Application of generalized hyperbolic Lévy motions to finance

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ABSTRACT   In standard mathematical finance Brownian motion plays the dominating role as driving process for modeling price movements. In order to achieve a better fit to real-life data it is, however, preferable to replace Brownian motion by a Lévy process. Generalized hyperbolic Lévy motions are processes which allow an almost perfect fit to financial data. We discuss in detail what the consequences for asset price modeling and interest rate theory are. We also touch on aspects of multivariate and intraday modeling.

1   Introduction

Since the overwhelming success of the valuation theory developed by Black, Scholes and Merton, stochastic models driven by Brownian motions have become predominant in finance (see e.g. Björk (1998)). To a certain degree this seems to be due to the fact that the technology for handling diffusion processes is widely known (see e.g. the books by Liptser and Shiryaev (1977) or Karatzas and Shreve (1991)), whereas stochastic analysis for wider classes of processes, in particular for semimartingales, is still a matter for specialists.

Lévy processes, the class of processes we consider here, are semimartingales but in order to treat them one does not need the full machinery of semimartingale theory. Some semimartingale components, like the continuous martingale part, the compensator of the random measure of jumps, and the drift part, simplify considerably. Nevertheless, Lévy processes constitute a very broad class of stochastic processes. They are generated by infinitely divisible distributions in the same way as Brownian motion is generated by the normal distribution. For distributions which have a finite first moment, the generated Lévy processes can be represented in the form

$$X_t = \sigma B_t + Z_t + \alpha t, \tag{1}$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion and $(Z_t)_{t \geq 0}$ a purely discontinuous martingale independent of $(B_t)_{t \geq 0}$. For further details on Lévy processes and an overview of the basic theory, we refer to the paper by K.-I. Sato (2001). It should be emphasized that Brownian motion itself is a Lévy process, but the class of Lévy processes which will be studied in more detail in the next sections, is in a certain sense at the opposite end of the spectrum. Whereas Brownian motion has continuous paths, apart from the drift term αt, generalized hyperbolic Lévy motions are purely discontinuous.

The purpose of this essay is to show that these processes are tractable from a mathematical point of view and lead to attractive models for financial time series.

2   Empirical facts

Financial data is typically provided in the form of a discrete time series $S_1, S_2, S_3, \ldots$, where S_n denotes the price (e.g. closing price or settlement price) of a certain security at time point
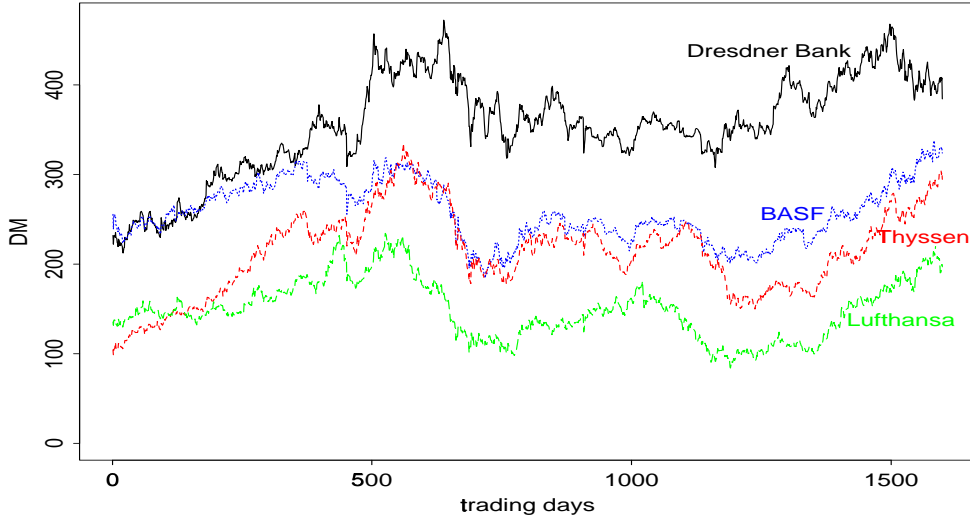
FIGURE 1. Daily stock prices (Dec. 30, 1987–May 24, 1994).

n (see Figure 1). As one of the consequences of the present transition from traditional floor trading to electronic markets, large data sets are produced routinely every day at the exchanges on an intraday level. These intraday records collect price quotes (bid and ask) as well as prices at which trades actually took place together with the corresponding time stamp. Time series corresponding to equidistant time points down to a certain grid size can easily be extracted from these intraday data sets.

In order to allow for the comparison of investments in different securities it is natural to look at relative price changes

$$Y_n = (S_n - S_{n-1})/S_{n-1}.$$

For a number of reasons most authors in the financial literature prefer a rate of return defined by log returns instead

$$Z_n = \log S_n - \log S_{n-1} = \log(1 + Y_n). \tag{2}$$

One reason is that the return over k periods, e.g. k days, is then the sum

$$Z_n + \cdots + Z_{n+k-1} = \log S_{n+k-1} - \log S_{n-1}.$$

We shall model asset prices by continuous time price processes $(S_t)_{t \geq 0}$. The discrete time series considered above corresponds to the values of the continuous time process at equidistant time points. Thus, a time series of daily prices will correspond to the values of the continuous time model at integer time points. For continuous time processes returns with continuous compounding, i.e. log returns, are the natural choice. Therefore, in the language of discrete time series we shall consider models of the form

$$S_n = S_o \exp\left(\sum_{i=1}^n Z_i\right).$$

Usually, real price series behave more in a multiplicative way than in an additive way. It is this property which is also reflected in models based on log returns. Numerically the difference
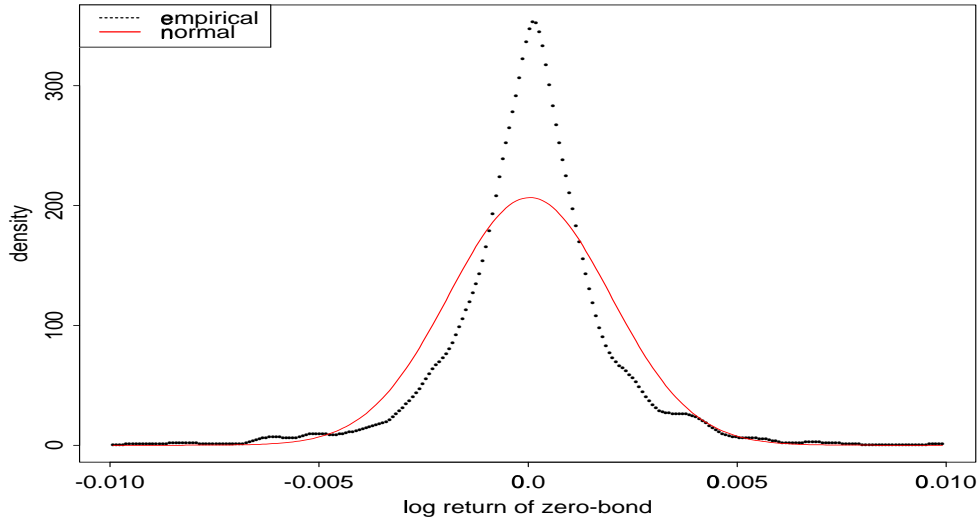
FIGURE 2. Log return density of German zero bonds (1985–95), 5 years to maturity.

between Y_n and Z_n in (2) is negligible since the functions $\log x$ and $x - 1$ have almost identical values near 1.

Looking at empirical densities of log returns from financial data one observes the following stylized features: compared to the normal distribution, which is used in classical models, there is more mass near the origin, less in the flanks and considerably more mass in the tails. This means that tiny price movements occur with higher frequency, small and middle sized movements with lower frequency and big changes are much more frequent than predicted by the normal law. Figure 2 shows a typical example. The points represent the kernel-smoothed density of log returns of zero coupon bonds with 5 years to maturity. The empirical data and the normal density have the same mean and variance. Given the rather small numerical values in the tails, the strong deviation there would only be visible, if one looked at the same graph with the y-axis in logarithmic scale. It is the class of generalized hyperbolic distributions which allows for an almost perfect statistical fit to these empirical distributions.

3   Generalized Hyperbolic Distributions

Generalized hyperbolic distributions were introduced by Barndorff-Nielsen (1977) in connection with the 'sand project', where in cooperation with geologists the physics of wind-blown sand was investigated. Their Lebesgue densities are given by:

$$d_{GH}(x; \lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta) \left(\delta^2 + (x - \mu)^2\right)^{(\lambda - \frac{1}{2})/2} \tag{3}$$
$$\times K_{\lambda - \frac{1}{2}}\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right) \exp(\beta(x - \mu)),$$

where

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\,\alpha^{\lambda - \frac{1}{2}}\,\delta^\lambda\,K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}$$

is the normalizing constant and K_ν denotes the modified Bessel function of the third kind with index ν. An integral representation of K_ν is given by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left(-\frac{1}{2}z\left(y + y^{-1}\right)\right) dy.$$

The densities above depend on five parameters: $\alpha > 0$ determines the shape, β with $0 \leq |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ is a scaling parameter comparable to σ in the normal distribution. Finally $\lambda \in \mathbb{R}$ characterizes certain sub-classes. It is essentially the heaviness of the tails which can be modified by changing λ. Compared to the normal density with only two parameters μ and σ, the class described by (3) is very flexible and therefore enables us to fit the empirical densities of log returns in an optimal way.

If X is a random variable which is generalized hyperbolically distributed with parameters $(\lambda, \alpha, \beta, \delta, \mu)$ then one can easily see that any affine transform $Y = aX + b$ with $a \neq 0$ is again generalized hyperbolically distributed with parameters $\widetilde{\lambda} = \lambda$, $\widetilde{\alpha} = |a|^{-1}\alpha$, $\widetilde{\beta} = |a|^{-1}\beta$, $\widetilde{\delta} = |a|\delta$, and $\widetilde{\mu} = a\mu + b$. From this result one can deduce that the following two alternative parametrizations are scale- and location-invariant, i.e. they do not change under affine transformations: $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$, $\varrho = \beta/\alpha$ and $\xi = (1 + \zeta)^{-1/2}$, $\chi = \xi\varrho$. Since $0 \leq |\chi| < \xi < 1$, the distributions parametrized by χ and ξ can be represented by the points of a triangle, the so called shape triangle.

Various special cases are of interest. For $\lambda = 1$ one gets the sub-class of hyperbolic distributions. Since $K_{1/2}(z) = (\pi/2z)^{1/2}e^{-z}$, the density (3) simplifies considerably. The argument x no longer appears in a Bessel function. The density is

$$d_H(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta\,K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right). \tag{4}$$

The name 'hyperbolic' is explained by this density. Taking the logarithm of d_H, instead of the parabola, which results from the normal distribution, we get a hyperbola from the term $\sqrt{\delta^2 + (x-\mu)^2}$. It is the hyperbolic case which was first used in finance (Eberlein and Keller (1995), see also Eberlein, Keller, and Prause (1998)). Figure 3 shows the effect of varying the shape parameter ζ in the case of the hyperbolic distribution. All other parameters are kept fixed.

Another special case is the normal inverse Gaussian distribution, which results when $\lambda = -1/2$. It was introduced to finance in Barndorff-Nielsen (1998) and has density

$$d_{NIG}(x) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right) \frac{K_1\left(\alpha\delta\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}}. \tag{5}$$

The normal inverse Gaussian distributions form the only sub-class of generalized hyperbolic laws which have the following convolution property

$$NIG(\alpha, \beta, \delta_1, \mu_1) * NIG(\alpha, \beta, \delta_2, \mu_2) = NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2). \tag{6}$$

In this sense it is close to the normal distribution, where means and variances of independent random variables add up too. (6) simplifies the numerics of the option pricing formula, which will be derived later and speeds up simulations of paths of processes.

Generalized hyperbolic distributions have a number of appealing analytic properties. Their moment generating function is given by

$$M_{GH}(u) = e^{\mu u}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2}\right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + u)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \tag{7}$$
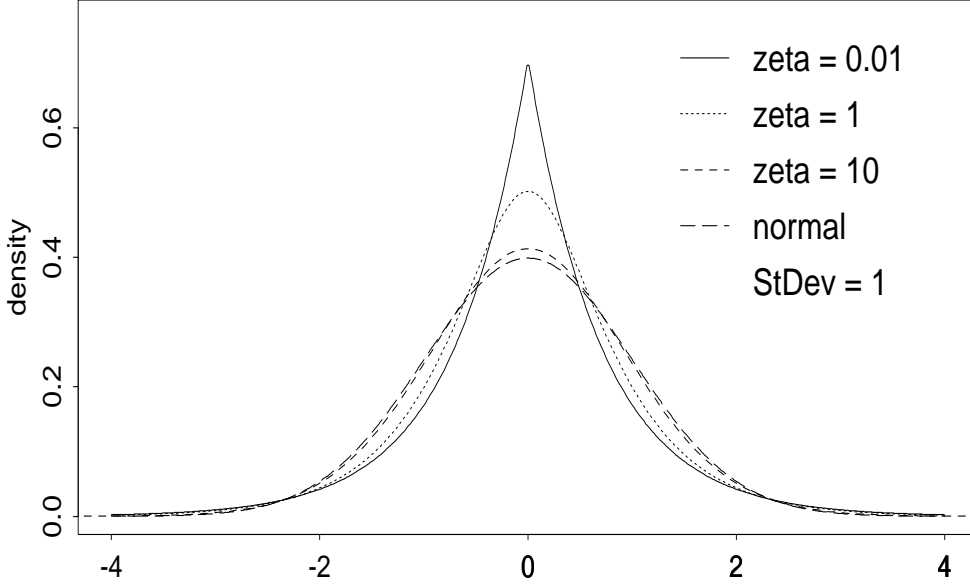
FIGURE 3. Hyperbolic densities with constant variance.

for $|\beta + u| < \alpha$. From this formula moments of any integer order can be derived. Using the parameter ζ defined above we derive the following expressions for the first moment

$$E[GH] = \mu + \frac{\beta\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \tag{8}$$

and for the variance

$$\mathrm{Var}[GH] = \frac{\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \beta^2\frac{\delta^4}{\zeta^2}\left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K_{\lambda+1}^2(\zeta)}{K_\lambda^2(\zeta)}\right). \tag{9}$$

Note that these expressions simplify considerably in the case of symmetry, i.e. for $\beta = 0$. Also the characteristic function ϕ_{GH} is easily obtained by exploiting the relation

$$\phi_{GH}(u) = M_{GH}(iu). \tag{10}$$

We shall need the Lévy-Khintchine representation of the characteristic function of generalized hyperbolic distributions. It is given by

$$\ln\left(\phi_{GH}(u)\right) = iu\,E[GH] + \int_{-\infty}^{+\infty}\left(e^{iux} - 1 - iux\right)g(x)dx \tag{11}$$

where the density $g(x)$ of the Lévy measure is

$$g(x) = \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp\left(-\sqrt{2y + \alpha^2}\,|x|\right)}{\pi^2 y\left(J_\lambda^2(\delta\sqrt{2y}) + Y_\lambda^2(\delta\sqrt{2y})\right)}\,dy + \lambda e^{-\alpha|x|}\right) \quad \text{if } \lambda \geq 0 \tag{12}$$

and

$$g(x) = \frac{e^{\beta x}}{|x|}\int_0^\infty \frac{\exp\left(-\sqrt{2y + \alpha^2}\,|x|\right)}{\pi^2 y\left(J_{-\lambda}^2(\delta\sqrt{2y}) + Y_{-\lambda}^2(\delta\sqrt{2y})\right)}\,dy \quad \text{if } \lambda < 0. \tag{13}$$

Here J_λ and Y_λ are the Bessel functions of the first and second kind, respectively.
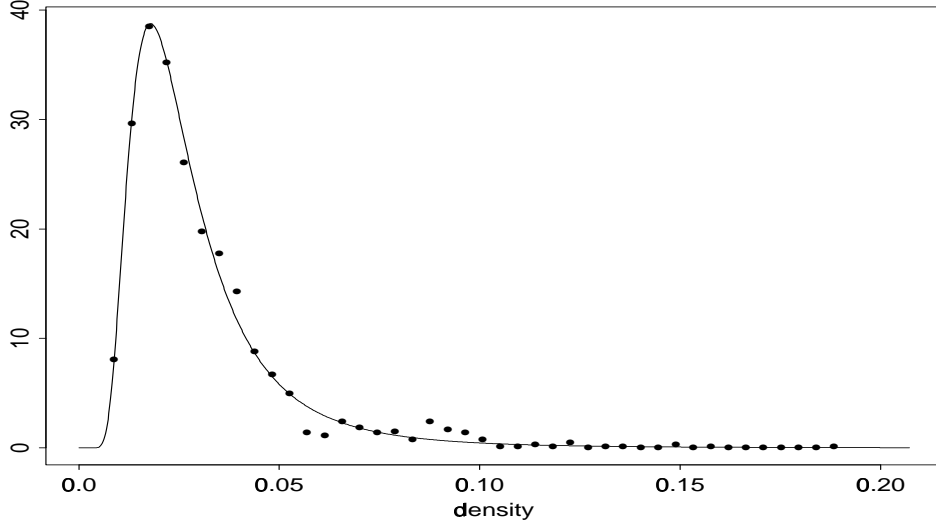
FIGURE 4. Empirical volatilities fitted by a generalized inverse Gaussian density.

4   Generalized Inverse Gaussian Distributions

There is an intuitive way for deriving the densities introduced above. Generalized hyperbolic distributions are variance-mean mixtures of normal distributions. Let d_{GIG} denote the density of a generalized inverse Gaussian distribution with parameters λ, δ, and γ, i.e.

$$d_{GIG}(x; \lambda, \delta, \gamma) = \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} \exp\left(-\frac{1}{2}\left(\frac{\delta^2}{x} + \gamma^2 x\right)\right) \quad \text{for } x > 0. \qquad (14)$$

The parameter space is given by

$$\begin{aligned} \delta \geq 0, \quad &\gamma > 0 \qquad \text{if } \lambda > 0, \\ \delta > 0, \quad &\gamma > 0 \qquad \text{if } \lambda = 0, \\ \delta > 0, \quad &\gamma \geq 0 \qquad \text{if } \lambda < 0. \end{aligned}$$

Then if $N(\mu + \beta y, y)$ denotes the normal distribution with mean $\mu + \beta y$ and variance y one can easily verify that

$$d_{GH}(x; \lambda, \alpha, \beta, \delta, \mu) = \int_0^{\infty} d_{N(\mu+\beta y, y)}(x)\, d_{GIG}(y; \lambda, \delta, \sqrt{\alpha^2 - \beta^2})\, dy. \qquad (15)$$

As a consequence one gets the following relation for moment generating functions

$$M_{GH}(u) = e^{\mu u} M_{GIG}\left(\frac{1}{2}u^2 + \beta u\right), \qquad (16)$$

where the generalized inverse Gaussian distribution on the right-hand side has parameters λ, δ, and $\gamma = \sqrt{\alpha^2 - \beta^2}$.

The points in Figure 4 show the empirical density of daily squared volatilities given by the German volatility index VDAX. By construction the VDAX measures an average implied

volatility where the time to maturity is roughly 45 days. The data covers the period from Jan. 2, 1992 to December 30, 1997. Figure 4 shows that the empirical distribution can be fitted by a generalized inverse Gaussian distribution.

Since Gamma distributions are often used in finance, in particular in the context of models for credit risk, let us point out that they represent the special case where $\lambda > 0$, $\delta = 0$, and $\gamma > 0$. The corresponding density reduces to

$$d_\Gamma(x) = \left(\frac{\gamma^2}{2}\right)^\lambda \frac{1}{\Gamma(\lambda)} x^{\lambda-1} \exp\left(-\frac{1}{2}\gamma^2 x\right) \quad \text{for } x > 0. \tag{17}$$

If one considers the parameters $\lambda < 0$, $\delta > 0$, and $\gamma = 0$, one gets the inverse Gamma distribution with density

$$d_{I\Gamma}(x) = \left(\frac{2}{\delta^2}\right)^\lambda \frac{1}{\Gamma(-\lambda)} x^{\lambda-1} \exp\left(-\frac{1}{2}\frac{\delta^2}{x}\right) \quad \text{for } x > 0. \tag{18}$$

For $\lambda = -1/2$ we get the following density

$$d_{IG}(x) = \left(\frac{\delta^2}{2\pi}\right)^{1/2} x^{-3/2} \exp\left(-\frac{\gamma^2}{2x}\left(x - \frac{\delta}{\gamma}\right)^2\right) \quad \text{for } x > 0. \tag{19}$$

This is the density of the inverse Gaussian distribution. It leads to the normal inverse Gaussian distribution introduced in (5).

5   Generalized Hyperbolic Lévy Motions

Barndorff-Nielsen and Halgreen (1977) showed that generalized hyperbolic distributions are infinitely divisible by proving infinite divisibility of the generalized inverse Gaussian distribution, which is used in the representation (15) as mixture of normals. Therefore every member of this family with parameters $(\lambda, \alpha, \beta, \delta, \mu)$ generates a Lévy process $(X_t)_{t\geq0}$, i.e. a process with stationary independent increments such that $X_0 = 0$ and $\mathcal{L}(X_1)$, the distribution of X_1, has density d_{GH}. We can choose a càdlàg version and call this process the generalized hyperbolic Lévy motion (with parameters $\lambda, \alpha, \beta, \delta, \mu$).

According to the construction, increments of length 1 have a generalized hyperbolic distribution, but in general none of the increments of length different from 1 has a distribution from the same class. This follows immediately from the explicit form of the characteristic function (see (7) and (10)) and the fact that the characteristic function ϕ_t of a distribution of an increment of length t is given by $(\phi_{GH})^t$.

The exception is the normal inverse Gaussian Lévy motion, i.e. the case $\lambda = -1/2$, since

$$\phi_{NIG}(u) = e^{i\mu u} \frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{e^{\delta\sqrt{\alpha^2-(\beta+iu)^2}}}. \tag{20}$$

The power t of this function produces parameters $t\delta$ and $t\mu$ for increments of length t.

The Lévy-Khintchine representation (11) – (13) has only a drift and a jump term, therefore $(X_t)_{t\geq0}$ does not have a continuous Gaussian component.

Analyzing the behavior of the densities g of the Lévy measure (12) – (13) more carefully for $x \to 0$ reveals that the Lévy measures have infinite mass in every neighborhood of the origin.

This means that the process $(X_t)_{t \geq 0}$ has an infinite number of small jumps in every finite time interval.

Let us consider the representation in (1) in more detail. By (7), $\mathcal{L}(X_1)$ has moments of any order, in particular $E[X_1] < \infty$. By stationarity $E[X_t] = t E[X_1]$, therefore we have the trivial decomposition

$$X_t = X_t - E[X_t] + t E[X_1]. \tag{21}$$

As mentioned above the martingale $(X_t - E[X_t])_{t \geq 0}$ does not have a continuous component. We write μ^X for the random measure of jumps associated with the process $(X_t)_{t \geq 0}$

$$\mu^X(\omega, dt, dx) = \sum_{s > 0} \mathbb{1}_{\{\Delta X_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta X_s(\omega))}(dt, dx),$$

where $\varepsilon_{(s, \Delta X_s(\omega))}$ denotes the Dirac measure at $(s, \Delta X_s(\omega))$. The compensator of the random measure of jumps is deterministic and of the form $du \, \nu(dx)$, since $(X_t)_{t \geq 0}$ has independent increments. The measure $\nu(dx)$ appearing here is the Lévy measure of the generalized hyperbolic distribution with Lebesgue density g as given in (12) – (13). With this notation (21) can be written in the form

$$X_t = \int_0^t \int_{\mathbb{R} \setminus \{0\}} x \left(\mu^X(\cdot, du, dx) - du \, \nu(dx) \right) + t E[X_1]. \tag{22}$$

Comparing this with the canonical representation of semimartingales in general as given on p. 84 in Jacod and Shiryaev (1987) (see also Theorem I.42 in Protter (1990)) we can see what particular case generalized hyperbolic Lévy motions are. First, since we have finite moments, we do not have to split off the big jumps. Second, the continuous martingale component, which would be a Brownian motion for Lévy processes, is zero. Third, the drift term is a linear process and finally the compensator of μ^X is of the simple product form $du \, \nu(dx)$.

For completeness let us mention that the generalized hyperbolic Lévy motion can also be introduced via subordination. This means that we use operational time or business time. Generalized inverse Gaussian distributions as introduced in Section 4 are infinitely divisible. Therefore they generate a Lévy process $(\tau(t))_{t \geq 0}$ such that $\tau(0) = 0$ and $\mathcal{L}(\tau(1))$ is given by d_{GIG} (see (14)).

Since d_{GIG} has only mass on $\mathbb{R}_+$, the increments of $\tau(t)$ are positive. Consequently $\tau(t)$ has increasing paths and is a subordinator. For a standard Brownian motion $(B_t)_{t \geq 0}$, independent of $\tau(t)$, we consider the process

$$X_t = \mu t + \beta \tau(t) + B_{\tau(t)}. \tag{23}$$

If the parameters of the generalized inverse Gaussian distribution defining $\tau(t)$ are those from the mixture representation (15), namely δ and $\gamma = \sqrt{\alpha^2 - \beta^2}$, $(X_t)_{t \geq 0}$ is the generalized hyperbolic Lévy motion with parameters $(\lambda, \alpha, \beta, \delta, \mu)$. This can be proved as follows. Write $\zeta_{GIG}(z) = \int_0^\infty e^{-zx} d_{GIG}(x) dx$ ($z \in \mathbb{C}$, Re $z > 0$) for the Laplace transform of the generalized inverse Gaussian distribution. Then

$$
\begin{aligned}
E[e^{iuX_t}] &= e^{iu\mu t} E[e^{iu\beta\tau(t)} E[e^{iuB_{\tau(t)}} | \tau(t)]] \\
&= e^{iu\mu t} E[e^{-(u^2/2 - iu\beta)\tau(t)}] \\
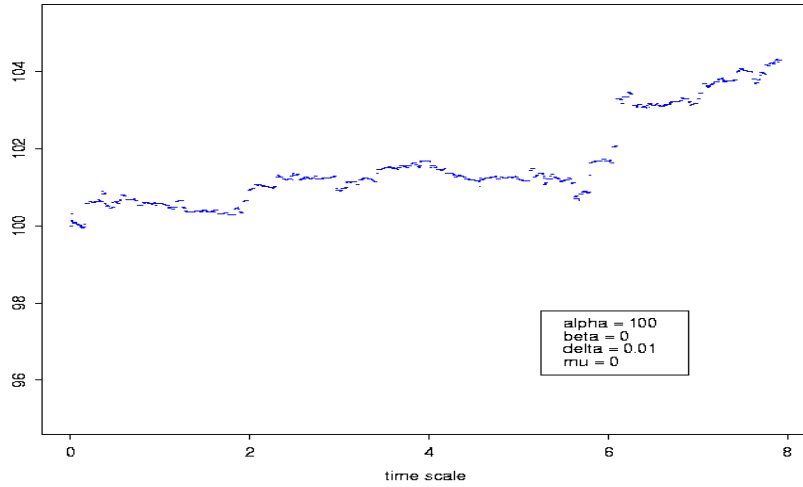&= \left(e^{iu\mu} \zeta_{GIG}(u^2/2 - iu\beta) \right)^t.
\end{aligned}
$$

FIGURE 5. Pathwise simulation of an asset price driven by a GH Lévy motion.

On the other hand if we write (16) not for the moment generating but for the characteristic function, we get the relation

$$\phi_{GH}(u) = e^{iu\mu}\zeta_{GIG}\left(u^2/2 - iu\beta\right). \tag{24}$$

This proves our statement. For a discussion of the subordination approach in finance see e.g. Hurst, Platen, and Rachev (1997).

6   The Asset Price Model

Given the empirical facts on log return distributions our goal is to model asset prices in such a way that log returns of the model produce exactly a generalized hyperbolic distribution along time intervals of a certain length, say one trading day. This can be achieved by setting

$$S_t = S_0 \exp(X_t) \tag{25}$$

where $(X_t)_{t \geq 0}$ is a generalized hyperbolic Lévy motion. Of course the generalized hyperbolic distribution with parameters $(\lambda, \alpha, \beta, \delta, \mu)$ which enters here is the one we estimated from the data set we want to model. Estimation is done by the maximum likelihood method. Let us note that if our data consists of price quotes of a particular stock (such as BMW) the estimated parameters $(\lambda, \alpha, \beta, \delta, \mu)$ are a label for this stock and the derived asset price process $(S_t)_{t \geq 0}$ is a specific process for this stock. Data from a different stock or an index, like the DAX, will yield different parameters and the corresponding model will be different.

It is necessary to emphasize that we discuss only the basic model here. This model assumes as the classical Black-Scholes model that log returns have independent increments, but there is a substantial literature observing that some of the features of the data are due to volatility clustering. More sophisticated versions of our model will have to take this into account. Any approach to stochastic volatility which has been investigated for models driven by Brownian motions can be implemented here as well.
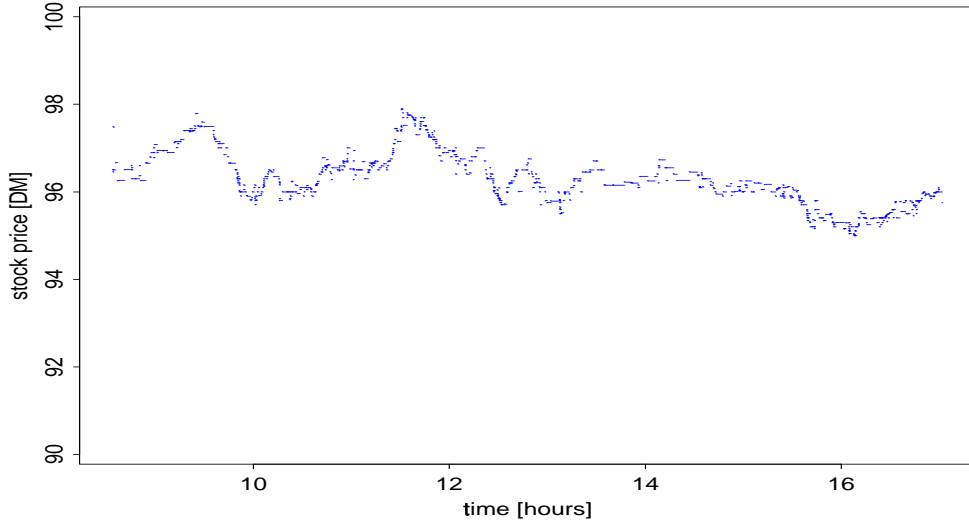
FIGURE 6. Intraday prices, Deutsche Bank, December 09, 1998.

By Itô's formula, $(S_t)_{t\geq 0}$ is the solution of the following stochastic differential equation

$$dS_t = S_{t-}(dX_t + e^{\Delta X_t} - 1 - \Delta X_t). \tag{26}$$

Here $\Delta X_t = X_t - X_{t-}$ denotes the jump at time t and X_{t-} the left hand limit of the path at time t. Of course the path properties of $(X_t)_{t\geq 0}$ carry over to $(S_t)_{t\geq 0}$. Thus $(S_t)_{t\geq 0}$ also changes its values by jumps only. Figure 5 shows a simulation for the case where $(X_t)_{t\geq 0}$ is a normal inverse Gaussian Lévy motion. Comparing this simulation with the evolution of stock prices on an intraday time scale (see Figure 6) one sees that our model (25) is able to catch to a certain degree the microstructure of price fluctuations. (25) is an incomplete model, i.e. there is a large set of equivalent martingale measures. In other words there are many candidates of measures for risk-neutral valuation. For an analysis and characterization of the set of equivalent martingale measures see Eberlein and Jacod (1997).

As far as the valuation of derivatives is concerned we choose the Esscher equivalent martingale measure. It is from a mathematical point of view the simplest candidate.

Remember that $M_{GH}(u)$ denotes the moment generating function. If r denotes the interest rate, there is a unique solution θ of the equation

$$r = \log M_{GH}(\theta+1) - \log M_{GH}(\theta). \tag{27}$$

For this unique θ consider the probability measure P^θ defined by the Radon-Nikodym density

$$dP^\theta = \exp\left(\theta X_t - t\log M_{GH}(\theta)\right) dP. \tag{28}$$

(27) is a different way of writing the martingale condition $S_0 = e^{-rt}E^\theta[S_t]$ (see Eberlein and Keller (1995)). Now if $H(S_T)$ is the payoff function of a derivative depending on the price $(S_t)_{t\geq 0}$ of the underlying at time T, then the value of the derivative is the discounted expectation of $H(S_T)$ with respect to P^θ, i.e.
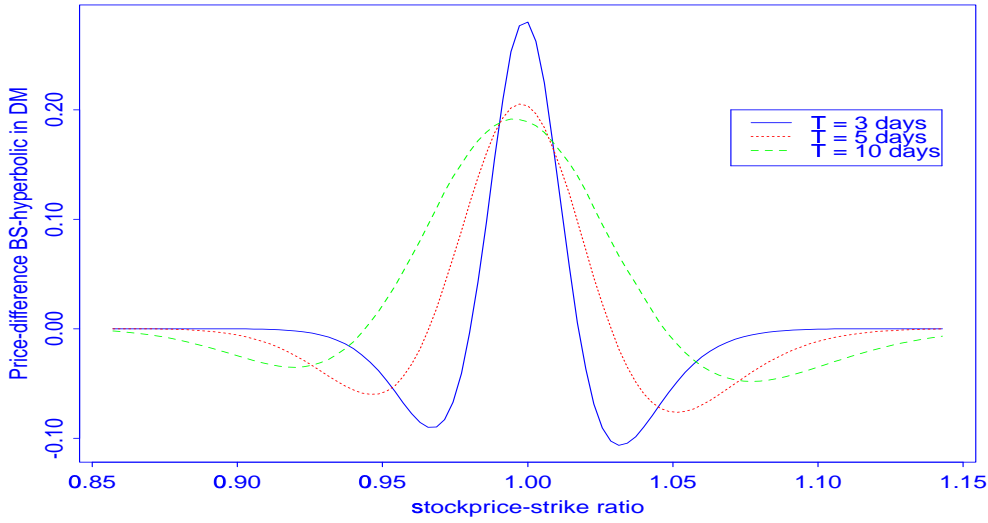
$$e^{-rT}E^\theta[H(S_T)]. \tag{29}$$

FIGURE 7. Difference of Black-Scholes minus hyperbolic option prices.

For a European call option with strike K the payoff is $H(S_T) = (S_T - K)^+$ and we obtain for the expectation under P^θ the following explicit expression which has the same structure as the Black-Scholes formula

$$S_0 \int_\gamma^\infty d_{GH}^{*T}(x; \theta+1)dx - e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta)dx, \tag{30}$$

where $\gamma = \ln(K/S_0)$ and

$$d_{GH}^{*t}(x; \theta) = \frac{e^{\theta x} d_{GH}^{*t}(x)}{\displaystyle\int_{-\infty}^\infty e^{\theta y} d_{GH}^{*t}(y)dy}$$

is the density of the distribution of X_t under the risk-neutral measure. The density d_{GH}^{*t} of the t-fold convolution of the generalized hyperbolic distribution can be computed by applying the Fourier inversion formula to the characteristic function. As pointed out in Section 5 this is simple in particular for normal inverse Gaussian distributions. Figure 7 shows the difference of the classical Black-Scholes price and the generalized hyperbolic price in the case $\lambda = 1$ for various maturities. We see the typical W-shape. At the money, where most of the volume is traded, the Black-Scholes price is too high. On the contrary, in the money and out of the money, the Black-Scholes price is too low. This is clear if one is aware of the fact that the Black-Scholes model does not see the risk of larger price movements. Note that very deep in the money and very deep out of the money the option price is essentially model independent. This follows from the fact that in these cases the integrals in (30) are close to 0 or 1.

There are various ways of looking at the performance of this new option pricing formula (30). An inconsistency of the classical Black-Scholes valuation is the so-called smile effect. This is the dependence of implicit volatilities on the moneyness, i.e. the stock price – strike ratio S_0/K, of the option. For a fixed time to maturity the resulting curve looks like a smiling mouth. As shown in Eberlein, Keller, and Prause (1998) for the hyperbolic model and in Eberlein and Prause (1998) for the generalized hyperbolic model, (30) leads to a reduction of the smile. The

reduction is stronger for the generalized hyperbolic model than for the hyperbolic or the normal inverse Gaussian model.

Another approach to pricing performance is to compare theoretical and observed prices directly. Comparing implicit volatilities only would not give a complete picture, since the same change in volatility has a larger effect on the price for an option with a longer time to maturity. This analysis was also done in the two papers mentioned above. Mispricing is somewhat reduced by (30).

As mentioned before there is a large set of equivalent martingale measures in the incomplete case. In particular it is shown in Eberlein and Jacod (1997) that the price range resulting from this set is the entire non-arbitrage interval. Although the question of which theoretical principle should be used to determine the martingale measure is discussed in a number of papers, there is still some arbitrariness for any particular choice. Consequently, instead of using some mathematical principle like minimizing the L^1- or L^2-distances or considering entropy one could ask the market for the right measure. This is the idea of the so-called statistical martingale measure (see e.g. Keller (1997), Eberlein, Keller and Prause (1998)). Denote by $\widehat{C} = \widehat{C}(S, K, T, r)$ the price of an option as observed at the exchange. Here S is the price of the underlying, K the strike, T the time to maturity and r the interest rate. For generalized hyperbolic parameters $\theta = (\lambda, \alpha, \beta, \delta, \mu)$ compute the prices $C(\theta)$ according to formula (30) and compare $\widehat{C}$ and $C(\theta)$. For a large number of prices $\widehat{C}_i$, such as $N \approx 40\,000$ quotes from the secondary market, we consider the functional

$$\min_{\theta} \sum_{i=1}^{N} \left(\widehat{C}_i - C_i(\theta) \right)^2 \tag{31}$$

under the restriction that the parameters θ describe a martingale measure. This is a numerically demanding optimization problem, but it can be solved in reasonable time. The statistical martingale measure P^{θ} derived as the solution of this problem is optimal in the sense that it minimizes the distance of hyperbolic prices derived from (30) to the real market prices.

7 Multivariate Modeling

In portfolio management one does not look at a single asset, but at a large universe of instruments. The problem is to select from this universe a dynamic portfolio such that the capital to be invested is used in an optimal way. Optimal means that either the risk is minimized given a certain return or the return maximized given a predetermined risk level. Various measures for risk and return can be used here. Since the instruments in the market such as stock, bonds, foreign currencies or derivatives are not independent, but typically highly correlated, it is natural to use multivariate distributions.

A straightforward way for introducing multivariate generalized hyperbolic distributions is via the mixture representation (15). Given n instruments, let Δ be a positive definite, symmetric $(n \times n)$-matrix. Δ can be chosen to have determinant 1. Now consider the multidimensional normal distribution with mean vector $\xi \in \mathbb{R}^n$ and covariance matrix $\sigma^2 \Delta$ for some $\sigma^2 > 0$. It has the density

$$d_{N(\xi, \sigma^2 \Delta)}(x) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(\det \sigma^2 \Delta)^{1/2}} \cdot \exp\left(-\frac{1}{2}(x - \xi)'(\sigma^2 \Delta)^{-1}(x - \xi) \right) \quad \text{for } x \in \mathbb{R}^n. \tag{32}$$

The Lebesgue density of the multivariate generalized hyperbolic distribution is then obtained by the formula

$$d_{GH,n}(x) = \int_0^\infty d_{N(\mu+(y\Delta)\beta, y\Delta)}(x)\, d_{GIG}(y) dy. \tag{33}$$

The parameters of the generalized inverse Gaussian distribution here are $\lambda \in \mathbb{R}$, $\delta > 0$, and $\gamma = \sqrt{\alpha^2 - \beta'\Delta\beta}$ for $\alpha \in \mathbb{R}$, and a vector $\beta \in \mathbb{R}^n$. As a result of the integration in (33) we get the following density

$$
\begin{aligned}
d_{GH,n}(x) = {} & a_n \left(\alpha^{-1} \sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)} \right)^{\lambda - \frac{n}{2}} \\
& \times K_{\lambda - \frac{n}{2}} \left(\alpha \sqrt{\delta^2 + (x-\mu)'\Delta^{-1}(x-\mu)} \right) \exp((x-\mu)'\beta) \quad \text{for } x \in \mathbb{R}^n
\end{aligned}
\tag{34}
$$

where the normalizing constant is

$$a_n = \frac{\left(\alpha^2 - \beta'\Delta\beta \right)^{\lambda/2}}{(2\pi)^{n/2} \delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta'\Delta\beta})}.$$

In addition to the positive definite, symmetric $(n \times n)$-matrix Δ with determinant 1 the parameters are $\lambda \in \mathbb{R}$, $\delta > 0$, $\alpha > 0$, and $\mu, \beta \in \mathbb{R}^n$ such that $\alpha^2 > \beta'\Delta\beta$. For $\lambda = (n+1)/2$ one obtains the multivariate hyperbolic and for $\lambda = -1/2$ the multivariate normal inverse Gaussian distribution.

Based on the density (34) the multivariate generalized hyperbolic Lévy motion is now constructed in the same way as the univariate one is constructed in Section 5.

8   Intraday Modeling

The data sets we use to calibrate generalized hyperbolic models typically consist of daily price quotes. On the other hand as indicated in Section 2, for some financial instruments time-stamped intraday data sets are available. Thus it is not a problem to analyze price changes along different time grids, e.g. one can consider one hour returns. A natural question is then, how the empirical distribution which one obtains from analyzing one hour returns compares to the corresponding increment distribution in the price process which was fitted to daily data. Traditional floor sessions last three hours, namely from 10.30 a.m. to 1.30 p.m. Our model would reflect intraday behavior in a perfect way, if the empirical one hour return distribution would be well approximated by the log returns from the model corresponding to time length 0.33. It is necessary to remember that one trading day corresponds to a time increment of length 1 in our model. Figure 8 shows the distance of the empirical one hour distribution to the log increments of the model corresponding to time lengths between 0 and 1. In other words we compare the empirical one hour returns with the elements $(\mu_t)_{0 \le t \le 1}$ of the convolution semigroup which is generated by the generalized hyperbolic distribution estimated from daily data. The distance is measured here as the Kolmogorov distance. If one took the χ^2-distance instead, the picture would be roughly the same. The minimum in Figure 8 is reached somewhere below $t = 0.3$ and not at $t = 0.33$. The reason for this is the following. The price change from one day to the next is not only the sum of the three one hour changes between 10.30 a.m. and 1.30 p.m. In addition there is the overnight jump, i.e. the change from the closing price of one day to the opening price of the next day. The distribution of the overnight jumps can be fitted as well by generalized hyperbolic distributions.
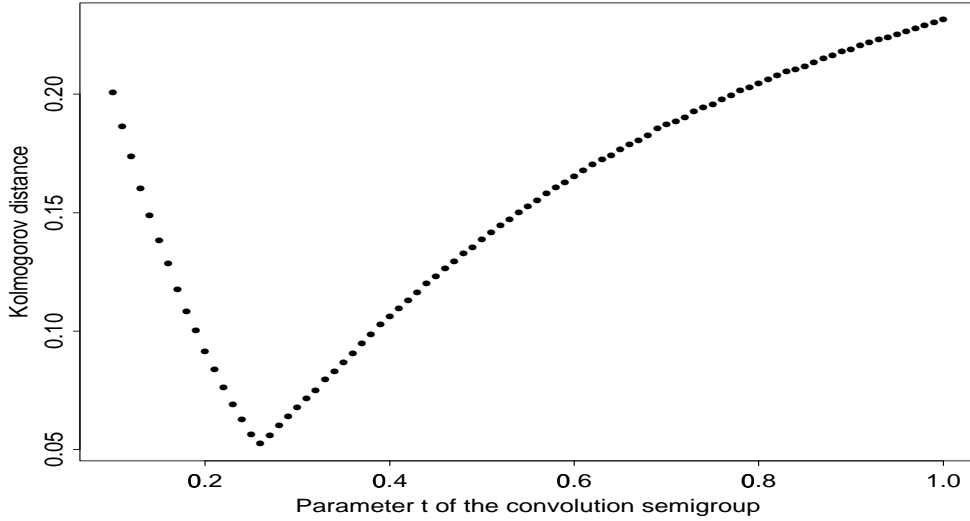
FIGURE 8. Distance of empirical one hour returns to the convolution semigroup.

Taking this overnight effect into account, Figure 8 shows that our model is highly consistent. The investigation of the microstructure of price fluctuations along a time grid corresponding to 60 minutes is somewhat arbitrary. Similar results are obtain for even denser time grids such as 30 minutes or 10 minutes returns. The latter grid makes only sense for instruments which are traded at a high frequency. Fitting a model on the basis of daily data and looking into the derived microstructure could be called a fold-down approach. We investigated also a fold-up approach where the model is fitted based on hourly price data for example. The distributions derived from this model are compared to empirical daily returns. The results of this intraday investigation will be published in a joint paper with Fehmi Özkan.

9   Interest Rate Theory

The shape of the distribution of returns is a key assumption in modeling financial time series. In the case of stock returns, the deviation from normality is widely known, although the various modifications and generalizations of classical models do not really take this into account. The deviation is much less known for returns from the bond market. It is for this reason that we chose interest rate data for Figure 2. Generalized hyperbolic distributions provide an excellent fit to these returns as shown in Figure 9. The theory we shall sketch in the following is for Lévy processes in general. We only require the existence of the moment generating function.

Historically it was always the short rate which was modelled as the basic process. Most of the models in the literature (see e.g. Björk (1998)) are so-called δ-root models

$$dr_t = (\theta(t) - ar_t)dt + \sigma r_t^\delta dB_t, \tag{35}$$

where $(B_t)_{t\geq 0}$ is again a standard Brownian motion. As Figure 10 shows, interest rates fluctuate around a long term mean $\theta(t)$. This behavior can be modeled through a proper choice of the drift term. The term $\theta(t) - ar_t$ has a mean reverting effect.
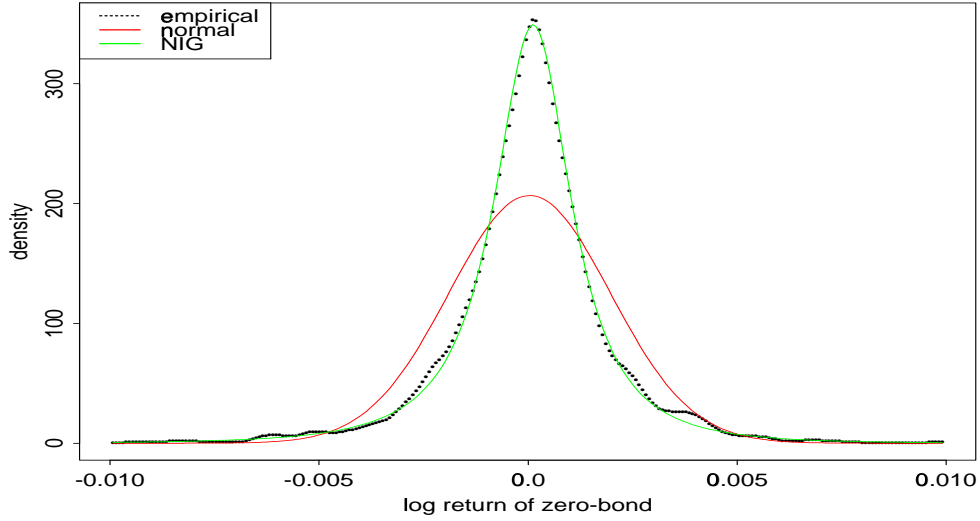
FIGURE 9. Fitted Log return density of German zero bonds (1985–95), 5 years to maturity.

The exponent δ in the random term forces the solution r_t to stay positive if $\delta \geq 1/2$. The case $\delta = 1/2$ is the widely used Cox-Ingersoll-Ross modell (Cox, Ingersoll, and Ross (1985)).

However, interest rates are not a one dimensional object. On the US bond market there are bonds with maturities between 0 and 30 years. The interest received depends on the time to maturity. Under normal conditions the interest paid for a bond with many years to maturity is higher than that for a bond which is close to maturity. Thus we have to consider a vector- or function-valued process. One assumes that there is a complete set of bonds with maturities T in the full time interval $[0, T^*]$. T^* can be 30 years for example. Mathematically it is simpler to consider zero coupon bonds. These are bonds which do not pay interest periodically, but given a certain face value which will be paid at maturity, the interest earned on this bond appears as a discount of the face value at the beginning.

Let $P(t, T)$ denote the price at time $t \in [0, T]$ of a zero coupon bond with maturity $T \in [0, T^*]$. We define

$$f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T). \tag{36}$$

$f(t, T)$ corresponds to the rate that one can contract for at time t on a riskless loan that begins at time T and is returned an instant later. $f(t, T)$ is called the (instantaneous) forward rate. Since

$$P(t, T) = \exp\left(-\int_t^T f(t, s)ds\right) \tag{37}$$

zero coupon bond prices and forward rates represent equivalent information. Note that the short rate r_t is contained in the forward rate structure since $r(t) = f(t, t)$. In 1992, Heath, Jarrow, and Morton introduced a model for the forward rate dynamics

$$df(t, T) = \alpha(t, T)dt + v(t, T)dB_t. \tag{38}$$

As explained before this is equivalent to modeling zero coupon bond prices in the form

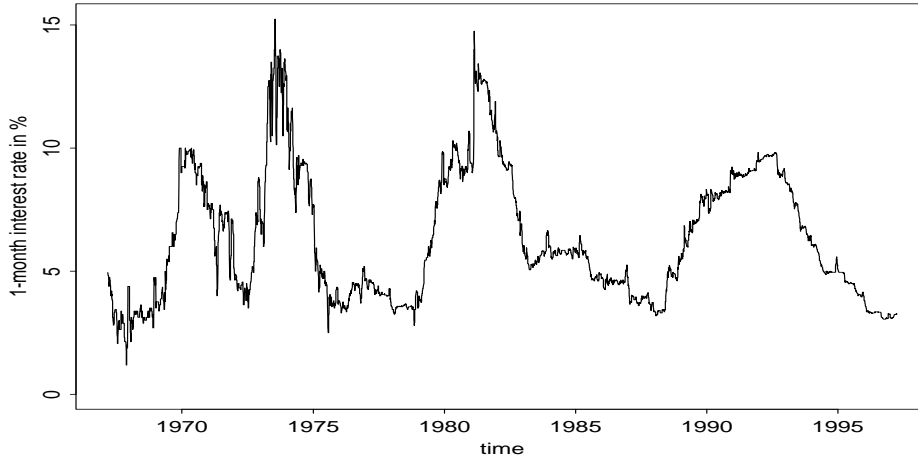$$dP(t, T) = P(t, T)(m(t, T)dt + \sigma(t, T)dB_t). \tag{39}$$

FIGURE 10. Interest rate fluctuations.

Under a risk-neutral measure the drift coefficient $m(t, T)$ is replaced by $r(t)$. Therefore the starting point for our generalization is the Heath-Jarrow-Morton model in the form

$$dP(t, T) = P(t, T)(r(t)dt + \sigma(t, T)dB_t). \tag{40}$$

As in the case of stock prices we do not replace the driving Brownian motion by a Lévy process in the stochastic differential equation. This would lead to a Doléans-Dade exponential as solution and thus to price processes which could have negative values as well. Instead we replace $(B_t)_{t \geq 0}$ in the solution of (40), which can be written in the form

$$P(t, T) = P(0, T) \exp \left(\int_0^t r(s)ds \right) \frac{\exp \left(\int_0^t \sigma(s, T)dB_s \right)}{E \left[\exp \left(\int_0^t \sigma(s, T)dB_s \right) \right]}. \tag{41}$$

Let $L = (L_t)_{t \geq 0}$ be a Lévy process and denote by F the Lévy measure of the infinitely divisible distribution $\mathcal{L}(L_1)$. In order to guarantee finiteness of the expectation in the denominator above in the case of general Lévy processes, we assume that

$$\int_{\{|x| > 1\}} \exp(vx)F(dx) < \infty \quad \text{for } |v| < (1 + \varepsilon)M, \tag{42}$$

where M is such that $0 \leq \sigma(s, T) \leq M$ for $0 \leq s \leq T \leq T^*$. Furthermore we assume that $P(0, T)$ as well as the (non-random) volatility $\sigma(s, T)$ are sufficiently smooth, namely at least C^2, and $\sigma(s, s) = 0$. Note that contrary to the case of stock price models in (39) it would not make any sense to consider a constant volatility σ. When a default-free bond approaches its maturity, the span of possible price fluctuations narrows. This is clear since at maturity the owner of the bond will get the face value with certainty. A volatility structure which is often used is the Vasicek structure given by

$$\sigma(t, T) = \frac{\widehat{\sigma}}{a} \left(1 - \exp(-a(T - t)) \right) \tag{43}$$

for parameters $\widehat{\sigma}$ and a. (43) defines a stationary volatility structure, i.e. $\sigma(t, T)$ depends only on the difference $T - t$.
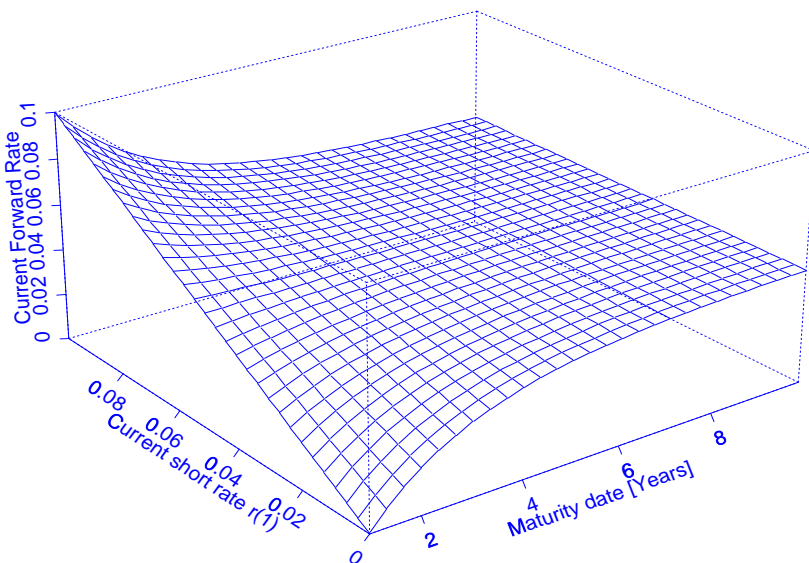
FIGURE 11. Forward rates driven by a centered and symmetric hyperbolic Lévy motion. Parameters: $\zeta = 0.01$; Vasicek volatility structure with $\hat{\sigma} = 0.015$, $a = 0.5$; flat initial forward rate with $f = 0.05$; current date = 1 year

The stochastic integral process $X_t = \int_0^t \sigma(s, T) dB_s$ is a process with independent increments as long as we consider a non-random volatility structure $\sigma(s, T)$, although it is no longer stationary. It can easily be seen that for processes $(X_t)_{t \geq 0}$ having independent increments, $(\exp(X_t)/E[\exp(X_t)])_{t \geq 0}$ is a martingale provided $E[\exp(X_t)]$ is finite. This fact explains why we wrote $P(t, T)$ in the form (41). The discounted bond price process

$$\left(\exp\left(-\int_0^t r(s) ds \right) P(t, T) \right)_{0 \leq t \leq T}$$

is a martingale. This is still true if we replace the Brownian motion $(B_t)_{t \geq 0}$ in (41) by a Lévy process $(L_t)_{t \geq 0}$ satisfying (42).

Going through the analysis as given in Eberlein and Raible (1999) one derives the following forward rate process

$$f(t, T) = f(0, T) + \int_0^t \theta'(\sigma(s, T)) \sigma_2(s, T) ds - \int_0^t \sigma_2(s, T) dL_s. \tag{44}$$

Here $\theta(u) = \log(E[\exp(uL_1)])$ denotes the logarithm of the moment generating function of $\mathcal{L}(L_1)$, θ' its derivative and $\sigma_2(s, T) = \frac{\partial}{\partial T} \sigma(s, T)$ as well as $f(0, T) = -\frac{\partial}{\partial T} \log P(0, T)$ the corresponding partial derivatives. Figure 11 shows the forward rates in the case where the driving process $(L_t)_{t \geq 0}$ is a hyperbolic Lévy motion. Finally the bond price process itself can be obtained in the form

$$P(t, T) = P(0, T) \exp\left[\int_0^t r(s) ds - \int_0^t \theta(\sigma(s, T)) ds + \int_0^t \sigma(s, T) dL_s \right]. \tag{45}$$

The classical Gaussian model follows from (45) if one chooses $\theta(u) = u^2/2$ and $L_s = B_s$. The

stochastic differential equation with solution (45) is

$$
\begin{aligned}
dP(t,T) \;=\; & P(t^-,T)\left(r(t)dt + \left(\frac{c}{2}\sigma^2(t,T) - \theta(\sigma(t,T))\right)dt\right. \\
& \left. + \sigma(t,T)dL_t + \left(e^{\sigma(t,T)\Delta L_t} - 1 - \sigma(t,T)\Delta L_t\right)\right).
\end{aligned}
\tag{46}
$$

According to the construction discounted bond prices are martingales in this term structure model. It is shown in Raible (1999) that the martingale measure is unique. As a consequence arbitrage-free prices of interest-rate derivatives are uniquely determined once the parameters of the driving Lévy process and the volatility structure are fixed, since these prices are given as expectations under an equivalent martingale measure. As an example let us consider a European call option on a bond maturing at time T with exercise date t and strike price K. The time 0 price of this option is $C(0,t,T,K) = E\left[\exp\left(-\int_0^t r(s)ds\right)(P(t,T) - K)^+\right]$. It is shown in Eberlein and Raible (1999) how one can evaluate this formula numerically.

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