_N) \right\rangle_t \\
&\quad + \sum_{s \le t} \Delta F_{B^i}(s,T_N,T^*)\,\Delta F_{B^0}(s,T^*,T_N).
\end{aligned}$$

From equations (A.1) and (A.2) we infer that the predictable covariation of the forward processes $F_{B^i}(t,T_N,T^*)$ and $F_{B^0}(t,T^*,T_N)$ is given by

$$\frac{\mathrm{d}\left\langle F^c_{B^i}(\cdot,T_N,T^*), F^c_{B^0}(\cdot,T^*,T_N) \right\rangle_t}{F_{B^i}(t-,T_N,T^*)\,F_{B^0}(t-,T^*,T_N)} = -\alpha^0(t,T_N,T^*)^\top \alpha^i(t,T_N,T^*)\,\mathrm{d}t, \quad i = 1, \ldots, m.$$

The jumps of $F_{B^i}(t,T_N,T^*)$ and $F_{B^0}(t,T^*,T_N)$ satisfy

$$\begin{aligned}
\Delta F_{B^i}(t,T_N,T^*) &= F_{B^i}(t-,T_N,T^*) \int_{\mathbb{R}^d} \left(\beta^i(t,x,T_N,T^*) - 1 \right) \mu(\{t\} \times \mathrm{d}x), \\
\Delta F_{B^0}(t,T^*,T_N) &= F_{B^0}(t-,T^*,T_N) \int_{\mathbb{R}^d} \left(\frac{1}{\beta^0(t,x,T_N,T^*)} - 1 \right) \mu(\{t\} \times \mathrm{d}x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\mathrm{d}[F_{B^i}(\cdot,T_N,T^*), F_{B^0}(\cdot,T^*,T_N)]_t}{F_{B^i}(t-,T_N,T^*)\,F_{B^0}(t-,T^*,T_N)} &= -\alpha^0(t,T_N,T^*)^\top \alpha^i(t,T_N,T^*)\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^d} \left(\beta^i(t,x,T_N,T^*) - 1 \right) \left(\frac{1}{\beta^0(t,x,T_N,T^*)} - 1 \right) \mu(\mathrm{d}t, \mathrm{d}x).
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
\frac{\mathrm{d}F_{\frac{B^i}{B^0}}(t,T_N,T^*)}{F_{\frac{B^i}{B^0}}(t-,T_N,T^*)} &= \left(\alpha^i(t,T_N,T^*) - \alpha^0(t,T_N,T^*) \right)^\top \left(\mathrm{d}W_t^{0,T^*} - \alpha^0(t,T_N,T^*)\,\mathrm{d}t \right) \\
&\quad - \alpha^i(t,T_N,T^*)^\top c_t\,\xi^i(t,T^*)\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^d} \left(\frac{\beta^i(t,x,T_N,T^*)}{\beta^0(t,x,T_N,T^*)} - 1 \right) (\mu - \nu_{0,T^*})(\mathrm{d}t, \mathrm{d}x) \qquad \text{(A.3)}
\end{aligned}$$

$$+\int_{\mathbb{R}^d}\Bigg(\exp\Big(\xi^i(t,T^*)^\top x\Big)-1+\beta^0(t,x,T_N,T^*)-1$$
$$+\beta^i(t,x,T_N,T^*)\left(\frac{1}{\beta^0(t,x,T_N,T^*)}-\exp\Big(\xi^i(t,T^*)^\top x\Big)\right)\Bigg)\nu_{0,T^*}(\mathrm{d}t,\mathrm{d}x).$$

Making use of assumption ($\mathbb{FXR}$.4), we can determine the foreign forward exchange rate $F_{X^i}(\cdot,T_N)$, applying the integration by parts rule to the product of the terminal forward exchange rate $F_{X^i}(\cdot,T^*)$ and the mixed forward price ratio $F_{\frac{B^i}{B^0}}(\cdot,T_N,T^*)$. More specifically,

$$\begin{aligned}\mathrm{d}F_{X^i}(t,T_N) &= F_{X^i}(t-,T^*)\,\mathrm{d}F_{\frac{B^i}{B^0}}(t,T_N,T^*)+F_{\frac{B^i}{B^0}}(t-,T_N,T^*)\,\mathrm{d}F_{X^i}(t,T^*)\\ &\quad+\mathrm{d}\Big[F_{X^i}(\cdot,T^*),F_{\frac{B^i}{B^0}}(\cdot,T_N,T^*)\Big]_t.\end{aligned}$$

From this we obtain finally

$$\begin{aligned}\frac{\mathrm{d}F_{X^i}(t,T_N)}{F_{X^i}(t-,T_N)} &= \big(\alpha^i(t,T_N,T^*)-\alpha^0(t,T_N,T^*)+c_t\,\xi^i(t,T^*)\big)^\top\mathrm{d}W_t^{0,T^*}\\ &\quad-\big(\alpha^i(t,T_N,T^*)-\alpha^0(t,T_N,T^*)+c_t\,\xi^i(t,T^*)\big)^\top\alpha^0(t,T_N,T^*)\,\mathrm{d}t\\ &\quad+\int_{\mathbb{R}^d}\left(\frac{\beta^i(t,x,T_N,T^*)}{\beta^0(t,x,T_N,T^*)}\exp\Big(\xi^i(t,T^*)^\top x\Big)-1\right)(\mu-\nu_{0,T^*})(\mathrm{d}t,\mathrm{d}x)\\ &\quad+\int_{\mathbb{R}^d}\Bigg[\exp\Big(\xi^i(t,T^*)^\top x\Big)\left(\frac{\beta^i(t,x,T_N,T^*)}{\beta^0(t,x,T_N,T^*)}-\beta^i(t,x,T_N,T^*)\right)\\ &\qquad\qquad+\beta^0(t,x,T_N,T^*)-1\Bigg]\nu_{0,T^*}(\mathrm{d}t,\mathrm{d}x).\end{aligned}$$

Applying equations (3.7), we can rewrite the last equation under the domestic forward measure $\mathbb{P}^{0,T_N}$ in the following way

$$\frac{\mathrm{d}F_{X^i}(t,T_N)}{F_{X^i}(t-,T_N)}=\zeta^i(t,T_N,T^*)^\top\,\mathrm{d}W_t^{0,T_N}+\int_{\mathbb{R}^d}\big(\overline{\zeta}^i(t,x,T_N,T^*)-1\big)(\mu-\nu_{0,T_N})(\mathrm{d}t,\mathrm{d}x),$$

where we have set

$$\begin{aligned}\zeta^i(t,T_N,T^*)^\top &:= \alpha^i(t,T_N,T^*)^\top-\alpha^0(t,T_N,T^*)^\top+\xi^i(t,T^*)^\top c_t,\\ \overline{\zeta}^i(t,x,T_N,T^*) &:= \frac{\beta^i(t,x,T_N,T^*)}{\beta^0(t,x,T_N,T^*)}\exp\Big(\xi^i(t,T^*)^\top x\Big).\end{aligned}$$

Analogously as for date T_N we derive the dynamics for the other forward exchange rates in an inductive way using in each step the product representation in ($\mathbb{FXR}$.4).

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