SYMMETRIES AND PRICING OF EXOTIC OPTIONS IN LÉVY MODELS

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Abstract. Standard models fail to reproduce observed prices of vanilla options because implied volatilities exhibit a term structure of smiles. We consider time-inhomogeneous Lévy processes to overcome these limitations. Then the scope of this paper is two-fold. On the one hand, we apply measure changes in the spirit of Geman et al., to simplify the valuation problem for various options. On the other hand, we discuss a method for the valuation of European options and survey valuation methods for exotic options in Lévy models.

1. Introduction

The efforts to calibrate standard Gaussian models to the empirically observed volatility surfaces very often do not produce satisfactory results. This phenomenon is not restricted to data from equity markets, but it is observed in interest rate and foreign exchange markets as well. There are two basic aspects to which the classical models cannot respond appropriately: the underlying distribution is not flexible enough to capture the implied volatilities either across different strikes or across different maturities. The first phenomenon is the so-called *volatility smile* and the second one the *term structure of smiles*; together they lead to the *volatility surface*, a typical example of which can be seen in Figure 1.1. One way to improve the calibration results is to use stochastic volatility models; let us just mention Heston (1993) for a very popular model, among the various stochastic volatility approaches.

A fundamentally different approach is to replace the driving process. Lévy processes offer a large variety of distributions that are capable of fitting the return distributions in the real world and the volatility smiles in the risk-neutral world. Nevertheless, they cannot capture the term structure of smiles adequately. In order to take care of the change of the smile across maturities, one has to go a step further and consider time-inhomogeneous Lévy processes —also called *additive* processes— as the driving processes. For term structure models this approach was introduced in Eberlein et al. (2005) and further investigated in Eberlein and Kluge (2004), where cap and swaption volatilities were calibrated quite successfully.

Key words and phrases. time-inhomogeneous Lévy processes, change of numéraire, change of measure, symmetry, homogeneity, vanilla and exotic options.

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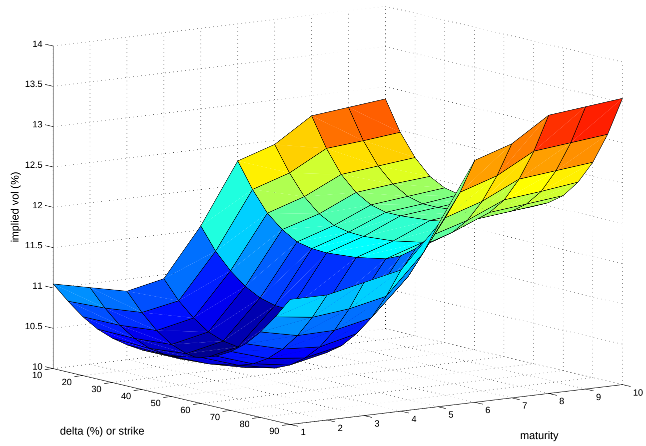

FIGURE 1.1. Implied volatilities of vanilla options on the Euro/Dollar rate; date: 5 November 2001. Data available at `http://www.mathfinance.de/FF/sampleinputdata.txt`

As far as plain vanilla options are concerned, a number of explicit pricing formulas is available for Lévy driven models, one of which is discussed in this article as well. The situation is much more difficult in the case of exotic options. The aim of this paper is to derive symmetries and to survey valuation methods for exotic options in Lévy models. By symmetries, we mean a relationship between pricing formulas for options of different type. Such a relation is of particular interest if it succeeds to derive the value of a complex payoff from that of a simpler one. A typical example is Theorem 5.1, where a floating strike Asian or lookback option can be priced via the formula for a fixed strike Asian or lookback option. Moreover, some symmetries are derived in situations where a put-call parity is not available.

The discussion here is rather general as far as the class of time-inhomogeneous Lévy processes is concerned. For implementation of these models a very convenient class are the processes generated by the Generalized Hyperbolic distributions (cf. Eberlein and Prause 2002).

The paper is organized as follows: in the next section we present time-inhomogeneous Lévy processes, the asset price model and some useful results. In section 3 we describe a method for exploring symmetries in option pricing. The next section contains symmetries and valuation methods for vanilla options while exotic options are tackled in the preceding section. Finally, in section 6 we present symmetries for options depending on two assets.

2. Model and Assumptions

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis in the sense of Jacod and Shiryaev (2003, I.1.3). Let $\bar{T} \in \mathbb{R}_+$ be a fixed time horizon and assume that $\mathcal{F} = \mathcal{F}_{\bar{T}}$. We shall consider $T \in [0, \bar{T}]$. The class of uniformly integrable martingales is denoted by $\mathcal{M}$; for further notation, we refer the reader to Jacod and Shiryaev (2003). Let $D = \{x \in \mathbb{R}^d : |x| > 1\}$.

Following Eberlein, Jacod, and Raible (2005) we use as driving process L a time-inhomogeneous Lévy process, more precisely, $L = (L^1, \dots, L^d)$ is a *process* with *independent increments* and *absolutely continuous* characteristics, in the sequel abbreviated PIIAC. The law of L_t is described by the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{i\langle u, L_t\rangle}\right] = \exp\int_0^t \Big[i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s, \tag{2.1}$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric non-negative definite $d \times d$ matrix and λ_t is a Lévy measure on $\mathbb{R}^d$, i.e. it satisfies $\lambda_t(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\lambda_t(\mathrm{d}x) < \infty$ for all $t \in [0, \bar{T}]$. The Euclidean scalar product on $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$, the corresponding norm by $|\cdot|$ and $\|\cdot\|$ denotes a norm on the set of $d \times d$ matrices. The transpose of a matrix or vector v is denoted by $v^\top$ and $\mathbf{1}$ denotes the unit vector, i.e. $\mathbf{1} = (1, \dots, 1)^\top$. The process L has càdlàg paths and $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,\bar{T}]}$ is the filtration generated by L; moreover, L satisfies Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$) given below.

Assumption ($\mathbb{AC}$). Assume that the triplets (b_t, c_t, λ_t) satisfy

$$\int_0^{\bar{T}} \Big[|b_t| + \|c_t\| + \int_{\mathbb{R}^d}(1 \wedge |x|^2)\lambda_t(\mathrm{d}x)\Big]\mathrm{d}t < \infty.$$

Assumption ($\mathbb{EM}$). Assume there exists a constant $M > 1$, such that the Lévy measures λ_t satisfy

$$\int_0^{\bar{T}}\int_D \exp\langle u, x\rangle\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty, \qquad \forall u \in [-M, M]^d.$$

Under these assumptions, L is a special semimartingale and its *triplet of semimartingale characteristics* (cf. Jacod and Shiryaev 2003, II.2.6) is given by

$$B_t = \int_0^t b_s\mathrm{d}s, \quad C_t = \int_0^t c_s\mathrm{d}s, \quad \nu([0,t] \times A) = \int_0^t\int_A \lambda_s(\mathrm{d}x)\mathrm{d}s, \tag{2.2}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. The triplet of semimartingale characteristics (B, C, ν) completely characterizes the distribution of L. Additionally, L is *exponentially special* (cf. Kallsen and Shiryaev 2002, 2.12, 2.13).

We model the asset price process as an exponential PIIAC

$$S_t = S_0 \exp L_t \tag{2.3}$$

with $(S^1, \dots, S^d) = (S_0^1\mathrm{e}^{L^1}, \dots, S_0^d\mathrm{e}^{L^d})$, where the superscript i refers to the i-th coordinate, $i \le d$. We assume that $\mathbb{P}$ is a risk neutral measure, i.e. the asset prices have mean rate of return $\mu^i \triangleq r - \delta^i$ and the auxiliary processes $\widehat{S}_t^i = \mathrm{e}^{\delta^i t} S_t^i$, once discounted at the rate r, are $\mathbb{P}$-martingales. Here, r is the risk-free rate and δ^i is the dividend yield of the i-th asset. Notice that finiteness of $\mathbb{E}[\widehat{S}_{\bar{T}}]$ is ensured by Assumption ($\mathbb{EM}$).

The driving process L has the canonical decomposition (cf. Jacod and Shiryaev 2003, II.2.38 and Eberlein et al. 2005)

$$L_t = \int_0^t b_s \mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x) \tag{2.4}$$

where, $c_t^{1/2}$ is a measurable version of the square root of c_t, W a $\mathbb{P}$-standard Brownian motion on $\mathbb{R}^d$, μ^L the random measure of jumps of the process L and $\nu(\mathrm{d}t, \mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$ is the $\mathbb{P}$-compensator of the jump measure μ^L.

Because S is modeled under a risk neutral measure, the drift characteristic B is completely determined by the other two characteristics (C, ν) and the rate of return of the asset. Therefore, the i-th component of B_t has the form

$$B_t^i = \int_0^t (r - \delta^i)\mathrm{d}s - \frac{1}{2}\int_0^t (c_s \mathbf{1})^i \mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} (\mathrm{e}^{x^i} - 1 - x^i)\nu(\mathrm{d}s, \mathrm{d}x). \tag{2.5}$$

In a foreign exchange context, δ^i can be viewed as the foreign interest rate.

In general, markets modeled by exponential time-inhomogeneous Lévy processes are incomplete and there exists a large class of risk neutral (equivalent martingale) measures. An exception occurs in interest rate models driven by Lévy processes, where —in certain cases— there is a unique martingale measure; we refer to Theorem 6.4 in Eberlein et al. (2005). Eberlein and Jacod (1997) provide a characterization of the class of equivalent martingale measures for exponential Lévy models in the time-homogeneous case; this was later extended to general semimartingales in Gushchin and Mordecki (2002).

In this article, we do not dive into the theory of choosing a martingale measure, we rather assume that the choice has already taken place. We refer to Eberlein and Keller (1995), Kallsen and Shiryaev (2002) for the Esscher transform, Frittelli (2000), Fujiwara and Miyahara (2003) for the minimal entropy martingale measure and Bellini and Frittelli (2002) for minimax martingale measures, to mention just a small part of the literature on this subject. A unifying exposition —in terms of f-divergences— of the different methods for selecting an equivalent martingale measure can be found in Goll and Rüschendorf (2001).

Alternatively, one can consider the choice of the martingale measure as the result of a calibration to the *smile* of the vanilla options market. Hakala and Wystup (2002) describe the calibration procedure in detail; we refer to Cont and Tankov (2004) for a numerically stable calibration method for Lévy driven models.

Remark 2.1. In the above setting, we can easily incorporate dynamic interest rates and dividend yields (or foreign and domestic rates). Let D_t denote the domestic and F_t the foreign savings account respectively, then they can have the form

$$D_t = \exp \int_0^t r_s \mathrm{d}s \qquad \text{and} \qquad F_t = \exp \int_0^t \delta_s \mathrm{d}s$$

and (2.5) has a similar form, taking r_s and δ_s into account.

Remark 2.2. The PIIAC L is an *additive* process, i.e. a process with independent increments, which is stochastically continuous and satisfies $L_0 = 0$ a.s. (Sato 1999, Definition 1.6).

Remark 2.3. If the triplet (b_t, c_t, λ_t) is not time-dependent, then the PIIAC L becomes a (homogeneous) Lévy process, i.e. a *process* with *independent* and *stationary increments* (PIIS). In that case, the distribution of L is described by the *Lévy triplet* (b, c, λ), where λ is the Lévy measure and the compensator of μ^L becomes a product measure of the form $\nu = \lambda \otimes \lambda\!\!\lambda^1$, where $\lambda\!\!\lambda^1$ denotes the Lebesgue measure. In that case, equation (2.1) takes the form $\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp[t \cdot \psi(u)]$ where

$$\psi(u) = i\langle u, b\rangle - \frac{1}{2}\langle u, cu\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\lambda(\mathrm{d}x) \tag{2.6}$$

which is called the *characteristic exponent* of L.

Lemma 2.4. *For fixed $t \in [0, \bar{T}]$ the distribution of L_t is infinitely divisible with Lévy triplet (b', c', λ'), given by*

$$b' := \int_0^t b_s \mathrm{d}s, \qquad c' := \int_0^t c_s \mathrm{d}s, \qquad \lambda'(\mathrm{d}x) := \int_0^t \lambda_s(\mathrm{d}x)\mathrm{d}s. \tag{2.7}$$

(The integrals should be understood componentwise.)

Proof. We refer to the proof of Lemma 1 in Eberlein and Kluge (2004). □

Remark 2.5. The PIIACs $L^1, \dots, L^d$ are independent if and only if the matrices C_t are diagonal and the Lévy measures λ_t are supported by the union of the coordinate axes; this follows directly from Exercise 12.10 in Sato (1999) or I.5.2 in Bertoin (1996) and Lemma 2.4. Describing the dependence is a more difficult task; we refer to Müller and Stoyan (2002) for a comprehensive exposition of various dependence concepts and their applications. We also refer to Kallsen and Tankov (2004), where a Lévy copula is used to describe the dependence of the components of multidimensional Lévy processes.

Remark 2.6. Assumption ($\mathbb{EM}$) is sufficient for all our considerations, but in general too strong. In the sequel we will replace ($\mathbb{EM}$), on occasion, by the *minimal* necessary assumptions. From a practical point of view though, it is not too restrictive to assume ($\mathbb{EM}$), since all examples of Lévy models we are interested in, e.g. the Generalized Hyperbolic model (cf. Eberlein and Prause 2002), the CGMY model (cf. Carr et al. 2002) or the Meixner model (cf. Schoutens 2002), possess moments of all order.

We can relate the finiteness of the *g-moment* of L_t for a PIIAC L and a *submultiplicative* function g, with an integrability property of its compensator measure ν. For the notions of the g-moment and submultiplicative function, we refer to Definitions 25.1 and 25.2 in Sato (1999).

Lemma 2.7 (g-Moment)**.** *Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$. Then the following statements are equivalent*

(1) $\int_0^{\bar{T}}\int_D g(x)\nu(\mathrm{d}t,\mathrm{d}x) < \infty$
(2) $\mathbb{E}\left[g(L_{\bar{T}})\right] < \infty$.

Proof. The result follows from Theorem 25.3 in Sato (1999) combined with Lemma 6 in Eberlein and Kluge (2004). □

Now, since $g(x) = \exp\langle u, x\rangle$ is a submultiplicative function, we immediately get the following equivalence concerning Assumption ($\mathbb{EM}$).

Corollary 2.8. *Let $M > 1$ be a constant. Then the following statements are equivalent*

(1) $\int_0^{\bar{T}}\int_D \exp\langle u, x\rangle\nu(\mathrm{d}t,\mathrm{d}x) < \infty, \qquad \forall u \in [-M, M]^d$
(2) $\mathbb{E}\left[\exp\langle u, L_{\bar{T}}\rangle\right] < \infty, \qquad \forall u \in [-M, M]^d$.

We can describe the characteristic triplet of the *dual* of a 1-dimensional PIIAC in terms of the characteristic triplet of the original process. First we introduce some necessary notation and the next lemma provides the result.

Notation. We denote by $-\lambda_t$ the *Lévy measure* defined by

$$-\lambda_t([a, b]) := \lambda_t([-b, -a])$$

for $a, b \in \mathbb{R}$, $a < b$, $t \in \mathbb{R}_+$. Thus, $-\lambda_t$ is a non-negative measure and the mirror image of the original measure with respect to the vertical axis. For a compensator of the form $\nu(\mathrm{d}t,\mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$, we denote by $-\nu$ the (non-negative) measure defined as

$$-\nu(\mathrm{d}t,\mathrm{d}x) := -\lambda_t(\mathrm{d}x)\mathrm{d}t.$$

Whenever we use the symbol "$-$" in front of a Lévy measure or a compensator, we will refer to measures defined as above.

Lemma 2.9 (dual characteristics)**.** *Let L be a PIIAC, as described above, with characteristic triplet (B, C, ν). Then $L^\star := -L$ is again a PIIAC with characteristic triplet $(B^\star, C^\star, \nu^\star)$, where $B^\star = -B$, $C^\star = C$ and $\nu^\star = -\nu$.*

Proof. From the Lévy-Khintchine representation we have that

$$\varphi_{L_t}(u) = \mathbb{E}\left[\mathrm{e}^{iuL_t}\right] = \exp\int_0^t \Big[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}}(\mathrm{e}^{iux} - 1 - iux)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s.$$

We get immediately

$$\begin{aligned}\varphi_{-L_t}(u) &= \varphi_{L_t}(-u)\\ &= \exp\int_0^t \Big[ib_s(-u) - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{i(-u)x} - 1 - i(-u)x)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s\\ &= \exp\int_0^t \Big[i(-b_s)u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iu(-x)} - 1 - iu(-x))\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s.\end{aligned}$$

Then $b_t^\star = -b_t, c_t^\star = c_t$, and $\lambda_t^\star = -\lambda_t$ clearly satisfy Assumption ($\mathbb{AC}$). Hence, we can conclude that $L^\star$ is also a PIIAC and has characteristics $B_t^\star = \int_0^t b_s^\star \mathrm{d}s = -B_t$, $C_t^\star = \int_0^t c_s^\star \mathrm{d}s = C_t$ and $\nu^\star(\mathrm{d}t, \mathrm{d}x) = \lambda_t^\star(\mathrm{d}x)\mathrm{d}t = -\nu(\mathrm{d}t, \mathrm{d}x)$. □

3. General description of the method

In this section, we give a brief and general description of the method we shall use to explore symmetries in option pricing. The method is based on the choice of a suitable numéraire and a subsequent change of the underlying probability measure; we refer to Geman et al. (1995) who pioneered this method.

The discounted asset price process, corrected for dividends, serves as the numéraire for a number of cases, in case the option payoff is homogeneous of degree one. Using the numéraire, evaluated at the time of maturity, as the Radon-Nikodym derivative, we form a new measure. Under the new measure, the numéraire asset is riskless while all other assets, including the savings account are now risky. In case the payoff is homogeneous of higher degree, say $\alpha \geq 1$, we have to modify the asset price process so that it serves as the numéraire. As a result, the asset dynamics under the new measure will depend on α as well.

We consider three cases for the driving process L and the asset price process(es):

- **(P1):** $L = L^1$ is a (1-d) PIIAC, $L^2 = k$ is constant and $S^1 = S_0^1 \exp L^1$, $S^2 = \exp L^2 = K$;
- **(P2):** $L = L^1$ is a (1-d) PIIAC, $S^1 = S_0^1 \exp L^1$ and $S^2 = h(S^1)$ is a functional of S^1;
- **(P3):** $L = (L^1, L^2)$ is a 2-dimensional PIIAC and $S^i = S_0^i \exp L^i$, $i = 1, 2$.

Consider a payoff function

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \tag{3.1}$$

which is homogeneous of degree $\alpha \geq 1$, that is for $\kappa, x, y \in \mathbb{R}_+^*$

$$f(\kappa x, \kappa y) = \kappa^\alpha f(x, y);$$

for simplicity we assume that $\alpha = 1$ and later —in the case of power options— we will treat the case of a more general α.

According to the general arbitrage pricing theory (Delbaen and Schachermayer 1994, 1998), the value V of an option on assets S^1, S^2 with payoff f is equal to its discounted expected payoff under an equivalent martingale measure. Throughout the paper, we will assume that options start at time 0 and mature at T, therefore we have

$$V = \mathrm{e}^{-rT}\mathbb{E}\left[f\left(S_T^1, S_T^2\right)\right]. \tag{3.2}$$

We choose asset S^1 as the numéraire and express the value of the option in terms of this numéraire, which yields

$$\begin{aligned}\widetilde{V} = \frac{V}{S_0^1} &= \mathrm{e}^{-rT}\mathbb{E}\left[\frac{f\left(S_T^1, S_T^2\right)}{S_0^1}\right] \\ &= \mathrm{e}^{-\delta^1 T}\mathbb{E}\left[\frac{\mathrm{e}^{-rT}S_T^1}{\mathrm{e}^{-\delta^1 T}S_0^1} f\left(1, \frac{S_T^2}{S_T^1}\right)\right]. \end{aligned}\tag{3.3}$$

Define a new measure $\widetilde{\mathbb{P}}$ via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{-rT}S_T^1}{\mathrm{e}^{-\delta^1 T}S_0^1} = \eta_T. \tag{3.4}$$

After the change of measure, the valuation problem, under the measure $\widetilde{\mathbb{P}}$, becomes

$$\widetilde{V} = \mathrm{e}^{-\delta^1 T}\widetilde{\mathbb{E}}\left[f\left(1, S_T^{1,2}\right)\right] \tag{3.5}$$

where we define the process $S^{1,2} := \frac{S^2}{S^1}$.

The measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related via the density process $\eta_t = \mathbb{E}[\eta_T|\mathcal{F}_t]$, therefore $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$ and we can apply Girsanov's theorem for semimartingales (cf. Jacod and Shiryaev 2003, III.3.24); this will allow us to determine the dynamics of $S^{1,2}$ under $\widetilde{\mathbb{P}}$.

After some calculations, which depend on the particular choice of L^2 or S^2, we can transform the original valuation problem into a simpler one.

4. Vanilla options

These results are motivated by Carr (1994), where a symmetry relationship between European call and put options in the Black and Scholes (1973), Merton (1973) model was derived. This result was later extended by Carr and Chesney (1996) to American options for the Black-Scholes case and for general diffusion models; see also McDonald and Schroder (1998) and Detemple (2001).

This relationship has an intuitive interpretation in foreign exchange markets (cf. Wystup 2002). Consider the Euro/Dollar market; then a call option on the Euro/Dollar exchange rate S_t with payoff $(S_T - K)^+$ has time-t value $V_c(S_t, K; r_d, r_e)$ in dollars and $V_c(S_t, K; r_d, r_e)/S_t$ in euros. This euro-call option can also be viewed as a dollar-put option on the Dollar/Euro rate with payoff $K(K^{-1} - S_T^{-1})^+$ and time-t value $KV_p(K^{-1}, S_T^{-1}; r_e, r_d)$ in euros. Since the processes S and S^{-1} have the same (Black-Scholes) volatility, by the absence of arbitrage opportunities, their prices must be equal.

4.1. **Symmetry.** For Vanilla options, the setting is that of (**P1**): $L^1 = L$ is the driving $\mathbb{R}$-valued PIIAC with triplet (B, C, ν), $S^1 = \exp L^1 = S$ and $L^2 = k$, such that $S^2 = \mathrm{e}^k = K$, the strike price of the option.

In accordance with the standard notation, we will use σ_s^2 instead of c_s, which corresponds to the volatility in the Black-Scholes model. Therefore, the characteristic C in (2.2) has the form $C_t = \int_0^t \sigma_s^2 \mathrm{d}s$.

We will prove a more general version of Carr's symmetry, namely a symmetry relating *power options*; the payoff of the power call and put option respectively is

$$\left[(S_T - K)^+\right]^\alpha \qquad \text{and} \qquad \left[(K - S_T)^+\right]^\alpha$$

where $\alpha \geq 1$, $\alpha \in \mathbb{N}$ (more generally $\alpha \in \mathbb{R}$). We introduce the following notation for the value of a power call option with strike K and power index α

$$V_c(S_0, K, \alpha; r, \delta, C, \nu) = \mathrm{e}^{-rT} \mathbb{E}\left[(S_T - K)^+\right]^\alpha$$

where the asset is modeled as an exponential PIIAC according to (2.3)–(2.5) and $x^+ = \max\{x, 0\}$. Similarly, for a power put option we set

$$V_p(S_0, K, \alpha; r, \delta, C, \nu) = \mathrm{e}^{-rT} \mathbb{E}\left[(K - S_T)^+\right]^\alpha.$$

Of course, for $\alpha = 1$ we recover the European plain vanilla option and the power index α will be omitted from the notation.

Assumption ($\mathbb{EM}$) can be replaced by the following weaker assumption, which is the minimal condition necessary for the symmetry results to hold. Let $D_+ = D \cap \mathbb{R}_+$ and $D_- = D \cap \mathbb{R}_-$.

Assumption ($\mathbb{M}$). The Lévy measures λ_t of the distribution of L_t satisfy

$$\int_0^{\bar{T}} \int_{D_-} |x| \lambda_t(\mathrm{d}x)\mathrm{d}t < \infty \qquad \text{and} \qquad \int_0^{\bar{T}} \int_{D_+} x\mathrm{e}^{\alpha x} \lambda_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

Theorem 4.1. *Assume that* ($\mathbb{M}$) *is in force and the asset price evolves as an exponential PIIAC according to equations (2.3)–(2.5). We can relate the power call and put option via the following symmetry:*

$$V_c(S_0, K, \alpha; r, \delta, C, \nu) = K^\alpha S_0^\alpha \mathcal{C}_T \mathrm{e}^{\alpha \mathcal{C}^*_T} V_p(S_0^{-1}, \mathcal{K}, \alpha; \delta, r, C, -f\nu) \tag{4.1}$$

where the constants $\mathcal{C}$ *and* $\mathcal{C}^*$ *are given by (4.3) and (4.10) respectively,* $\mathcal{K} = K^{-1}\mathrm{e}^{-\mathcal{C}^*_T}$ *and* $f(x) = \mathrm{e}^{\alpha x}$.

Proof. Firstly, we note that $[\mathrm{e}^{(\delta - r)t} S_t]^\alpha = S_0^\alpha \exp(\alpha(\delta - r)t + \alpha L_t)$ is not a $\mathbb{P}$-martingale; we denote by L^α the martingale part of the exponent, hence

$$L_t^\alpha = \int_0^t \alpha \sigma_s \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} \alpha x (\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x).$$

Since L^α is exponentially special, with Theorem 2.18 in Kallsen and Shiryaev (2002) we have that its exponential compensator, denoted CL^α, has the form

$$CL_t^\alpha = \frac{1}{2} \int_0^t \alpha^2 \sigma_s^2 \mathrm{d}s + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^{\alpha x} - 1 - \alpha x) \nu(\mathrm{d}s, \mathrm{d}x)$$

and $\exp(L^\alpha - CL^\alpha) \in \mathcal{M}$.

The price of the power call option expressed in units of the *numéraire* yields

$$
\begin{aligned}
\widetilde{V}_c := \frac{V_c}{S_0^\alpha} &= \frac{\mathrm{e}^{-rT}}{S_0^\alpha}\mathbb{E}\left[(S_T - K)^+\right]^\alpha \\
&= \mathrm{e}^{-\delta T}\mathbb{E}\left[\frac{\mathrm{e}^{-rT}S_T^\alpha K^\alpha}{\mathrm{e}^{-\delta T}S_0^\alpha}\left[(K^{-1} - S_T^{-1})^+\right]^\alpha\right] \\
&= \mathrm{e}^{-\delta T}K^\alpha\mathbb{E}\Bigg[\exp\Big((\delta - r)T + \alpha\int_0^T b_s\mathrm{d}s + CL_T^\alpha\Big) \\
&\qquad\qquad \times \exp\left(L_T^\alpha - CL_T^\alpha\right)\left[(K^{-1} - S_T^{-1})^+\right]^\alpha\Bigg] \\
&= \mathrm{e}^{-\delta T}K^\alpha\mathcal{C}_T\mathbb{E}\Big[\exp\left(L_T^\alpha - CL_T^\alpha\right)\left[(K^{-1} - S_T^{-1})^+\right]^\alpha\Big]
\end{aligned}
\tag{4.2}
$$

where, using (2.5) and (2.2), we have that

$$
\begin{aligned}
\log\mathcal{C}_T &= (\delta - r)T + \alpha B_T + CL_T^\alpha \\
&= (\alpha - 1)(r - \delta)T + \frac{\alpha(\alpha - 1)}{2}\int_0^T \sigma_s^2\mathrm{d}s \\
&\qquad + \int_0^T\int_{\mathbb{R}}(\mathrm{e}^{\alpha x} - \alpha\mathrm{e}^x + \alpha - 1)\nu(\mathrm{d}s, \mathrm{d}x).
\end{aligned}
\tag{4.3}
$$

Define a new measure $\widetilde{\mathbb{P}}$ via its Radon-Nikodym derivative

$$
\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(L_T^\alpha - CL_T^\alpha\right) = \eta_T \tag{4.4}
$$

and the valuation problem (4.2) becomes

$$
\widetilde{V}_c = \mathrm{e}^{-\delta T}K^\alpha\mathcal{C}_T\widetilde{\mathbb{E}}\left[(\widetilde{K} - \widetilde{S}_T)^+\right]^\alpha \tag{4.5}
$$

where $\widetilde{K} = K^{-1}$ and $\widetilde{S}_t := S_t^{-1}$.

Since the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related via the density process (η_t), which is a positive martingale with $\eta_0 = 1$, we immediately deduce that $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$ and we can apply Girsanov's theorem for semimartingales (cf. Jacod and Shiryaev 2003, III.3.24). The density process can be represented in the usual form

$$\eta_t = \mathbb{E}\left[\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{F}_t\right] = \exp\left(L_t^\alpha - CL_t^\alpha\right)$$
$$= \exp\Bigg[\int_0^t \alpha\sigma_s \mathrm{d}W_s + \int_0^t\int_{\mathbb{R}} \alpha x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$
$$- \frac{1}{2}\int_0^t \alpha^2\sigma_s^2 \mathrm{d}s - \int_0^t\int_{\mathbb{R}} (\mathrm{e}^{\alpha x} - 1 - \alpha x)\nu(\mathrm{d}s, \mathrm{d}x)\Bigg]. \tag{4.6}$$

Consequently, we can identify the tuple (β, Y) of predictable processes

$$\beta(t) = \alpha \quad \text{and} \quad Y(t, x) = \exp(\alpha x)$$

that characterizes the change of measure.

From Girsanov's theorem combined with Theorem II.4.15 in Jacod and Shiryaev (2003), we deduce that a PIIAC remains a PIIAC under the measure $\widetilde{\mathbb{P}}$, because the processes β and Y are deterministic and the resulting characteristics under $\widetilde{\mathbb{P}}$ satisfy Assumption ($\mathbb{AC}$).

As a consequence of Girsanov's theorem for semimartingales, we infer that $\widetilde{W}_t = W_t - \int_0^t \alpha\sigma_s \mathrm{d}s$ is a $\widetilde{\mathbb{P}}$-Brownian motion and $\widetilde{\nu} = Y\nu$ is the $\widetilde{\mathbb{P}}$ compensator of the jumps of L. Furthermore, as a corollary of Girsanov's theorem, we can calculate the canonical decomposition of L under $\widetilde{\mathbb{P}}$;

$$L_t = \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\widetilde{W}_s + \int_0^t\int_{\mathbb{R}} x(\mu^L - \widetilde{\nu})(\mathrm{d}s, \mathrm{d}x) \tag{4.7}$$

where

$$\widetilde{B}_t = \int_0^t \widetilde{b}_s \mathrm{d}s = (r - \delta)t + \left(\alpha - \frac{1}{2}\right)\int_0^t \sigma_s^2 \mathrm{d}s$$
$$+ \int_0^t\int_{\mathbb{R}} (\mathrm{e}^{-\alpha x} - \mathrm{e}^{(1-\alpha)x} + x)\widetilde{\nu}(\mathrm{d}s, \mathrm{d}x) \tag{4.8}$$

hence, its triplet of characteristics is $(\widetilde{B}, C, \widetilde{\nu})$. Define its dual process, $L^\star := -L$ and by Lemma 2.9, we get that its triplet is $(B^\star, C^\star, \nu^\star) = (-\widetilde{B}, C, -\widetilde{\nu})$. The canonical decomposition of $L^\star$ is

$$L_t^\star = -\int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s^\star + \int_0^t\int_{\mathbb{R}} x(\mu^{L^\star} - \nu^\star)(\mathrm{d}s, \mathrm{d}x) \tag{4.9}$$

and we can easily deduce that $\mathrm{e}^{(r-\delta)t} S_t^\star$ is not a $\widetilde{\mathbb{P}}$-martingale for $\alpha \neq 1$.

Adding the appropriate terms, we can re-write $L^\star$ as $L^\star := \mathcal{C}^* + \overline{L}$, where

$$\mathcal{C}^* = (1 - \alpha)\int_0^\cdot \sigma_s^2 \mathrm{d}s - \int_0^\cdot\int_{\mathbb{R}} (\mathrm{e}^{-\alpha x} - \mathrm{e}^{(1-\alpha)x} + 1 - \mathrm{e}^{-x})\widetilde{\nu}(\mathrm{d}s, \mathrm{d}x) \tag{4.10}$$

and $\overline{L}$ is such that $\mathrm{e}^{(r-\delta)t}\overline{S}_t$ is a $\widetilde{\mathbb{P}}$-martingale. The characteristic triplet of $\overline{L}$ is $(B^\star - \mathcal{C}^*, C, \nu^\star)$ and $\overline{S}_t = S_0^{-1}\exp \overline{L}_t$.

Therefore, we can conclude the proof

$$\begin{aligned}
\widetilde{V}_c &= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \widetilde{\mathbb{E}}\left[(\widetilde{K} - \widetilde{S}_T)^+\right]^\alpha \\
&= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \widetilde{\mathbb{E}}\left[(\widetilde{K} - \mathrm{e}^{\mathcal{C}_T^*}\overline{S}_T)^+\right]^\alpha \\
&= \mathrm{e}^{-\delta T} K^\alpha \mathcal{C}_T \mathrm{e}^{\alpha \mathcal{C}_T^*} \widetilde{\mathbb{E}}\left[(\mathcal{K} - \overline{S}_T)^+\right]^\alpha
\end{aligned}$$

where $\mathcal{K} = \widetilde{K}\mathrm{e}^{-\mathcal{C}_T^*} = K^{-1}\mathrm{e}^{-\mathcal{C}_T^*}$. □

Setting $\alpha = 1$ in the previous Theorem, we immediately get a symmetry between European plain vanilla call and put options.

Corollary 4.2. *Assuming that* $(\mathbb{M})$ *is in force and the asset price evolves as an exponential PIIAC, we can relate the European call and put option via the following symmetry:*

$$V_c\left(S_0, K; r, \delta, C, \nu\right) = K S_0 V_p\left(S_0^{-1}, K^{-1}; \delta, r, C, -f\nu\right) \tag{4.11}$$

where $f(x) = \mathrm{e}^x$.

This symmetry relating European and also American plain vanilla call and put options, in exponential Lévy models, was proved independently in Fajardo and Mordecki (2003). Schroder (1999) proved similar results in a general semimartingale model; however, using a Lévy or PIIAC as the driving motion allows for the explicit calculation of the distribution under the new measure.

A different symmetry, again relating European and American call and put options, in the Black-Scholes model was derived by Peskir and Shiryaev (2002), where they use the mathematical concept of *negative* volatility; their main result states that

$$V_c(S_T, K; \sigma) = V_p(-S_T, -K; -\sigma). \tag{4.12}$$

See also the discussion —and the corresponding cartoon— in Haug (2002).

In this framework, one can derive symmetry relationships between self-quanto and European plain vanilla options. This result is, of course, a special case of Theorem 6.4; nevertheless, we give a short proof since it simplifies considerably because the driving process is 1-dimensional.

The payoff of the self-quanto call and put option is

$$S_T(S_T - K)^+ \qquad \text{and} \qquad S_T(K - S_T)^+$$

respectively. Introduce the following notation for the value of the self-quanto call option

$$V_{qc}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\big[S_T(S_T - K)^+\big]$$

and similarly, for the self-quanto put option we set

$$V_{qp}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\big[S_T(K - S_T)^+\big].$$

Assumption $(\mathbb{EM})$ can be replaced by the following weaker assumption, which is the minimal condition necessary for the symmetry results to hold.

Assumption ($\mathbb{M}'$). The Lévy measures λ_t of the distribution of L_t satisfy

$$\int_0^{\bar{T}}\int_{D_-}|x|\lambda_t(\mathrm{d}x)\mathrm{d}t<\infty \qquad \text{and} \qquad \int_0^{\bar{T}}\int_{D_+}\mathrm{e}^{2x}\lambda_t(\mathrm{d}x)\mathrm{d}t<\infty.$$

Theorem 4.3. *Assume that the asset price evolves as an exponential PIIAC and* ($\mathbb{M}'$) *is in force. We can relate the self-quanto and European plain vanilla call and put options via the following symmetry:*

$$V_{qc}(S_0,K;r,\delta,C,\nu)=S_0\mathrm{e}^{\mathcal{C}_T^*}V_c(S_0,K^*;\delta,r,C,f\nu) \tag{4.13}$$

$$V_{qp}(S_0,K;r,\delta,C,\nu)=S_0\mathrm{e}^{\mathcal{C}_T^*}V_p(S_0,K^*;\delta,r,C,f\nu) \tag{4.14}$$

where $\mathcal{C}^*$ is given by (4.16), $K^*=K\mathrm{e}^{-\mathcal{C}_T^*}$ and $f(x)=\mathrm{e}^x$.

Proof. Expressing the value of the self-quanto call option in units of the *numéraire* as described in Section 3, we define a new measure $\widetilde{\mathbb{P}}$ via its Radon-Nikodym derivative given by (3.4) and the original valuation problem becomes

$$\widetilde{V}_{qc}=\mathrm{e}^{-\delta T}\widetilde{\mathbb{E}}\left[(S_T-K)^+\right]. \tag{4.15}$$

Now it suffices to calculate the characteristic triplet of L under $\widetilde{\mathbb{P}}$. Arguing as in the proof of Theorem 4.1, the density process η has the form (4.6) for $\alpha=1$ hence, the tuple (β,Y) of predictable processes that describes the change of measure is

$$\beta(t)=1 \qquad \text{and} \qquad Y(t,x)=\exp(x).$$

Therefore, L has the canonical decomposition under $\widetilde{\mathbb{P}}$

$$L_t=\int_0^t\widetilde{b}_s\mathrm{d}s+\int_0^t\sigma_s\mathrm{d}\widetilde{W}_s+\int_0^t\int_{\mathbb{R}}x(\mu^L-\widetilde{\nu})(\mathrm{d}s,\mathrm{d}x)$$

where

$$\widetilde{b}_t=r-\delta+\frac{\sigma_t^2}{2}+\int_{\mathbb{R}}(\mathrm{e}^{-x}-1+x)\mathrm{e}^x\lambda_t(\mathrm{d}x).$$

Notice that $\mathrm{e}^{(r-\delta)t}\mathrm{e}^{L_t}$ is not a $\widetilde{\mathbb{P}}$-martingale, but if we define L^* as

$$\begin{aligned}L_t^*:=(\delta-r)t&+\int_0^t\sigma_s\mathrm{d}\widetilde{W}_s+\int_0^t\int_{\mathbb{R}}x(\mu^L-\widetilde{\nu})(\mathrm{d}s,\mathrm{d}x)\\&-\int_0^t\frac{\sigma_t^2}{2}\mathrm{d}s-\int_0^t\int_{\mathbb{R}}(\mathrm{e}^x-1-x)\mathrm{e}^x\nu(\mathrm{d}s,\mathrm{d}x)\end{aligned}$$

then $\mathrm{e}^{(r-\delta)t}\mathrm{e}^{L_t^*}\in\mathcal{M}$. Next, we re-express L as $L=L^*+\mathcal{C}^*$, where

$$\mathcal{C}_T^*=\exp\left[2(r-\delta)T+\int_0^T\sigma_s^2\mathrm{d}s+\int_0^T\int_{\mathbb{R}}(\mathrm{e}^x+\mathrm{e}^{-x}-2)\mathrm{e}^x\nu(\mathrm{d}s,\mathrm{d}x)\right]. \tag{4.16}$$

By re-arranging the terms in (4.15), the result follows. $\square$

4.2. **Valuation of European options.** We outline a method for the valuation of vanilla options, based on bilateral Laplace transforms, that was developed in the PhD thesis of Sebastian Raible; see Chapter 3 in Raible (2000). The method is extremely fast and allows for the valuation not only of plain vanilla European derivatives, but also of more complex payoffs, such as digital, self-quanto and power options; in principle, every *European* payoff can be priced using this method. Moreover, a large variety of driving processes can be handled, including Lévy and additive processes.

The main idea of Raible's method is to represent the option price as a *convolution* of two functions and consider its bilateral Laplace transform; then, using the property that, *the Laplace transform of a convolution equals the product of the Laplace transforms of the factors*, we arrive at two Laplace transforms that are easier to calculate analytically than the original one. Inverting this Laplace transform yields the option price.

A similar method, in Fourier space, can be found in Lewis (2001). See also Carr and Madan (1999) for some preliminary results that motivated this research. Lee (2004) unifies and generalizes the existing Fourier-space methods and develops error bounds for the discretized inverse transforms.

We first state the necessary Assumptions regarding the distribution of the asset price process and the option payoff respectively.

- **(L1):** Assume that $\varphi_{L_T}(z)$, the extended characteristic function of L_T, exists for all $z \in \mathbb{C}$ with $\Im z \in I_1 \supset [0,1]$.
- **(L2):** Assume that $\mathbb{P}_{L_T}$, the distribution of L_T, is absolutely continuous w.r.t. the Lebesgue measure $\lambda\!\!\!\lambda^1$ with density ρ.
- **(L3):** Consider a European-style payoff function $f(S_T)$ that is integrable.
- **(L4):** Assume that $x \mapsto \mathrm{e}^{-Rx}|f(\mathrm{e}^{-x})|$ is bounded and integrable for all $R \in I_2 \subset \mathbb{R}$.

In order to price a European option with payoff function $f(S_T)$, we proceed as follows.

$$\begin{aligned} V = \mathrm{e}^{-rT}\mathbb{E}[f(S_T)] &= \mathrm{e}^{-rT}\int_\Omega f(S_T)\mathrm{d}\mathbb{P} \\ &= \mathrm{e}^{-rT}\int_\mathbb{R} f(S_0\mathrm{e}^x)\mathrm{d}\mathbb{P}_{L_T}(x) \\ &= \mathrm{e}^{-rT}\int_\mathbb{R} f(S_0\mathrm{e}^x)\rho(x)\mathrm{d}x \qquad (4.17) \end{aligned}$$

because of absolute continuity. Define $\zeta = -\log S_0$ and $g(x) = f(\mathrm{e}^{-x})$, then

$$V = \mathrm{e}^{-rT}\int_\mathbb{R} g(\zeta - x)\rho(x)\mathrm{d}x = \mathrm{e}^{-rT}(g * \rho)(\zeta) \qquad (4.18)$$

which is a convolution at point ζ. Applying bilateral Laplace transforms on both sides of (4.18) and using Theorem B.2 in Raible (2000), we get

$$\begin{aligned}\mathfrak{L}_V(z) &= \mathrm{e}^{-rT}\int\limits_{\mathbb{R}} \mathrm{e}^{-zx}(g*\rho)(x)\mathrm{d}x \\ &= \mathrm{e}^{-rT}\int\limits_{\mathbb{R}} \mathrm{e}^{-zx}g(x)\mathrm{d}x \int\limits_{\mathbb{R}} \mathrm{e}^{-zx}\rho(x)\mathrm{d}x \\ &= \mathrm{e}^{-rT}\mathfrak{L}_g(z)\mathfrak{L}_\rho(z) \qquad (4.19)\end{aligned}$$

where $\mathfrak{L}_h(z)$ denotes the bilateral Laplace transform of a function h at $z \in \mathbb{C}$, i.e. $\mathfrak{L}_h(z) := \int_{\mathbb{R}} \mathrm{e}^{-zx}h(x)\mathrm{d}x$. The Laplace transform of g is very easy to compute analytically and the Laplace transform of ρ can be expressed as the extended characteristic function φ_{L_T} of L_T. By numerically inverting this Laplace transform, we recover the option price.

The next Theorem gives us an explicit expression for the price of an option with payoff function f and driving PIIAC L.

Theorem 4.4. *Assume that* (**L1**)–(**L4**) *are in force and let* $g(x) := f(\mathrm{e}^{-x})$ *denote the modified payoff function of an option with payoff* $f(x)$ *at time* T. *Assume that* $I_1 \cap I_2 \neq \emptyset$ *and choose an* $R \in I_1 \cap I_2$. *Letting* $V(\zeta)$ *denote the price of this option, as a function of* $\zeta := -\log S_0$, *we have*

$$V(\zeta) = \frac{\mathrm{e}^{\zeta R - rT}}{2\pi}\int\limits_{\mathbb{R}} \mathrm{e}^{iu\zeta}\mathfrak{L}_g(R+iu)\varphi_{L_T}(iR-u)du, \qquad (4.20)$$

whenever the integral on the r.h.s. exists.

Proof. The claim can be proved using the arguments of the proof of Theorem 3.2 in Raible (2000); there, no explicit statement is made about the driving process L, hence it directly transfers to the case of a time-inhomogeneous Lévy process. □

Remark 4.5. In order to apply this method, validity of the necessary assumptions has to be verified. (**L1**), (**L3**) and (**L4**) are easy to certify, while (**L2**) is the most demanding one. Let us mention that the distributions underlying the most popular Lévy processes, such as the Generalized Hyperbolic Lévy motion (cf. Eberlein and Prause 2002), possess a known Lebesgue density.

Remark 4.6. The method of Raible for the valuation of European options can be applied to general driving processes that satisfy Assumptions (**L1**)–(**L4**). Therefore it can also be applied to stochastic volatility models based on Lévy processes that have attracted much interest lately; we refer to Barndorff-Nielsen and Shephard (2001), Eberlein et al. (2003) and Carr et al. (2003) for an account of different models.

4.3. **Valuation of American options.** The method of Raible presented in the previous section, can be used for pricing several types of European derivatives, but not path-dependent ones. The valuation of American options in Lévy driven models is quite a hard task and no analytical solution exists for the finite horizon case.

For perpetual American options, i.e. options with infinite time horizon, Mordecki (2002) derived formulas in the general case in terms of the law of the extrema of the Lévy process, using a random walk approximation to the process. He also provides explicit solutions for the case of a jump-diffusion with exponential jumps. Alili and Kyprianou (2005) recapture the results of Mordecki making use of excursion theory. Boyarchenko and Levendorskiĭ (2002c) obtained formulas for the price of the American put option in terms of the Wiener-Hopf factors and derive some more explicit formulas for these factors. Asmussen et al. (2004) find explicit expressions for the price of American put options for Lévy processes with two-sided phase-type jumps; the solution uses the Wiener-Hopf factorization and can also be applied to regime-switching Lévy processes with phase-type jumps.

For the valuation of finite time horizon American options one has to resort to numerical methods. Denote by $x = \ln S$ the log price, $\tau = T - t$ the time to maturity and $v(\tau, x) = f(\mathrm{e}^x, T - \tau)$ the time-t value of an option with payoff function $g(\mathrm{e}^x) = \phi(x)$. One approach is to use numerical schemes for solving the corresponding partial integro-differential inequality (PIDI),

$$\frac{\partial v}{\partial \tau} - \mathcal{A}v + rv \geq 0 \qquad \text{in } (0,T)\times\mathbb{R} \tag{4.21}$$

subject to the conditions

$$\begin{cases} v(\tau,x) \geq \phi(x), & \text{a.e. in } [0,T]\times\mathbb{R} \\ (v(\tau,x)-\phi(x))\left(\frac{\partial v}{\partial\tau} - \mathcal{A}v + rv\right) = 0, & \text{in } (0,T)\times\mathbb{R} \\ v(0,x) = \phi(x) \end{cases} \tag{4.22}$$

where

$$\begin{aligned} \mathcal{A}v(x) = \Big(r - \delta - \frac{\sigma^2}{2}\Big)\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\ + \int_{\mathbb{R}} \Big(v(x+y) - v(x) - (\mathrm{e}^y - 1)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\Big)\lambda(\mathrm{d}y) \end{aligned} \tag{4.23}$$

is the infinitesimal generator of the transition semigroup of L; see Matache et al. (2003, 2005) for all the details and numerical solution of the problem using wavelets. Almendral (2004) solves the problem numerically using implicit-explicit methods in case the CGMY is the driving process. Equation (4.21) is a backward PIDE in spot and time *to* maturity; Carr and Hirsa (2003) develop a forward PIDE in strike and time *of* maturity and solve it using finite-difference methods.

Another alternative is to employ Monte Carlo methods adapted for optimal stopping problems such as the American option; we refer to Rogers (2002) or Glasserman (2003). Këllezi and Webber (2004) constructed a lattice for Lévy driven assets and applied it to the valuation of Bermudan options. Levendorskiĭ (2004) develops a non-Gaussian analog of the method of lines and uses Carr's randomization method in order to formulate an approximate algorithm for the valuation of American options. Chesney and Jeanblanc (2004) revisit the perpetual American problem and obtain formulas for the optimal boundary when jumps are either only positive or only negative. Using these results, they approximate the finite horizon problem in a fashion similar to Barone-Adesi and Whaley (1987). Empirical tests

Option type	Asian Payoff	Lookback payoff
Fixed Strike call	$(\Sigma_T - K)^+$	$(M_T - K)^+$
Fixed Strike put	$(K - \Sigma_T)^+$	$(K - N_T)^+$
Floating Strike call	$(S_T - \Sigma_T)^+$	$(S_T - N_T)^+$
Floating Strike put	$(\Sigma_T - S_T)^+$	$(M_T - S_T)^+$

TABLE 5.1. Types of payoffs for Asian and Lookback options

show that this approximation provides good results only when the process is continuous at the exercise boundary.

5. EXOTIC OPTIONS

The work on this topic follows along the lines of Henderson and Wojakowski (2002); they proved an equivalence between the price of floating and fixed strike Asian options in the Black-Scholes model. We also refer to Vanmaele et al. (2005) for a generalization of these results to forward-start options and discrete averaging in the Black-Scholes model.

5.1. **Symmetry.** For Exotic options, the setting is that of (**P2**): $L^1 = L$ is the driving $\mathbb{R}$-valued PIIAC with triplet (B, C, ν), $S^1 = S^1_0 \exp L^1 = S$ and $S^2 = h(S)$ is a functional of S. The most prominent candidates for functionals are the maximum, the minimum and the (arithmetic) average; let $0 = t_1 < t_2 < \cdots < t_n = T$ be equidistant time points, then the resulting processes, in case of discrete monitoring, are

$$M_T = \max_{0\leq t_i\leq T} S_{t_i}, \quad N_T = \min_{0\leq t_i\leq T} S_{t_i} \quad \text{and} \quad \Sigma_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}.$$

Therefore, we can exploit symmetries between floating and fixed strike Asian and lookback options in this framework; the different types of payoffs of the Asian and lookback option are summarized in Table 5.1.

We introduce the following notation for the value of the floating strike call option, be it Asian or lookback

$$V_c(S_T, h(S); r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[(S_T - h(S)_T)^+\right]$$

and similarly, for the fixed strike put option we set

$$V_p(K, h(S); r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[(K - h(S)_T)^+\right];$$

similar notation will be used for the other two cases.

Now we can state a result that relates the value of floating and fixed strike options. Notice that because stationarity of the increments plays an important role in the proof, the result is valid only for Lévy processes.

Theorem 5.1. *Assuming that the asset price evolves as an exponential Lévy process, we can relate the floating and fixed strike Asian or lookback option via the following symmetry:*

$$V_c\left(S_T, h(S); r, \delta, \sigma^2, \lambda\right) = V_p\left(S_0, h(S); \delta, r, \sigma^2, -f\lambda\right) \tag{5.1}$$

$$V_p\left(h(S), S_T; r, \delta, \sigma^2, \lambda\right) = V_c\left(h(S), S_0; \delta, r, \sigma^2, -f\lambda\right) \tag{5.2}$$

where $f(x) = \mathrm{e}^x$.

Proof. We refer to the proof of Theorems 3.1 and 4.1 in Eberlein and Papapantoleon (2005). The minimal assumptions necessary for the results to hold are also stated there. □

Remark 5.2. These results also hold for *forward-start* Asian and lookback options, for *continuously* monitored options, for *partial* options and for Asian options on the *geometric* and *harmonic* average; see Eberlein and Papapantoleon (2005) for all the details. Note that the equivalence result is not valid for *in-progress* Asian options.

5.2. **Valuation of Barrier and Lookback options.** The valuation of barrier and lookback options for assets driven by general Lévy processes is another hard mathematical problem. The difficulty stems from the fact that (a) the distribution of the supremum or infimum of a Lévy process is not known explicitly and (b) the overshoot distribution associated with the passage of a Lévy process across a barrier is also not known explicitly.

Various authors have treated the problem in case the driving process is a spectrally positive/negative Lévy process, see for example Rogers (2000), Schürger (2002) and Avram et al. (2004). Kou and Wang (2003, 2004) have derived explicit formulas for the values of barrier and lookback options in a jump diffusion model where the jumps are double-exponentially distributed; they make use of a special property of the exponential distribution, namely the *memoryless* property, that allows them to explicitly calculate the overshoot distribution. Lipton (2002) derives similar formulas for the same model, making use of fluctuation theory.

Fluctuation theory and the Wiener-Hopf factorization of Lévy processes play a crucial role in every attempt to derive closed form solutions for the value of barrier and lookback options in Lévy driven models. Introduce the notation

$$M_t = \sup_{0\le s\le t} L_s \qquad \text{and} \qquad N_t = \inf_{0\le s\le t} L_s$$

and let θ denote a random variable exponentially distributed with parameter q, independent of L. Then, the celebrated *Wiener-Hopf factorization* of the Lévy process L states that

$$\mathbb{E}[\exp(izL_\theta)] = \mathbb{E}[\exp(izM_\theta)] \cdot \mathbb{E}[\exp(izN_\theta)] \tag{5.3}$$

or equivalently

$$q(q-\psi(z))^{-1} = \varphi_q^+(z)\cdot\varphi_q^-(z), \qquad z\in\mathbb{R}, \tag{5.4}$$

where ψ denotes the characteristic exponent of L. The functions φ_q^+ and φ_q^- have the following representations

$$\varphi_q^+(z) = \exp\Big[\int_0^\infty t^{-1}\mathrm{e}^{-qt}\mathrm{d}t\int_0^\infty(\mathrm{e}^{izx}-1)\mu^t(\mathrm{d}x)\Big] \tag{5.5}$$

$$\varphi_q^-(z) = \exp\Big[\int_0^\infty t^{-1}\mathrm{e}^{-qt}\mathrm{d}t\int_{-\infty}^0(\mathrm{e}^{izx}-1)\mu^t(\mathrm{d}x)\Big] \tag{5.6}$$

where $\mu^t(\mathrm{d}x) = \mathbb{P}_{L_t}(\mathrm{d}x)$ is the probability measure of L_t. These results where first proved for Lévy processes in Bingham (1975) —where an approximation of Lévy processes by random walks is employed— and subsequently by Greenwood and Pitman (1980) —where excursion theory is applied. See also the recent books by Sato (1999, Chapter 9) and Bertoin (1996, Chapter VI) respectively, for an account of these two methods.

Building upon these results, various authors have derived formulas for the valuation of barrier and lookback options; Boyarchenko and Levendorskiĭ (2002a) apply methods from potential theory and pseudodifferential operators to derive formulas for barrier and touch options, while Nguyen-Ngoc and Yor (2005) use a probabilistic approach based on excursion theory. Recently, Nguyen-Ngoc (2003) takes a similar probabilistic approach, motivated from Carr and Madan (1999) and derives quite simple formulas for the value of barrier and lookback options, that can be numerically evaluated with the use of Fourier inversion algorithms in 2 and 3 dimensions.

More specifically, let us denote by $V_c(M_T, K; T)$ the price of a fixed strike lookback option with payoff $(M_T - K)^+$, where $M_T = \max_{0\le t\le T} S_t$ and S is an exponential Lévy process. Choose $\gamma > 1$ and $\alpha > 0$ such that $\mathbb{E}[\mathrm{e}^{2L_1}] < \mathrm{e}^{r+\alpha}$ and set $V_c^{\alpha,\gamma}(M_T, K; T) = \mathrm{e}^{-\alpha T-\gamma k} V_c(M_T, K; T)$ where $k = \log(K/S_0)$. Then, we have the following result.

Proposition 5.3. *If $k > 0$, then for all $q, u > 0$ we have:*

$$\int_0^\infty \mathrm{e}^{-qT}\mathrm{d}T \int_0^\infty \mathrm{e}^{-uk} V_c^{\alpha,\gamma}(M_T, S_0\mathrm{e}^k; T)\mathrm{d}k =$$
$$= S_0 \frac{1}{q+r+\alpha} \frac{1}{z(z-1)} [\varphi^+_{q+r+\alpha}(i(z-1)) + (z-1)\varphi^+_{q+r+\alpha}(-i) - z] \quad (5.7)$$

where $z = u + \gamma$.

Proof. We refer to the proof of Proposition 3.9 in Nguyen-Ngoc (2003). □

The formula for the value of the floating strike lookback option is —as one could easily foresee— a lot more complicated than (5.7). Using the symmetry result of Theorem 5.1, this case can be dealt with via a change of the Lévy triplet and strike in the previous Proposition.

The Wiener-Hopf factors are not known explicitly in the general case and numerical computation could be extremely time-consuming. Boyarchenko and Levendorskiĭ (2002b) provide some more efficient formulas for the Wiener-Hopf factors of —what they call— regular Lévy processes of exponential type (RLPE); for the definition refer to chapter 3 in the above mentioned reference. Given that L is an RLPE, $\varphi_q^+(z)$ has an analytic continuation on the half plane $\Im z > \omega$ and

$$\varphi_q^+(z) = \exp\Big[\frac{z}{2\pi i}\int_{-\infty+i\omega}^{+\infty+i\omega} \frac{\ln(q+\psi(u))}{u(z-u)}du\Big]. \quad (5.8)$$

The family of RLPEs contains many popular —in mathematical finance— Lévy motions such as the Generalized Hyperbolic and Variance Gamma models, see Boyarchenko and Levendorskiĭ (2002b).

Discretely monitored options have received much less attention in the literature than their continuous time counterparts. Borovkov and Novikov (2002) use Fourier methods and Spitzer's identity to derive formulas for fixed strike lookback options.

Various numerical methods have been applied for the valuation of barrier and lookback options in Lévy driven models. Cont and Voltchkova (2005a, 2005b) study finite-difference methods for the solution of the corresponding PIDE, see also Matache et al. (2004). Ribeiro and Webber (2003, 2004) have developed fast Monte Carlo methods for the valuation of exotic options in models driven by the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) Lévy motions; their method is based on the construction of Gamma and Inverse Gaussian bridges respectively, to speed up the Monte Carlo simulation. The recent book of Schoutens (2003) contains a detailed account of Monte Carlo methods for Lévy processes, also allowing for stochastic volatility.

5.3. **Valuation of Asian and Basket options.** An explicit solution for the value of the arithmetic Asian or Basket option is not known in the Black-Scholes model and, of course, the situation is similar for Lévy models. The difficulty is that the distribution of the arithmetic sum of log-normal random variables —more generally random variables drawn from some log-infinitely divisible distribution— is not know in closed-form.

Večeř and Xu (2004) formulated a PIDE for all types of Asian options —including in-progress options— in a model driven by a process with independent increments (PII) or, more generally, a special semimartingale. Their derivation is based on the construction of a suitable self-financing trading strategy to replicate the average and then a change of numéraire —which is essentially the one we use— in order to reduce the number of variables in the equation. Their PIDE is relatively simple and can be solved using numerical techniques such as finite-differences.

Albrecher and Predota (2002, 2004) use moment-matching methods to derive approximate formulas for the value of Asian options in some popular Lévy models such as the NIG and VG models; they also derive bounds for the option price in these models. See also the survey paper Albrecher (2004) for a detailed account of the above mentioned results. Hartinger and Predota (2002) apply Quasi Monte-Carlo methods for the valuation of Asian options in the Hyperbolic model. Their method can be extended to the class of Generalized Hyperbolic Lévy motions, which contains the VG motion as a special case; see Eberlein and v. Hammerstein (2004). Benhamou (2002), building upon the work of Carverhill and Clewlow (1992), uses the Fast Fourier transform and a transformation of dependent variables into independent ones, in order to value discretely monitored fixed strike Asian options. As he points out, this method can be applied when the return distribution is fat-tailed, with Lévy processes being prominent candidates.

Henderson et al. (2004) derive an upper bound for in-progress floating strike Asian options in the Black-Scholes model, using the symmetry result of Henderson and Wojakowski (2002) and valuation methods for fixed strike ones. Their pricing bound relies on a model-dependent symmetry result and a model-independent decomposition of the floating-strike Asian option into a fixed-strike one and a vanilla option. Therefore, given the symmetry result of Theorem 5.1, their general methodology can also be applied to Lévy models.

Albrecher et al. (2004) derive static super-hedging strategies for fixed strike Asian options in Lévy models; these results where extended to Lévy models with stochastic volatility in Albrecher and Schoutens (2004). The method is based on super-replicating the Asian payoff with a portfolio of plain vanilla calls, using the following upper bound

$$\Big(\sum_{j=1}^{n} S_{t_j} - nK\Big)^+ \leq \sum_{j=1}^{n} (S_{t_j} - nK_j)^+ \tag{5.9}$$

and then optimizing the hedge, i.e. the choice of K_js, using results from co-monotonicity theory.

Similar ideas appear in Hobson et al. (2004) for the static super-hedging of Basket options. The payoff of the basket option is super-replicated by a portfolio of plain vanilla calls on each individual asset, using the upper bound

$$\Big(\sum_{i=1}^{n} w_i S_T^i - K\Big)^+ \leq \sum_{i=1}^{n} \big(w_i S_T^i - l_i K\big)^+ \tag{5.10}$$

where $l_i \geq 0$ and $\sum_{i=1}^{n} l_i = 1$; subsequently, the portfolio is optimized using co-monotonicity theory. Moreover, no distribution is assumed about the asset dynamics, since all the information needed are the marginal distributions which can be deduced from the volatility smile; we refer to Breeden and Litzenberger (1978). This is also observed by Albrecher and Schoutens (2004).

6. Margrabe-type options

In this section we derive symmetry results between options involving two assets —such as Margrabe or Quanto options— and European plain vanilla options; therefore, we generalize results by Margrabe (1978) and Fajardo and Mordecki (2003) to the case of time-inhomogeneous Lévy processes. Schroder (1999) provides similar results for semimartingale models; the advantage of using a Lévy process or a PIIAC instead of a semimartingale as the driving motion, is that the distribution of the asset returns under the new measure can be deduced from the distribution of the returns of each individual asset under the risk-neutral measure.

For Margrabe-type options, the setting is that of (**P3**): $L = (L^1, L^2)$ is the driving $\mathbb{R}^2$-valued PIIAC with triplet (B, C, ν) and $S = (S^1, S^2)$ is the asset price process. For convenience, we set

$$S_t^i = S_0^i \exp\big[(r - \delta^i)t + L_t^i\big], \qquad i = 1, 2, \tag{6.1}$$

modifying the characteristic triplet (B, C, ν) accordingly.

With Theorem 25.17 in Sato (1999) and Lemma 2.4, Assumption ($\mathbb{EM}$) guarantees the existence of the moment generating function M_{L_t} of L_t for $u \in \mathbb{C}^d$ such that $\Re u \in [-M, M]^d$. Furthermore, for $u \in \mathbb{C}^d$ with $\Re u \in$

$[-M,M]^d$, we have that

$$\begin{aligned} M_{L_t}(u) = \varphi_{L_t}(-iu) &= \mathbb{E}\left[\mathrm{e}^{\langle u,L_t\rangle}\right] \\ &= \exp\int_0^t \Big[\langle u,b_s\rangle + \frac{1}{2}\langle u,c_s u\rangle \\ &\qquad + \int_{\mathbb{R}^d}(\mathrm{e}^{\langle u,x\rangle}-1-\langle u,x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s. \end{aligned} \tag{6.2}$$

The next result will allow us to calculate the characteristic triplet of a 1-dimensional process, defined as a scalar product of a vector with the d-dimensional process L, from the characteristics of L under an equivalent change of probability measure.

Proposition 6.1. *Let L be a d-dimensional PIIAC with triplet (B,C,ν) under $\mathbb{P}$, let u,v be vectors in $\mathbb{R}^d$ and $v\in[-M,M]^d$. Moreover let $\widetilde{\mathbb{P}} \overset{\mathrm{loc}}{\sim} \mathbb{P}$, with density*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{\langle v,L_{\bar{T}}\rangle}}{\mathbb{E}[\mathrm{e}^{\langle v,L_{\bar{T}}\rangle}]}.$$

Then, the 1-dimensional process $\widehat{L} := \langle u,L\rangle$ is a $\widetilde{\mathbb{P}}$-PIIAC and its characteristic triplet is $(\widehat{B},\widehat{C},\widehat{\nu})$ with

$$\begin{aligned} \widehat{b}_s &= \langle u,b_s\rangle + \frac{1}{2}\big(\langle u,c_s v\rangle + \langle v,c_s u\rangle\big) + \int_{\mathbb{R}^d}\langle u,x\rangle\big(\mathrm{e}^{\langle v,x\rangle}-1\big)\lambda_s(\mathrm{d}x) \\ \widehat{c}_s &= \langle u,c_s u\rangle \\ \widehat{\lambda}_s &= \mathcal{T}(\kappa_s) \end{aligned}$$

where $\mathcal{T}$ is a mapping $\mathcal{T}:\mathbb{R}^d\to\mathbb{R}$ such that $x\mapsto\mathcal{T}(x)=\langle u,x\rangle$ and κ_s is a measure defined by

$$\kappa_s(A) = \int_A \mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x).$$

Proof. Because the density process (η_t) is given by $\eta_t = \mathrm{e}^{\langle v,L_t\rangle}\mathbb{E}[\mathrm{e}^{\langle v,L_t\rangle}]^{-1}$, using (6.2) we get

$$\begin{aligned} \widetilde{\mathbb{E}}\left[\mathrm{e}^{z\langle u,L_t\rangle}\right] &= \mathbb{E}\left[\mathrm{e}^{z\langle u,L_t\rangle}\eta_t\right] \\ &= \mathbb{E}\left[\mathrm{e}^{z\langle u,L_t\rangle}\mathrm{e}^{\langle v,L_t\rangle}\mathbb{E}\big[\mathrm{e}^{\langle v,L_t\rangle}\big]^{-1}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{\langle zu+v,L_t\rangle}\right]\mathbb{E}\left[\mathrm{e}^{\langle v,L_t\rangle}\right]^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \exp \int_0^t \Big[\langle zu+v, b_s\rangle + \frac{1}{2}\langle zu+v, c_s(zu+v)\rangle \\
&\qquad + \int_{\mathbb{R}^d} (\mathrm{e}^{\langle zu+v,x\rangle} - 1 - \langle zu+v, x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s \\
&\quad \times \exp \int_0^t -\Big[\langle v, b_s\rangle + \frac{1}{2}\langle v, c_s v\rangle \\
&\qquad + \int_{\mathbb{R}^d} (\mathrm{e}^{\langle v,x\rangle} - 1 - \langle v, x\rangle)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s \\
&= \exp \int_0^t \Big[z\big\{ \langle u, b_s\rangle + \frac{1}{2}\big(\langle u, c_s v\rangle + \langle v, c_s u\rangle\big) \\
&\qquad + \int_{\mathbb{R}^d} \langle u, x\rangle \big(\mathrm{e}^{\langle v,x\rangle} - 1\big)\lambda_s \mathrm{d}s \big\} + \frac{1}{2} z^2 \langle u, c_s u\rangle \\
&\qquad + \int_{\mathbb{R}^d} \big(\mathrm{e}^{z\langle u,x\rangle} - 1 - z\langle u, x\rangle\big)\mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s. \qquad (6.3)
\end{aligned}
$$

If we write κ_s for the measure on $\mathbb{R}^d$ given by

$$
\kappa_s(A) = \int_A \mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x) \qquad (6.4)
$$

$A \in \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{T}$ for the linear mapping $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}$ given by $\mathcal{T}(x) = \langle u, x\rangle$, then we get for the last term in the exponent of (6.3)

$$
\int_{\mathbb{R}^d} \big(\mathrm{e}^{z\langle u,x\rangle} - 1 - z\langle u, x\rangle\big)\mathrm{e}^{\langle v,x\rangle}\lambda_s(\mathrm{d}x) = \int_{\mathbb{R}} \big(\mathrm{e}^{zy} - 1 - zy\big)\mathcal{T}(\kappa_s)(\mathrm{d}y)
$$

by the change-of-variable formula. The resulting characteristics satisfy Assumption ($\mathbb{AC}$), thus the result follows. □

The valuation of options depending on two assets modeled by a 2-dimensional PIIAC can now be simplified —using the technique described in section 3 and Proposition 6.1— to the valuation of an option on a 1-dimensional asset. Subsequently, this option can be priced using bilateral Laplace transforms, as described in section 4.2.

The payoff of a Margrabe option, or option to exchange one asset for another, is

$$
\big(S_T^1 - S_T^2\big)^+
$$

and we denote its value by

$$
V_m(S_0^1, S_0^2; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\Big[\big(S_T^1 - S_T^2\big)^+\Big]
$$

where $\delta = (\delta^1, \delta^2)$. The payoff of the Quanto call and put option is

$$
S_T^1\big(S_T^2 - K\big)^+ \qquad \text{and} \qquad S_T^1\big(K - S_T^2\big)^+
$$

respectively and we will use the following notation for the value of the Quanto call option

$$V_{qc}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1\left(S_T^2 - K\right)^+\right]$$

and similarly for the Quanto put option

$$V_{qp}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1\left(K - S_T^2\right)^+\right].$$

The different variants of the Quanto option traded in Foreign Exchange markets are explained in Musiela and Rutkowski (1997). The payoff of a cash-or-nothing and a 2-dimensional asset-or-nothing option is

$$1\!\!1_{\{S_T > K\}} \qquad \text{and} \qquad S_T^1 1\!\!1_{\{S_T^2 > K\}}.$$

The holder of a 2-dimensional asset-or-nothing option receives one unit of asset S^1 at expiration, if asset S^2 ends up in the money; of course, this is a generalization of the (standard) asset-or-nothing option, where the holder receives one unit of the asset if it ends up in the money. We denote the value of the cash-or-nothing option by

$$V_{cn}(S_0, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[1\!\!1_{\{S_T > K\}}\right]$$

and the value of the 2-dimensional asset-or-nothing option by

$$V_{an}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathrm{e}^{-rT}\mathbb{E}\left[S_T^1 1\!\!1_{\{S_T^2 > K\}}\right].$$

Notice that in the the first case r, δ, C and ν correspond to a 1-dimensional driving process, while in the second case to a 2-dimensional one.

Theorem 6.2. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (2.3)–(2.5). We can relate the value of a Margrabe and a European plain vanilla option via the following symmetry:*

$$V_m(S_0^1, S_0^2; r, \delta, C, \nu) = \mathbb{E}[S_T^1]\mathrm{e}^{\widehat{\mathcal{C}}_T} V_p\big(S_0^2/S_0^1, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\big) \tag{6.5}$$

where $\mathcal{K} = \mathrm{e}^{-\widehat{\mathcal{C}}_T}$, $\widehat{\mathcal{C}}$ *is given by (6.9) and the characteristics* $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (-1, 1)$.

Proof. Expressing the value of the Margrabe option in units of the *numéraire*, we get

$$\begin{aligned}\widetilde{V} := \frac{V_m}{S_0^1} &= \frac{\mathrm{e}^{-rT}}{S_0^1}\mathbb{E}\left[\left(S_T^1 - S_T^2\right)^+\right] \\ &= \mathrm{e}^{-\delta^1 T}\mathbb{E}\left[\frac{\mathrm{e}^{-rT} S_T^1}{\mathrm{e}^{-\delta^1 T} S_0^1}\frac{\eta_T^1}{\eta_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right]\end{aligned}$$

where $\eta^1 = \mathbb{E}[\exp(L^1)] = \mathbb{E}[\exp\langle v, L\rangle]$, for $v = (1, 0)$ and using (6.1) we get

$$= \mathrm{e}^{-\delta^1 T}\eta_T^1\mathbb{E}\left[\frac{\mathrm{e}^{L_T^1}}{\eta_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right]. \tag{6.6}$$

Define a new measure $\widetilde{\mathbb{P}}$ via its Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{L_T^1}}{\mathbb{E}[\mathrm{e}^{L_T^1}]}$$

and the valuation problem takes the form

$$\widetilde{V} = \mathrm{e}^{-\delta^1 T}\eta_T^1 \widetilde{\mathbb{E}}\left[\left(1 - \widehat{S}_T\right)^+\right]$$

where, using (6.1) we get

$$\widehat{S}_t := \frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1}\mathrm{e}^{(\delta^1-\delta^2)t+L_t^2-L_t^1} =: \widehat{S}_0 \exp\left[(\delta^1-\delta^2)t + \widehat{L}_t\right] \tag{6.7}$$

and $\widehat{L} := L^2 - L^1 = \langle u, L\rangle$ for $u = (-1,1)$. The characteristic triplet of $\widehat{L}$, $(\widehat{B}, \widehat{C}, \widehat{\nu})$ under $\widetilde{\mathbb{P}}$, is given by Proposition 6.1 for $v = (1,0)$ and $u = (-1,1)$.

Observe that $\mathrm{e}^{(r-\delta^1)t}\widehat{S}_t$ is not a $\widetilde{\mathbb{P}}$-martingale. However, if we define

$$\begin{aligned}\overline{L}_t := (\delta^1 - r)t - \frac{1}{2}\int_0^t \widehat{c}_s \mathrm{d}s - \int_0^t\int_{\mathbb{R}}(\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s,\mathrm{d}x)\\ + \int_0^t \widehat{c}_s^{1/2}\mathrm{d}\widetilde{W}_s + \int_0^t\int_{\mathbb{R}} x(\mu^{\widehat{L}} - \widehat{\nu})(\mathrm{d}s,\mathrm{d}x)\end{aligned} \tag{6.8}$$

where $\widetilde{W}$ is a $\widetilde{\mathbb{P}}$-standard Brownian motion and $\mu^{\widehat{L}}$ is the random measure of jumps of $\widehat{L}$, then $\mathrm{e}^{(r-\delta^1)t}\mathrm{e}^{\overline{L}_t} \in \mathcal{M}$. Therefore, we re-express the exponent of (6.7) as $\widehat{L}_t + (\delta^1 - \delta^2)t = \overline{L}_t + \widehat{\mathfrak{C}}_t$ where

$$\widehat{\mathfrak{C}}_t = (r-\delta^2)t + \int_0^t \widehat{b}_s\mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s\mathrm{d}s + \int_0^t\int_{\mathbb{R}}(\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s,\mathrm{d}x) \tag{6.9}$$

and define $\overline{S}_t := \widehat{S}_0 \exp \overline{L}_t$.

Now the result follows, because

$$\begin{aligned}\widetilde{V} &= \mathrm{e}^{-\delta^1 T}\eta_T^1 \widetilde{\mathbb{E}}\left[\left(1 - \widehat{S}_T\right)^+\right]\\ &= \mathrm{e}^{-\delta^1 T}\eta_T^1 \widetilde{\mathbb{E}}\left[\left(1 - \overline{S}_T\mathrm{e}^{\widehat{\mathfrak{C}}_T}\right)^+\right]\\ &= \mathrm{e}^{-\delta^1 T}\eta_T^1\mathrm{e}^{\widehat{\mathfrak{C}}_T} \widetilde{\mathbb{E}}\left[\left(\mathrm{e}^{-\widehat{\mathfrak{C}}_T} - \overline{S}_T\right)^+\right].\end{aligned}$$

□

Theorem 6.3. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (2.3)–(2.5). We can relate the value of a Quanto and a European plain vanilla call option via the following symmetry:*

$$V_{qc}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathbb{E}[S_T^1]\mathrm{e}^{\widehat{\mathfrak{C}}_T} V_p\big(S_0^2, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\big) \tag{6.10}$$

where $\mathcal{K} = \mathrm{e}^{-\widehat{\mathfrak{C}}_T}$*, the constant* $\widehat{\mathfrak{C}}$ *is given by*

$$\widehat{\mathfrak{C}}_t = (2r - \delta^1 - \delta^2)t + \int_0^t \widehat{b}_s \mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s \mathrm{d}s + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x)$$

and the characteristics $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (0, 1)$*. A similar relationship holds for the Quanto and European plain vanilla put options.*

Proof. The proof follows along the lines of that of Theorem 6.2. □

Theorem 6.4. *Let Assumption* ($\mathbb{EM}$) *be in force and assume that the asset price evolves as an exponential PIIAC according to equations (2.3)–(2.5). We can relate the value of a 2-dimensional asset-or-nothing and a cash-or-nothing option via the following symmetry:*

$$V_{an}(S_0^1, S_0^2, K; r, \delta, C, \nu) = \mathbb{E}[S_T^1] V_{cn}\big(S_0^2, \mathcal{K}; \delta^1, r, \widehat{C}, \widehat{\nu}\big) \tag{6.11}$$

where $\mathcal{K} = K\mathrm{e}^{-\widehat{\mathfrak{C}}_T}$*, the constant* $\widehat{\mathfrak{C}}$ *is given by*

$$\widehat{\mathfrak{C}}_t = (2r - \delta^1 - \delta^2)t + \int_0^t \widehat{b}_s \mathrm{d}s + \frac{1}{2}\int_0^t \widehat{c}_s \mathrm{d}s + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\widehat{\nu}(\mathrm{d}s, \mathrm{d}x)$$

and the characteristics $(\widehat{C}, \widehat{\nu})$ *are given by Proposition 6.1 for* $v = (1, 0)$ *and* $u = (0, 1)$*. A similar relationship holds for the corresponding put options.*

Proof. The proof follows along the lines of that of Theorem 6.2. □

Remark 6.5. Notice that the factor $\mathbb{E}[S_T^1]$ is the forward price of the asset S^1, the *numéraire* asset.

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